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A circular elastic cylinder under its own weight

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ABSTRACT

An exact analysis of deformation and stress field in a finite circular elastic cylinder under its own weight is presented, with emphasis on the end effect. The problem is formulated on the basis of the state space formalism for axisymmetric deformation of a transversely isotropic body. Upon delineating the Hamiltonian characteristics of the formulation, a rigorous solution which satisfies the end conditions is determined by using eigenfunction expansion. The results show that the end effect is significant but confined to a local region near the base where the displacement and stress distributions are remarkably different from those according to the simplified solution that gives a uniaxial stress state. It is more pronounced in the cylinder with the bottom plane being perfectly bonded than in smooth contact with a rigid base.

1. Introduction

Analysis of an elastic cylinder under its own weight is often used as a simple example to illustrate the solution of a problem of elasticity requiring consideration of the body force (Timoshenko and Goodier, 1970; Sokolnikoff, 1956). Suppose that the top plane of the cylinder is free of traction and the bottom plane is supported in a suitable manner, by assuming a uniaxial stress state and disregarding the boundary conditions (BC) on the base plane, one can easily obtain a simplified solution that satisfies the traction-free BC on the top plane and on the lateral surface of the cylinder. The solution is valid provided that the base plane is free to distort in compliance with the uniaxial stress state. Yet, in real situations as in the case of a standing column, the base plane of the cylinder is not free to distort and the end conditions are prescribed rather than assigned a posterior. An exact analysis of this problem is not so simple as it appears.

When a cylinder is bonded or in contact with a rigid base, the interface between the bottom plane and the rigid base imposes certain constraints on the body, which dictate the end conditions of the problem. Obviously, the simplified solution assuming a uniaxial stress state fails to provide valid deformation and stress distribution on the base of the cylinder where the largest stress under the force of gravity is expected to occur. When the end condition on the bottom plane involves displacements, the end effects cannot be dismissed on the ground of Saint-Venant's principle. This seemingly simple and elementary problem has not been revisited and the applicability of the simplified solution is not clear.

In this paper, we present an exact analysis of the displacement and stress fields in a finite circular elastic cylinder under its own weight. When the force of gravity is the only external load and the material is transversely isotropic with the z axis being the axis of rotational symmetry, the problem is axially symmetric. It is well known that the axisymmetric problem can be formulated by introducing a displacement potential or a stress function (Love, 1944; Flügge, 1962; Lur'e, 1964; Lekhnitskii, 1981; Ding et al., 2006) to reduce the basic equations to a biharmonic equation, yet the BC in terms of the unknown function are intricate and working with the stress or displacement components often encounters great difficulty in satisfying the exact end conditions. In the present work, the equations of elasticity for axisymmetric deformation of a transversely isotropic body are formulated into a state equation and an output equation in terms of the state vector composed of the displacement and associated stress components as the dual variables. We show that the simplified solution is merely a particular solution of the state equation and it alone cannot satisfy the end conditions. To determine an exact solution of the problem, it is necessary to find a complete solution of the state equation and make it satisfy the prescribed BC.

While the characteristics of certain constant system matrices, such as a real symmetric matrix or a Hermitian matrix, are well known (Hildebrand, 1965; Pease, 1965), the characteristics of a Hamiltonian system matrix (Zhong, 1995, 2006) are less recognized, particularly when the matrix involves differential operators. In this paper, we delineate the Hamiltonian characteristics of the

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formulation for axisymmetric deformations of transversely isotropic and isotropic materials. Symplectic orthogonality relations of the eigenvectors are derived in particular. By means of eigenfunction expansion, an exact analysis of the displacement and stress fields in the cylinder under its own weight, with the bottom plane perfectly bonded or in smooth contact with the rigid base, is carried out. The end effect is evaluated with reference to the simplified solution.

2. State space formulation

Consider a finite circular cylinder of uniform cross-section resting on a rigid base under its own weight. Let the origin of the cylindrical coordinates (r, θ, z) be located at the center of the bottom plane, with the z axis pointing upward. When the force of gravity is the only external force acting on the circular cylinder with appropriate BC on the bottom plane, the deformation and stress fields are independent of θ. The basic equations for the problem are given below.

The strain–displacement relations

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\quad \varepsilon_{\theta\theta}=\frac{u_r}{r},\quad \varepsilon_{zz}=\frac{\partial u_z}{\partial z},\quad \varepsilon_{\theta z}=\frac{1}{2}\frac{\partial u_\theta}{\partial z},$$
$$\varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right),\quad \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right). \tag{1}$$

The equilibrium equations

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0,$$
$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{2\sigma_{r\theta}}{r}=0, \tag{2}$$
$$\frac{\partial\sigma_{rz}}{\partial r}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}-\rho g=0,$$

where ρg is the weight per unit volume of the body.

Constitutive equations for transversely isotropic materials

$$\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \sigma_{rz}\\ \sigma_{\theta z}\\ \sigma_{r\theta}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&0\\ c_{12}&c_{11}&c_{13}&0&0&0\\ c_{13}&c_{13}&c_{33}&0&0&0\\ 0&0&0&c_{44}&0&0\\ 0&0&0&0&c_{44}&0\\ 0&0&0&0&0&c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{zz}\\ 2\varepsilon_{rz}\\ 2\varepsilon_{\theta z}\\ 2\varepsilon_{r\theta}\end{bmatrix}, \tag{3}$$

where, it is understood that the indices 1, 2, 3 stand for r, θ, and z, respectively, c_{ij} are the elastic constants with reference to the cylindrical coordinates. With $c_{66}=(c_{11}-c_{12})/2$, there are five independent elastic constants for transversely isotropic materials. When the material is isotropic, $c_{33}=c_{11}, c_{13}=c_{12}$, $c_{44}=c_{66}=(c_{11}-c_{12})/2$.

On the basis of the state space formalism for anisotropic elasticity (Tarn, 2002a,b,c), Eqs. (1)–(3) can be formulated into two uncoupled sets of equations as follows:

$$\frac{\partial}{\partial z}\begin{bmatrix}u_r\\ u_z\\ r\sigma_{rz}\\ r\sigma_{zz}\end{bmatrix}=\begin{bmatrix}0&-\partial_r&r^{-1}c_{44}^{-1}&0\\ l_{31}&0&0&r^{-1}c_{33}^{-1}\\ l_{41}&0&0&l_{46}\\ 0&0&-\partial_r&0\end{bmatrix}\begin{bmatrix}u_r\\ u_z\\ r\sigma_{rz}\\ r\sigma_{zz}\end{bmatrix}+\begin{bmatrix}0\\0\\0\\r\rho g\end{bmatrix}, \tag{4}$$

$$\begin{bmatrix}r\sigma_{rr}\\ r\sigma_{\theta\theta}\end{bmatrix}=\begin{bmatrix}\tilde{c}_{11}(r\partial_r+1)-2c_{66}&\hat{c}_{13}\\ \tilde{c}_{12}(r\partial_r+1)+2c_{66}&\hat{c}_{13}\end{bmatrix}\begin{bmatrix}u_r\\ r\sigma_{zz}\end{bmatrix}; \tag{5}$$

$$\frac{\partial}{\partial z}\begin{bmatrix}u_\theta\\ r\sigma_{\theta z}\end{bmatrix}=\begin{bmatrix}0&r^{-1}c_{44}^{-1}\\ -c_{66}\left[\partial_r(r\partial_r)-r^{-1}\right]&0\end{bmatrix}\begin{bmatrix}u_\theta\\ r\sigma_{\theta z}\end{bmatrix}, \tag{6}$$

$$r\sigma_{r\theta}=c_{66}(r\partial_r-1)u_\theta, \tag{7}$$

where ∂_r, ∂_θ, ∂_z denote partial differentiation with respect to r, θ, and z, respectively, and

$$l_{31}=-\hat{c}_{13}(\partial_r+r^{-1}),\quad l_{41}=-\tilde{c}_{11}\left[\partial_r(r\partial_r)-r^{-1}\right],$$
$$l_{46}=-\hat{c}_{13}(\partial_r-r^{-1}),\quad \tilde{c}_{ij}=c_{ij}-c_{i3}c_{33}^{-1}c_{j3},\quad \hat{c}_{ij}=c_{ij}c_{33}^{-1}.$$

The cylindrical surface at $r=a$ and the upper plane at $z=l$ are traction-free:

$$[\sigma_{rr}\ \sigma_{r\theta}\ \sigma_{rz}]_{r=a}=[0\ 0\ 0], \tag{8}$$
$$[\sigma_{rz}\ \sigma_{\theta z}\ \sigma_{zz}]_{z=l}=[0\ 0\ 0] \tag{9}$$

The bottom plane is assumed to be perfectly bonded or in smooth contact with the rigid base. The end conditions at $z=0$ are

$$[u_r\ u_\theta\ u_z]_{z=0}=[0\ 0\ 0] \tag{10}$$

for the bottom plane perfectly bonded with the rigid base (the fixed end), and

$$[\sigma_{rz}\ \sigma_{\theta z}\ u_z]_{z=0}=[0\ 0\ 0] \tag{11}$$

for the bottom plane smoothly contact with the rigid base (the sliding-contact end).

By inspection, the solution of Eqs. (6) and (7) under the prescribed BC is trivial,

$$u_\theta=\sigma_{\theta z}=\sigma_{r\theta}=0, \tag{12}$$

so there is no need to treat Eqs. (6) and (7) in the sequel. We remark in passing that Eqs. (6) and (7) are useful for problems of torsion of circular cylinders (Tarn and Chang, 2008).

3. Particular solution

The particular solution of Eqs. (4) and (5) is

$$\begin{bmatrix}u_r\\ u_z\\ r\sigma_{rz}\\ r\sigma_{zz}\end{bmatrix}=\rho g\begin{bmatrix}a_{13}r(z-l)\\ \frac{1}{2}\left[a_{33}(z^2-2lz)-a_{13}r^2\right]\\ 0\\ r(z-l)\end{bmatrix},\quad \begin{bmatrix}r\sigma_{rr}\\ r\sigma_{\theta\theta}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}, \tag{13}$$

where the elastic compliances a_{ij} are related to the elastic constants c_{ij} by

$$a_{13}=\frac{c_{13}(c_{12}-c_{11})}{\Delta},\quad a_{33}=\frac{c_{11}^2-c_{12}^2}{\Delta},\quad \Delta=\begin{vmatrix}c_{11}&c_{12}&c_{13}\\ c_{12}&c_{11}&c_{13}\\ c_{13}&c_{13}&c_{33}\end{vmatrix}.$$

It can be verified that Eq. (13), which represents a uniaxial stress state, satisfies the equations of elasticity of transversely isotropic materials, the traction-free BC on the top plane and on the lateral surface of the cylinder, but not the end conditions at $z=0$. In fact, let $c_{33}=c_{11}$, $c_{13}=c_{12}$, Eq. (13) reduces to the simplified solution for an isotropic circular cylinder under its own weight (Timoshenko and Goodier, 1970; Sokolnikoff, 1956). To account for the end conditions, the first thing to do is to determine a complete solution of Eq. (4), which consists of the homogeneous solution in addition to the particular solution.

4. Complete solution

4.1. Transversely isotropic materials

We seek the homogeneous solution of Eq. (4) of the form

$$[u_r\ u_z\ r\sigma_{rz}\ r\sigma_{zz}]=[U_r\ U_z\ T_{rz}\ T_{zz}]e^{\mu z}, \tag{14}$$

where μ is a parameter to be determined; U_r, U_θ, T_{rz} and T_{zz} are unknown functions of r, which reduces Eq. (4) to

$$\begin{bmatrix} 0 & -d_r & r^{-1}c_{44}^{-1} & 0 \\ -\widehat{c}_{13}(d_r+r^{-1}) & 0 & 0 & r^{-1}c_{33}^{-1} \\ -\widetilde{c}_{11}[d_r(rd_r)-r^{-1}] & 0 & 0 & -\widehat{c}_{13}(d_r-r^{-1}) \\ 0 & 0 & -d_r & 0 \end{bmatrix}\begin{bmatrix} U_r \\ U_z \\ T_{rz} \\ T_{zz}\end{bmatrix} = \mu \begin{bmatrix} U_r \\ U_z \\ T_{rz} \\ T_{zz}\end{bmatrix}. \tag{15}$$

Eq. (15) can be solved by letting

$$[U_r\ U_r\ T_{rz}\ T_{zz}] = [C_1J_1(\lambda r)\ C_2J_0(\lambda r)\ C_3rJ_1(\lambda r)\ C_4rJ_0(\lambda r)], \tag{16}$$

where $J_0(\lambda r)$ and $J_1(\lambda r)$ are the Bessel functions of the first kind, of order *0* and *1*, respectively; λ and C_i (i = 1, 2, 3, 4) are constants to be determined.

Substituting Eq. (16) into Eq. (15), making use of the derivative formulas (Hildebrand, 1976; Watson, 1995)

$$\frac{d}{dr}J_0(\lambda r) = -\lambda J_1(\lambda r), \quad \frac{d}{dr}J_1(\lambda r) = \lambda J_0(\lambda r) - \frac{1}{r}J_1(\lambda r), \tag{17}$$

we arrive at

$$\begin{bmatrix} 0 & \lambda & c_{44}^{-1} & 0 \\ -\widehat{c}_{13}\lambda & 0 & 0 & c_{33}^{-1} \\ \widetilde{c}_{11}\lambda^2 & 0 & 0 & \widehat{c}_{13}\lambda \\ 0 & 0 & -\lambda & 0 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4\end{bmatrix} = \mu\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4\end{bmatrix}, \tag{18}$$

to which non-trivial solutions exist if and only if the determinant of the coefficient matrix equals to zero. This condition yields

$$c_{11}c_{44}\kappa^4 - [c_{11}c_{33} - c_{13}(c_{13}+2c_{44})]\kappa^2 + c_{33}c_{44} = 0 \quad (\kappa = \lambda/\mu) \tag{19}$$

which has four roots, given by

$$\kappa_1 = -\kappa_3 = \left[\frac{c_{11}c_{33} - c_{13}(c_{13}+2c_{44}) + \sqrt{(c_{11}c_{33}-c_{13}^2)\left[c_{11}c_{33}-(c_{13}+2c_{44})^2\right]}}{2c_{11}c_{44}}\right]^{1/2}, \tag{20}$$

$$\kappa_2 = -\kappa_4 = \left[\frac{c_{11}c_{33} - c_{13}(c_{13}+2c_{44}) - \sqrt{(c_{11}c_{33}-c_{13}^2)\left[c_{11}c_{33}-(c_{13}+2c_{44})^2\right]}}{2c_{11}c_{44}}\right]^{1/2}. \tag{21}$$

To each $\kappa_j(=\lambda_j/\mu)$ there corresponds a solution of Eq. (18) determined within a constant A_j as

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4\end{bmatrix} = A_j\begin{bmatrix} \kappa_j^2c_{44} - c_{33} \\ \kappa_j(c_{44}+c_{13}) \\ -c_{44}\mu(\kappa_j^2c_{13}+c_{33}) \\ c_{44}\mu\kappa_j(\kappa_j^2c_{13}+c_{33})\end{bmatrix}, \quad (j = 1,2,3,4) \tag{22}$$

It can be shown on account of positive-definiteness of the stiffness matrix that κ_j (j = 1, 2, 3, 4) cannot be purely imaginary. Since $J_0(-\kappa_j\mu r) = J_0(\kappa_j\mu r)$ and $J_1(-\kappa_j\mu r) = -J_1(\kappa_j\mu r)$, there are two linearly independent solutions derivable from Eqs. (16)–(22) The complete solution of Eqs. (4) and (5) is

$$\begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz}\end{bmatrix} = \sum_{j=1}^{2} A_je^{\mu z}\begin{bmatrix} f_1(\kappa_j,\mu,r) \\ f_2(\kappa_j,\mu,r) \\ f_3(\kappa_j,\mu,r) \\ f_4(\kappa_j,\mu,r)\end{bmatrix} + \rho g\begin{bmatrix} a_{13}r(z-l) \\ \frac{1}{2}\left[a_{33}(z^2-2lz) - a_{13}r^2\right] \\ 0 \\ r(z-l)\end{bmatrix}, \tag{23}$$

$$\begin{bmatrix} r\sigma_{rr} \\ r\sigma_{\theta\theta}\end{bmatrix} = \sum_{j=1}^{2} A_je^{\mu z}\begin{bmatrix} f_5(\kappa_j,\mu,r) \\ f_6(\kappa_j,\mu,r)\end{bmatrix}, \tag{24}$$

where

$$\begin{bmatrix} f_1(\kappa_j,\mu,r) \\ f_2(\kappa_j,\mu,r) \\ f_3(\kappa_j,\mu,r) \\ f_4(\kappa_j,\mu,r)\end{bmatrix} = \begin{bmatrix} (\kappa_j^2c_{44}-c_{33})J_1(\kappa_j\mu r) \\ \kappa_j(c_{44}+c_{13})J_0(\kappa_j\mu r) \\ -c_{44}\mu(\kappa_j^2c_{13}+c_{33})rJ_1(\kappa_j\mu r) \\ c_{44}\mu\kappa_j(\kappa_j^2c_{13}+c_{33})rJ_0(\kappa_j\mu r)\end{bmatrix},$$

$$f_5(\kappa_j,\mu,r) = \left[c_{44}(\kappa_j^2c_{11}+c_{13}) - \widetilde{c}_{11}c_{33}\right]\kappa_j\mu rJ_0(\kappa_j\mu r) - 2c_{66}(\kappa_j^2c_{44}-c_{33})J_1(\kappa_j\mu r),$$

$$f_6(\kappa_j,\mu,r) = \left[c_{44}(\kappa_j^2c_{12}+c_{13}) - \widetilde{c}_{12}c_{33}\right]\kappa_j\mu rJ_0(\kappa_j\mu r) + 2c_{66}(\kappa_j^2c_{44}-c_{33})J_1(\kappa_j\mu r).$$

4.2. Isotropic materials

The isotropic material is a special case of transversely isotropic materials. With $c_{33} = c_{11}$, $c_{13} = c_{12}$, $c_{44} = c_{66} = (c_{11} - c_{12})/2$ for isotropic materials, the parameters κ_j given by Eqs. (20) and (21) are

$$\kappa_1 = \kappa_2 = -\kappa_3 = -\kappa_4 = 1. \tag{25}$$

Since $J_0(-\mu r) = J_0(\mu r)$ and $J_1(-\mu r) = -J_1(\mu r)$, only one eigenvector is determined from Eqs. (23) and (24), which is

$$\begin{bmatrix} g_{11}(\mu,r) \\ g_{21}(\mu,r) \\ g_{31}(\mu,r) \\ g_{41}(\mu,r)\end{bmatrix} = \begin{bmatrix} f_1(\kappa_j,\mu,r) \\ f_2(\kappa_j,\mu,r) \\ f_3(\kappa_j,\mu,r) \\ f_4(\kappa_j,\mu,r)\end{bmatrix}\Bigg|_{\kappa_j=1} = \begin{bmatrix} -J_1(\mu r) \\ J_0(\mu r) \\ -(c_{11}-c_{12})\mu rJ_1(\mu r) \\ (c_{11}-c_{12})\mu rJ_0(\mu r)\end{bmatrix}. \tag{26}$$

Knowing that a linear combination of the eigenvectors for $\kappa_2 \neq \kappa_1$ is the eigenvector of the linear system as well, thus, by letting $\kappa_2 \to \kappa_1 = 1$ and through a limiting process, we obtain the other linearly independent eigenvector:

$$\begin{bmatrix} g_{12}(\mu,r) \\ g_{22}(\mu,r) \\ g_{32}(\mu,r) \\ g_{42}(\mu,r)\end{bmatrix} = \frac{d}{d\kappa}\begin{bmatrix} f_1(\kappa,\mu,r) \\ f_2(\kappa,\mu,r) \\ f_3(\kappa,\mu,r) \\ f_4(\kappa,\mu,r)\end{bmatrix}\Bigg|_{\kappa=1} = \begin{bmatrix} \frac{3c_{11}-c_{12}}{c_{11}+c_{12}}J_1(\mu r) - \mu rJ_0(\mu r) \\ J_0(\mu r) - \mu rJ_1(\mu r) \\ (c_{11}-c_{12})\mu r\left[\frac{c_{11}-c_{12}}{c_{11}+c_{12}}J_1(\mu r) - \mu rJ_0(\mu r)\right] \\ (c_{11}-c_{12})\mu r\left[\frac{c_{11}+3c_{12}}{c_{11}+c_{12}}J_0(\mu r) - \mu rJ_1(\mu r)\right]\end{bmatrix}. \tag{27}$$

As a a result, the complete solution of Eqs. (4) and (5) for the isotropic material is

$$\begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz}\end{bmatrix} = \sum_{j=1}^{2} A_je^{\mu z}\begin{bmatrix} g_{1j}(\mu,r) \\ g_{2j}(\mu,r) \\ g_{3j}(\mu,r) \\ g_{4j}(\mu,r)\end{bmatrix} + \rho g\begin{bmatrix} a_{13}r(z-l) \\ \frac{1}{2}\left[a_{33}(z^2-2lz) - a_{13}r^2\right] \\ 0 \\ r(z-l)\end{bmatrix}, \tag{28}$$

$$\begin{bmatrix} r\sigma_{rr} \\ r\sigma_{\theta\theta}\end{bmatrix} = \sum_{j=1}^{2} A_je^{\mu z}\begin{bmatrix} g_{5j}(\mu,r) \\ g_{6j}(\mu,r)\end{bmatrix}, \tag{29}$$

where

$$\begin{bmatrix} g_{51}(\mu,r) \\ g_{61}(\mu,r) \end{bmatrix} = (c_{11}-c_{12}) \begin{bmatrix} J_1(\mu r) - \mu r J_0(\mu r) \\ -J_1(\mu r) \end{bmatrix},$$

$$\begin{bmatrix} g_{52}(\mu,r) \\ g_{62}(\mu,r) \end{bmatrix} = (c_{11}-c_{12}) \begin{bmatrix} \frac{2c_{11}}{c_{11}+c_{12}} \mu r J_0(\mu r) + \left(\mu^2 r^2 - \frac{3c_{11}-c_{12}}{c_{11}+c_{12}}\right) J_1(\mu r) \\ -\frac{c_{11}-c_{12}}{c_{11}+c_{12}} \mu r J_0(\mu r) + \frac{3c_{11}-c_{12}}{c_{11}+c_{12}} J_1(\mu r) \end{bmatrix}.$$

5. Satisfaction of the BC at *r* = *a*

5.1. Transversely isotropic materials

Imposing Eq. (8) on Eq. (23) gives

$$\begin{bmatrix} f_3(\kappa_1,\mu,a) & f_3(\kappa_2,\mu,a) \\ f_5(\kappa_1,\mu,a) & f_5(\kappa_2,\mu,a) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{30}$$

Non-trivial solution of Eq. (30) exists if and only if the determinant of the coefficient matrix is zero. This condition leads to a transcendental equation

$$\eta_1 J_1(\kappa_1\mu a) J_1(\kappa_2\mu a) + \mu a[\eta_2 J_1(\kappa_1\mu a) J_0(\kappa_2\mu a) - \eta_3 J_0(\kappa_1\mu a) J_1(\kappa_2\mu a)] = 0, \tag{31}$$

where

$$\eta_1 = 2c_{66}c_{33}(c_{13}+c_{44})(\kappa_1^2-\kappa_2^2),$$
$$\eta_2 = \kappa_2(\kappa_1^2 c_{13}+c_{33})[\kappa_2^2 c_{44}c_{11} + c_{13}(c_{44}+c_{13}) - c_{11}c_{33}],$$
$$\eta_3 = \kappa_1(\kappa_2^2 c_{13}+c_{33})[\kappa_1^2 c_{44}c_{11} + c_{13}(c_{44}+c_{13}) - c_{11}c_{33}].$$

By inspection, if μ is a root of Eq. (31), so is $-\mu$. Moreover, $\mu = 0$ is a repeated root, which can be verified by replacing μ by $-\mu$ in Eq. (31) and using the fact that $\mu = 0$is a zero of $J_1(\kappa\mu a)$ and $J_0(-\kappa_j\mu r) = J_0(\kappa_j\mu r)$, $J_1(-\kappa_j\mu r) = -J_1(\kappa_j\mu r)$. As will be shown in Section 7, it is a natural result of a Hamiltonian system.

Substituting the eigenvalues μ_i ($i = 1, 2, \ldots$) determined from Eq. (31) back to Eq. (30), expressing A_{2i} in terms of A_{1i}, we obtain

$$A_{2i} = -\frac{f_3(\kappa_1,\mu_i,a)}{f_3(\kappa_2,\mu_i,a)} A_{1i} \equiv \alpha_i A_i. \tag{32}$$

The case of $\mu_0 = 0$ requires special consideration. Since $\mu_0 = 0$ is a repeated root of Eq. (31), only one of the eigenvectors associated with $\mu_0 = 0$ is obtained from Eqs. (30)–(32):

$$[u_r \; u_z \; r\sigma_{rz} \; r\sigma_{zz}] = [0 \; 1 \; 0 \; 0], \quad [r\sigma_{rr} \; r\sigma_{\theta\theta}] = [0 \; 0], \tag{33}$$

which is a rigid body translation in *z* axis.

Another linearly independent eigenvector associated with $\mu_0 = 0$ can be determined by considering the Jordan-chain solution given by Eqs. (66)–(69) as

$$[u_r \; u_z \; r\sigma_{rz} \; r\sigma_{zz}] = [\beta_1 r \; z \; 0 \; \beta_2 r], \quad [r\sigma_{rr} \; r\sigma_{\theta\theta}] = [0 \; 0], \tag{34}$$

where

$$\beta_1 = -\frac{c_{13}}{c_{11}+c_{12}}, \quad \beta_2 = \frac{c_{33}(c_{11}+c_{12}) - 2c_{13}^2}{c_{11}+c_{12}}.$$

A linear combination of the eigensolutions and the particular solution obtained above produces

$$\begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz} \end{bmatrix} = \sum_{i=1}^{\infty} \left(A_i e^{\mu_i z} \begin{bmatrix} h_1(\mu_i,r) \\ h_2(\mu_i,r) \\ h_3(\mu_i,r) \\ h_4(\mu_i,r) \end{bmatrix} + A_{-i} e^{-\mu_i z} \begin{bmatrix} h_1(-\mu_i,r) \\ h_2(-\mu_i,r) \\ h_3(-\mu_i,r) \\ h_4(-\mu_i,r) \end{bmatrix} \right) + A_0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + B_0 \begin{bmatrix} \beta_1 r \\ z \\ 0 \\ \beta_2 r \end{bmatrix} + \rho g \begin{bmatrix} a_{13} r(z-l) \\ \frac{1}{2}[a_{33}(z^2-2lz) - a_{13}r^2] \\ 0 \\ r(z-l) \end{bmatrix}, \tag{35}$$

$$\begin{bmatrix} r\sigma_{rr} \\ r\sigma_{\theta\theta} \end{bmatrix} = \sum_{i=1}^{\infty} \left(A_i e^{\mu_i z} \begin{bmatrix} h_5(\mu_i,r) \\ h_6(\mu_i,r) \end{bmatrix} + A_{-i} e^{-\mu_i z} \begin{bmatrix} h_5(-\mu_i,r) \\ h_6(-\mu_i,r) \end{bmatrix} \right), \tag{36}$$

where A_0, B_0, A_i and A_{-i} are constants of linear combination, and $h_j(\mu_i,r) = f_j(\kappa_1,\mu_i,r) + \alpha_i f_j(\kappa_2,\mu_i,r), (j = 1,2,\ldots,6)$.

5.2. Isotropic materials

Imposing Eq. (8) on Eq. (28) gives

$$\begin{bmatrix} g_{31}(\mu,a) & g_{32}(\mu,a) \\ g_{51}(\mu,a) & g_{52}(\mu,a) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{37}$$

which has a non-trivial solution if and only if the determinant of the coefficient matrix is zero, leading to

$$\left[(\mu a)^2 - \frac{2c_{11}}{c_{11}+c_{12}}\right] J_1^2(\mu a) + (\mu a)^2 J_0^2(\mu a) = 0. \tag{38}$$

Eq. (38) is even in μ, hence μ is a root of Eq. (38), so is $-\mu$, in accordance with the Hamiltonian characteristics to be described in Section 7. Substituting the eigenvalues μ_i ($i = 1, 2, \ldots$) determined from Eq. (38) into Eq. (37), and expressing A_{2i} in terms of A_{1i}, we have

$$A_{2i} = -\frac{g_{31}(\mu_i,a)}{g_{32}(\mu_i,a)} A_{1i} \equiv \alpha_i A_i. \tag{39}$$

By inspection, $\mu_0 = 0$ is a repeated root of Eq. (38) since $\mu = 0$ is a zero of $J_1(\mu a)$. The associated linearly independent eigenvectors are given by Eqs. (33) and (34) with $c_{13} = c_{12}$ and $c_{33} = c_{11}$.

As a result, the solution of Eq. (4) for isotropic materials takes the same form as Eqs. (35) and (36), in which $c_{33} = c_{11}$, $c_{13} = c_{12}$, $c_{44} = c_{66} = (c_{11} - c_{12})/2$, and

$$h_j(\mu_i,r) = g_{j1}(\mu_i,r) + \alpha_i g_{j2}(\mu_i,r) \quad (j = 1,2,\ldots,6). \tag{40}$$

6. Satisfaction of the end conditions

The solution of Eq. (4) is required to satisfy the end conditions at $z = 0$ and $z = l$. To this end, imposing Eqs. (9)–(11) on Eq. (35), we obtain

(1) Fixed end at $z = 0$ and free end at $z = l$

$$\mathbf{F}_0(r) = \sum_{i=1}^{\infty} [A_i \boldsymbol{\psi}_i(r) + A_{-i}\boldsymbol{\psi}_{-i}(r)] + A_0 \boldsymbol{\psi}_0^{(0)} + B_0 \boldsymbol{\psi}_0^{(1)}(r) + \mathbf{G}_0(r), \tag{41}$$

$$\mathbf{F}_l(r) = \sum_{i=1}^{\infty} [A_i e^{\mu_i l} \boldsymbol{\psi}_i(r) + A - i e^{-\mu_i l} \boldsymbol{\psi}_{-i}(r)] + (A_0 + B_0 l) \boldsymbol{\psi}_0^{(0)} + B_0 \boldsymbol{\psi}_0^{(1)}(r) + \mathbf{G}_l(r), \tag{42}$$

where

$$\boldsymbol{\psi}_i(r) = \begin{bmatrix} h_1(\mu_i,r) \\ h_2(\mu_i,r) \\ h_3(\mu_i,r) \\ h_4(\mu_i,r) \end{bmatrix}, \quad \boldsymbol{\psi}_{-i}(r) = \begin{bmatrix} h_1(-\mu_i,r) \\ h_2(-\mu_i,r) \\ h_3(-\mu_i,r) \\ h_4(-\mu_i,r) \end{bmatrix}, \quad \boldsymbol{\psi}_0^{(0)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\boldsymbol{\psi}_0^{(1)}(r) = \begin{bmatrix} \beta_1 r \\ 0 \\ 0 \\ \beta_2 r \end{bmatrix}, \quad \mathbf{F}_0(r) = \begin{bmatrix} 0 \\ 0 \\ r\sigma_{rz}(r,0) \\ r\sigma_{zz}(r,0) \end{bmatrix}, \quad \mathbf{F}_l(r) = \begin{bmatrix} u_r(r,l) \\ u_z(r,l) \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{G}_0(r) = -\rho g \begin{bmatrix} a_{13} rl \\ \frac{1}{2} a_{13} r^2 \\ 0 \\ rl \end{bmatrix}, \quad \mathbf{G}_l(r) = -\frac{\rho g}{2} \begin{bmatrix} 0 \\ (a_{33} l^2 + a_{13} r^2) \\ 0 \\ 0 \end{bmatrix},$$

in which A_0, B_0, A_i and A_{-i} are constants to be determined, $\mathbf{F}_0(r)$ contains the unknown tractions $\sigma_{rz}(r,0)$ and $\sigma_{zz}(r,0)$ at the fixed end, $\mathbf{F}_l(r)$ contains the unknown displacements $u_r(r,l)$ and $u_z(r,l)$ at the free end, which, in turn, can be expressed in terms of A_0, B_0, A_i and A_{-i} by using Eqs. (41) and (42) as

$$r\sigma_{rz}(r,0)=\sum_{i=1}^{\infty}\left[A_ih_3(\mu_i,r)+A_{-i}h_3(-\mu_i,r)\right], \tag{43}$$

$$r\sigma_{zz}(r,0)=\sum_{i=1}^{\infty}\left[A_ih_4(\mu_i,r)+A_{-i}h_4(-\mu_i,r)\right]+B_0\beta_2r-\rho grl, \tag{44}$$

$$u_r(r,l)=\sum_{i=1}^{\infty}\left[A_ie^{\mu_il}h_1(\mu_i,r)+A_{-i}e^{-\mu_il}h_1(-\mu_i,r)\right]+B_0\beta_1r, \tag{45}$$

$$u_z(r,l)=\sum_{i=1}^{\infty}\left[A_ie^{\mu_il}h_2(\mu_i,r)+A_{-i}e^{-\mu_il}h_2(-\mu_i,r)\right]+A_0+B_0l-\frac{1}{2}\rho g(a_{33}l^2+a_{13}r^2). \tag{46}$$

(2) Sliding-contact end at $z = 0$ and free end at $z = l$

$$\mathbf{H}_0(r)=\sum_{i=1}^{\infty}[A_i\psi_i(r)+A_{-i}\psi_{-i}(r)]+A_0\psi_0^{(0)}+B_0\psi_0^{(1)}(r)+\mathbf{G}_0(r), \tag{47}$$

$$\mathbf{F}_l(r)=\sum_{i=1}^{\infty}\left[A_ie^{\mu_il}\psi_i(r)+A_{-i}e^{-\mu_il}\psi_{-i}(r)\right]+(A_0+B_0l)\psi_0^{(0)}+B_0\psi_0^{(1)}(r)+\mathbf{G}_l(r), \tag{48}$$

where A_0, B_0, A_i and A_{-i} are constants to be determined, and the vector

$$\mathbf{H}_0(r)=[\,u_r(r,0)\quad 0\quad 0\quad r\sigma_{zz}(r,0)\,]^T$$

contains the unknown $u_r(r,0)$ and $\sigma_{zz}(r,0)$ at the fixed end, $\mathbf{F}_l(r)$ contains the unknown $u_r(r,l)$ and $u_z(r,l)$ at the free end. These unknowns can be expressed in terms of A_0, B_0, A_i and A_{-i} by using Eqs. (47) and (48) as

$$u_r(r,0)=\sum_{i=1}^{\infty}\left[A_ih_1(\mu_i,r)+A_{-i}h_1(-\mu_i,r)\right]+B_0\beta_1r-\rho ga_{13}rl, \tag{49}$$

$$r\sigma_{zz}(r,0)=\sum_{i=1}^{\infty}\left[A_ih_4(\mu_i,r)+A_{-i}h_4(-\mu_i,r)\right]+B_0\beta_2r-\rho grl, \tag{50}$$

$$u_r(r,l)=\sum_{i=1}^{\infty}\left[A_ie^{\mu_il}h_1(\mu_i,r)+A_{-i}e^{-\mu_il}h_1(-\mu_i,r)\right]+B_0\beta_1r, \tag{51}$$

$$u_z(r,l)=\sum_{i=1}^{\infty}A_ie^{\mu_il}h_2(\mu_i,r)+A_{-i}e^{-\mu_il}h_2(-\mu_i,r)]+A_0+B_0l-\frac{1}{2}\rho g(a_{33}l^2+a_{13}r^2). \tag{52}$$

At this stage, it remains to determine the constants A_0, B_0, A_i and A_{-i} in Eqs. (41)–(52). For this purpose, it is necessary to derive the orthogonality relations among the eigenvectors so as to express a given vector function in terms of the eigenvectors by using eigenfunction expansion.

7. Symplectic orthogonality

To facilitate exposition, let us express Eq. (15) with the homogeneous BC at $r = 0$ and $r = a$ as

$$\mathbf{H}\psi=\mu\psi,\quad \mathbf{B}(\psi)=\mathbf{0}, \tag{53}$$

where

$$\mathbf{H}=\begin{bmatrix}\mathbf{L}_{11} & \mathbf{L}_{12}\\ \mathbf{L}_{21} & -\mathbf{L}_{11}^*\end{bmatrix},\quad \psi=\begin{bmatrix}\mathbf{u}\\ \tau\end{bmatrix},\quad \mathbf{u}=\begin{bmatrix}U_r\\ U_z\end{bmatrix},\quad \tau=\begin{bmatrix}T_{rz}\\ T_{zz}\end{bmatrix},$$

$$\mathbf{L}_{11}=\begin{bmatrix}0 & -d_r\\ -\widehat{c}_{13}(d_r+r^{-1}) & 0\end{bmatrix},\quad \mathbf{L}_{12}=\begin{bmatrix}r^{-1}c_{44}^{-1} & 0\\ 0 & r^{-1}c_{33}^{-1}\end{bmatrix},$$

$$\mathbf{L}_{11}^*=\begin{bmatrix}0 & \widehat{c}_{13}(d_r-r^{-1})\\ d_r & 0\end{bmatrix},\quad \mathbf{L}_{21}=\begin{bmatrix}-\widetilde{c}_{11}[d_r(rd_r)-r^{-1}] & 0\\ 0 & 0\end{bmatrix}.$$

The matrix $\mathbf{H}$ in Eq. (53) exhibits the characteristics of a Hamiltonian matrix such that $\mathbf{L}_{11}$, $\mathbf{L}_{11}^*$ are adjoint matrix operators and $\mathbf{L}_{12}$, $\mathbf{L}_{21}$ are symmetric matrices. Such characteristics are of fundamental significance in the formulation.

First of all, the adjoint system of Eq. (53) is

$$\mathbf{H}\phi=-\mu\phi,\quad \mathbf{B}(\phi)=\mathbf{0}, \tag{54}$$

where ϕ is the eigenvector associated with the eigenvalue $-\mu$, and

$$\phi=\begin{bmatrix}\mathbf{v}\\ \mathbf{s}\end{bmatrix},\quad \mathbf{v}=\begin{bmatrix}V_r\\ V_z\end{bmatrix},\quad \mathbf{s}=\begin{bmatrix}S_{rz}\\ S_{zz}\end{bmatrix}.$$

Derivation of Eq. (54) begins with the equality

$$\int_0^a\phi^T\mathbf{J}\mathbf{H}\psi dr=\int_0^a\left[\mathbf{v}^T(\mathbf{L}_{21}\mathbf{u}-\mathbf{L}_{11}^*\tau)-\mathbf{s}^T(\mathbf{L}_{11}\mathbf{u}+\mathbf{L}_{12}\tau)\right]dr, \tag{55}$$

where

$$\mathbf{J}=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{I} & \mathbf{0}\end{bmatrix}_{4\times4},\quad \mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

Integrating by parts the right-hand side of Eq. (55), after manipulation we arrive at

$$\begin{aligned}\int_0^a\phi^T\mathbf{J}\mathbf{H}\psi dr&=\int_0^a[\mathbf{u}^T(\mathbf{L}_{21}\mathbf{v}-\mathbf{L}_{11}^*\mathbf{s})-\tau^T(\mathbf{L}_{11}\mathbf{v}+\mathbf{L}_{12}\mathbf{s})]dr\\&\quad+(\mathbf{u}^T\mathbf{s}-\mathbf{v}^T\tau)|_{r=0}^{r=a}\\&=\int_0^a\psi^T\mathbf{J}\mathbf{H}\phi dr+(\mathbf{u}^T\mathbf{s}-\mathbf{v}^T\tau)|_{r=0}^{r=a},\end{aligned} \tag{56}$$

which shows that $\mathbf{JH}$ is self-adjoint such that $(\mathbf{JH})^* = \mathbf{JH}$, where the star * denotes the adjoint.

Pre-multiplying Eq. (53) by $\phi^T\mathbf{J}$, integrating it over $(0,a)$ and taking transpose, with $\mathbf{J}^T = -\mathbf{J}$, we obtain

$$\int_0^a\phi^T\mathbf{J}\mathbf{H}\psi dr=\mu\int_0^a\phi^T\mathbf{J}\psi dr=-\mu\int_0^a\psi^T\mathbf{J}\phi dr, \tag{57}$$

which, upon substituting into Eq. (56), results in

$$\int_0^a\psi^T\mathbf{J}(\mathbf{H}\phi+\mu\phi)dr+(\mathbf{u}^T\mathbf{s}-\mathbf{v}^T\tau)|_{r=0}^{r=a}=0. \tag{58}$$

There follows Eq. (54) under the homogeneous BC at $r = 0$ and $r = a$.

Eqs. (53) and (54) reveal that if μ is an eigenvalue, so is $-\mu$. It follows that $\mu = 0$ must be a repeated eigenvalue. Indeed, the eigenvalues determined from Eqs. (31) and (38) verify this important property of an axisymmetric Hamiltonian system.

Since the eigenvalues appear in $(\mu_i, -\mu_i)$ pairs, it is expedient to arrange the order of the eigenvectors in the following sequence:

(1) the eigenvectors ψ_i ($i = 1, 2, \ldots$) associated with the eigenvalues μ_i.
(2) the eigenvectors $\psi_{-i} \triangleq \phi$ ($i = 1, 2, \ldots$) associated with the eigenvalues μ_{-i} ($=-\mu_i$).
(3) the eigenvectors ψ_0 associated with the repeated eigenvalue $\mu_0 = 0$.

Consider two linearly independent eigenvectors ψ_i and ψ_j associated with the eigenvalues μ_i and μ_j, respectively,

$$\mathbf{H}\psi_i = \mu_i\psi_i, \quad \mathbf{B}(\psi_i) = \mathbf{0}, \tag{59}$$

$$\mathbf{H}\psi_j = \mu_j\psi_j, \quad \mathbf{B}(\psi_j) = \mathbf{0}. \tag{60}$$

Pre-multiplying Eq. (59) by $\psi_j^T\mathbf{J}$ and Eq. (60) by $\psi_i^T\mathbf{J}$, integrating the resulting equations with respect to r over $(0,a)$, we obtain

$$\int_0^a \psi_j^T\mathbf{JH}\psi_i dr = \mu_i \int_0^a \psi_j^T\mathbf{J}\psi_i dr, \tag{61}$$

$$\int_0^a \psi_i^T\mathbf{JH}\psi_j dr = \mu_j \int_0^a \psi_i^T\mathbf{J}\psi_j dr. \tag{62}$$

Integrating by parts the left-hand side of (61) as was done in Eq. (56) by setting $\phi = \psi_j$, $\psi = \psi_i$, using the fact that $\mathbf{JH}$ is self-adjoint and $\mathbf{J}^T = -\mathbf{J}$, we obtain

$$\int_0^a \psi_i^T\mathbf{JH}\psi_j dr = -\mu_i \int_0^a \psi_i^T\mathbf{J}\psi_j dr. \tag{63}$$

Subtracting Eqs. (62) and (63) gives

$$(\mu_i + \mu_j)\int_0^a \psi_i^T\mathbf{J}\psi_j dr = 0. \tag{64}$$

Since $\mu_i + \mu_j \neq 0$ $(i \neq -j)$ according to the sequence of the eigenvectors defined before, there follows the orthogonal relation for the eigenvectors associated with $\mu_i \neq 0$ $(i = 1, 2, \ldots)$:

$$\int_0^a \psi_i^T\mathbf{J}\psi_j dr \begin{cases} = 0, & (j \neq -i) \\ \neq 0, & (j = -i) \end{cases}. \tag{65}$$

The case of the zero eigenvalue needs special consideration. It has been shown that $\mu_0 = 0$ is a repeated eigenvalue. The two linearly independent eigenvectors $\psi_{0(1)}$ and $\psi_{0(2)}$ can be determined from the Jordan chain associated with $\mu_0 = 0$:

$$\mathbf{H}\psi_0^{(0)} = \mathbf{0}, \quad \mathbf{B}(\psi_0^{(0)}) = \mathbf{0}, \tag{66}$$

$$\mathbf{H}\psi_0^{(1)} = \psi_0^{(0)}, \quad \mathbf{B}(\psi_0^{(1)}) = \mathbf{0}, \tag{67}$$

from which we obtain

$$\psi_{0(1)} = \psi_0^{(0)} = [0 \quad 1 \quad 0 \quad 0], \tag{68}$$

$$\psi_{0(2)} = \psi_0^{(1)} + z\psi_0^{(0)} = [\beta_1 r \quad 0 \quad 0 \quad \beta_2 r] + z[0 \quad 1 \quad 0 \quad 0]. \tag{69}$$

Consider the non-homogeneous system

$$\mathbf{H}\varphi = \psi_0^{(1)}, \quad \mathbf{B}(\varphi) = \mathbf{0}, \tag{70}$$

which is unsolvable because the Jordan chain of $\mu_0 = 0$ breaks at the second order.

Pre-multiplying Eq. (70) by $(\psi_0^{(0)})^T\mathbf{J}$ and integrating it over $(0,a)$ gives

$$\int_0^a (\psi_0^{(0)})^T\mathbf{JH}\varphi dr = \int_0^a (\psi_0^{(0)})^T\mathbf{J}\psi_0^{(1)} dr, \tag{71}$$

in which, on setting $\varphi = \psi_0^{(0)}$, the left-hand side is zero because of Eq. (66), so the solvability condition of Eq. (70) is

$$\int_0^a (\psi_0^{(0)})^T\mathbf{J}\psi_0^{(1)} dr = 0. \tag{72}$$

Knowing that Eq. (70) is unsolvable, so the above solvability condition is not satisfied. It follows

$$\int_0^a (\psi_0^{(0)})^T\mathbf{J}\psi_0^{(1)} dr = -\int_0^a (\psi_0^{(1)})^T\mathbf{J}\psi_0^{(0)} dr \neq 0, \tag{73}$$

in which the equality is reached by taking the transpose and using $\mathbf{J}^T = -J$.

By the same token,

$$\int_0^a (\psi_0^{(\alpha)})^T\mathbf{J}\psi_0^{(\alpha)} dr = -\int_0^a (\psi_0^{(\alpha)})^T\mathbf{J}\psi_0^{(\alpha)} dr \quad (\alpha = 0, 1) \tag{74}$$

which is possible if and only if

$$\int_0^a (\psi_0^{(\alpha)})^T\mathbf{J}\psi_0^{(\alpha)} dr = 0, (\alpha = 0, 1). \tag{75}$$

In summary, the orthogonal relations of the eigenvectors associated with $\mu_0 = 0$ are

$$\int_0^a (\psi_0^{(\alpha)})^T\mathbf{J}\psi_0^{(\beta)} dr \begin{cases} = 0 & (\beta = \alpha;\ \alpha, \beta = 0, 1) \\ \neq 0 & (\beta \neq \alpha;\ \alpha, \beta = 0, 1) \end{cases}. \tag{76}$$

Before passing on, a few remarks on the orthogonality and coefficient determination are in order. Unlike a conventional formulation in which orthogonality of the eigenvectors involves either the displacement vector or the stress vector, symplectic orthogonality in the Hamiltonian state space holds for the eigenvector derived from the state vector which is composed of the displacement vector and the associated stress vector. The eigenvector consists of two vector components, namely, the vector $\mathbf{u}$ and the vector $\boldsymbol{\tau}$ as defined in Eq. (53). Without fully grasping the Hamiltonian characteristics of the system, one is likely to impose the fixed end conditions on the displacement components and attempt to determine the unknown constants from the resulting scalar equations. A difficult situation then arises – there are four arbitrary scalar functions to be expanded into series in terms of the given functions with the same set of unknown constants. Consequently, the number of systems of equations is larger than the number of unknowns, rendering the equations inconsistent and unsolvable in general. In few restrictive cases where the system of equations is consistent and solvable, one often has to resort to an approximate method to determine the unknown constants. It is well known that using an approximate method to expand an arbitrary function into a series in terms of given functions often leads to non-convergent results. There is no guarantee that the method of least square or Gram-Schmidt orthogonalization (Kantorovich and Krylov, 1964; Hildebrand, 1965) will produce a valid solution. A solution scheme along this line is limited in its use.

8. Determining the unknown constants

Returning to the task of determining A_0, B_0, A_i and A_{-i} in Eqs. (41) and (42). With the symplectic orthogonality given by Eqs. (65) and (76), it is possible to express an arbitrary integrable state vector in terms of a sequence of orthogonal eigenvectors in the state space by means of eigenfunction expansion.

Let us consider the case of the fixed end at $z = 0$ in detail. Multiplying both sides of Eq. (41) by $\psi_{-i}^T\mathbf{J}$, $\psi_i^T\mathbf{J}$, $[\psi_0^{(1)}]^T\mathbf{J}$, and $[\psi_0^{(0)}]^T\mathbf{J}$, respectively, and integrating them over $(0,a)$, making use of the symplectic orthogonality, we obtain

$$A_i = \int_0^a \psi_{-i}^T(r)\mathbf{J}[\mathbf{F}_0(r) - \mathbf{G}_0(r)]dr, \tag{77}$$

$$A_{-i} = -\int_0^a \psi_i^T(r)\mathbf{J}[\mathbf{F}_0(r) - \mathbf{G}_0(r)]dr, \tag{78}$$

$$A_0 = \int_0^a [\psi_0^{(1)}(r)]^T\mathbf{J}[\mathbf{F}_0(r) - \mathbf{G}_0(r)]dr, \tag{79}$$

$$B_0 = -\int_0^a [\psi_0^{(0)}]^T\mathbf{J}[\mathbf{F}_0(r) - \mathbf{G}_0(r)]dr, \tag{80}$$

Likewise, it follows from Eq. (42)

$$A_i = e^{-\mu_i l}\int_0^a \psi_{-i}^T(r)\mathbf{J}[\mathbf{F}_l(r) - \mathbf{G}_l(r)]dr, \tag{81}$$

$$A_{-i} = -e^{\mu_i l}\int_0^a \psi_i^T(r)\mathbf{J}[\mathbf{F}_l(r) - \mathbf{G}_l(r)]dr, \tag{82}$$

$$A_0 + B_0 l = \int_0^a [\psi_0^{(1)}(r)]^T\mathbf{J}[\mathbf{F}_l(r) - \mathbf{G}_l(r)]dr, \tag{83}$$

$$B_0 = -\int_0^a [\psi_0^{(0)}]^T \mathbf{J}[\mathbf{F}_l(r) - \mathbf{G}_l(r)]dr. \tag{84}$$

In Eqs. (77)–(84) each of the vectors $\mathbf{F}_0(r)$ and $\mathbf{F}_l(r)$ contains four components, of which the tractions $\sigma_{rz}(r,0)$ and $\sigma_{zz}(r,0)$ are unknown at the fixed end $z = 0$, the displacements $u_r(r,l)$ and $u_z(r,l)$ are unknown at the free end $z = l$. Substituting Eqs. (77)–(80) in Eqs. (81)–(84) gives

$$\int_0^a \psi_{-i}^T(r)\mathbf{J}[\mathbf{F}_0(r) - e^{-\mu_i l}\mathbf{F}_l(r)]dr = \int_0^a \psi_{-i}^T(r)\mathbf{J}[\mathbf{G}_0(r) - e^{-\mu_i l}\mathbf{G}_l(r)]dr, \tag{85}$$

$$\int_0^a \psi_i^T(r)\mathbf{J}[\mathbf{F}_0(r) - e^{\mu_i l}\mathbf{F}_l(r)]dr = \int_0^a \psi_i^T(r)\mathbf{J}[\mathbf{G}_0(r) - e^{\mu_i l}\mathbf{G}_l(r)]dr, \tag{86}$$

$$\begin{aligned}&\int_0^a \left\{ \left[\psi_0^{(1)}(r)\right]^T \mathbf{J}[\mathbf{F}_0(r) - \mathbf{F}_l(r)] - l\left[\psi_0^{(0)}\right]^T \mathbf{J}\mathbf{F}_0(r) \right\} dr \\ &= \int_0^a \left\{ \left[\psi_0^{(1)}(r)\right]^T \mathbf{J}[\mathbf{G}_0(r) - \mathbf{G}_l(r)] - l\left[\psi_0^{(0)}\right]^T \mathbf{J}\mathbf{G}_0(r) \right\} dr,\end{aligned} \tag{87}$$

$$\int_0^a [\psi_0^{(0)}]^T \mathbf{J}[\mathbf{F}_0(r) - \mathbf{F}_l(r)]dr = \int_0^a [\psi_0^{(0)}]^T \mathbf{J}[\mathbf{G}_0(r) - \mathbf{G}_l(r)]dr, \tag{88}$$

in which the four unknowns in $\mathbf{F}_0(r)$ and $\mathbf{F}_l(r)$ are related to the constants A_0, B_0, A_i and A_{-i} by Eqs. (43)–(46); the terms on the right-hand sides are completely known.

Eqs. (85)–(88) are, in fact, a system of four integral equations for the four unknown functions of r (two stress components at the fixed end: $r\sigma_{rz}(r,0)$ and $r\sigma_{zz}(r,0)$; two displacement components at the free end: $u_r(r,l)$ and $u_z(r,l)$, which in turn are given by Eqs. (43)–(46) in series forms), solvable by using the method of collocation for solving the integral equations (Kantorovich and Krylov, 1964; Hildebrand, 1965). Thus, substituting Eqs. (43)–(46) in Eqs. (85)–(88), taking n terms of the series for computation, we obtain a system of $2n + 2$ algebraic equations ($2n$ equations from Eqs. (85) and (86) for $i = 1, 2, \ldots n$; two equations from Eqs. (87) and (88) for $i = 0$) for the $2n + 2$ unknowns: A_0, B_0, A_i and A_{-i} ($i = 1, 2, \ldots, n$), which can be solved by using a standard method for simultaneous linear algebraic equations.

For the case of the sliding-contact end at $z = 0$ the constants A_0, B_0, A_i and A_{-i} in Eqs. (47) and (48) are determined in a similar way.

9. Results and discussions

To evaluate the end effects on the deformation and stress distribution in the cylinder, the results are compared with the corresponding ones according to the simplified solution that gives a uniaxial stress state. The cylinders with a fixed end and with a sliding-contact end are considered. For numerical calculations using MATLAB, the following dimensionless elastic constants for isotropic and transversely isotropic materials are taken: $c_{11}/c_{33} = 1, 1/5, 1/10, 1/20$; $c_{12}/c_{11} = c_{13}/c_{11} = 1/3$. These values are typical for engineering materials, ranging from aluminum (isotropic material) to unidirectional fiber-reinforced composites such as glass/epoxy and graphite/epoxy (transversely isotropic material). The dimensionless length of the cylinder is taken to be $l/a = 10; 20$. In view of the solution form given by Eq. (35), the stress decay from the end in a sufficiently long cylinder is essentially dictated by the terms associated with $e^{-\mu_i z}$ since the terms of $e^{\mu_i z}$ which produce unbounded stresses as $z \to \infty$ should be dropped. Accordingly, the smallest eigenvalue may serve as an indication of the general trend of the stress decay behavior. Table 1 shows the first 25 dimensionless eigenvalues $\mu_i a$ for the material parameters considered, in which the eigenvalues in the first column are found from Eq. (38) and those in the other columns from Eq. (31). Note that the eigenvalues in the present analysis are independent of the prescribed end conditions. The first eigenvalue in all cases is zero and it is a repeated eigenvalue, with which associated linearly z–dependent displacements and z–independent stresses. The non-zero eigenvalues are related to the stress disturbance by the end conditions. All the eigenvalues for the isotropic material are complex conjugate, and the eigenvalues for transversely isotropic materials are mostly real, suggesting that the series solution will yield results oscillating more in the case of isotropy than in the case of transverse isotropy. This is reflected in the figures showing the displacement and stress distributions.

Variations of the displacements and stresses in the cylinders are evaluated by taking n terms in the eigenfunction expansion. The number n can be chosen as large as necessary to achieve a required accuracy. For the material constants used for computation, by taking less than 20 terms of the series, convergent results were obtained at locations away from the base plane. Stress results at the rim on the base plane of the cylinder converge rather slowly due to geometric discontinuity, it is necessary to take up to 40 terms to achieve steady results. Fig. 1 depicts variations of u_r, u_z, σ_{rr} and $\sigma_{\theta\theta}$ at $r = a$ (where $\sigma_{rz} = \sigma_{rr} = 0$) in the axial direction within a diameter from the fixed end for $l/a = 10$. Results for $l/a = 20$ are essentially similar to those of $l/a = 10$ after non-dimensionalization with respect to ρgl. The dot lines in the figure are the results given by the simplified solution, Eq. (13). The points at $r = a$ are around the rim of the cylinder where results significantly different from the simplified ones are to be expected. Indeed, remarkable differences of u_r, σ_{zz} and $\sigma_{\theta\theta}$ from those of the simplified solution exist in the region within a distance of the cylinder diameter near the fixed end. The fixed end effects diminish rapidly. At a distance of a diameter from the base plane, the simplified solution produces essentially the same results as those obtained from the present analysis. This implies that the end effect is confined to a local region near the fixed end for cylinders of typical engineering materials under the force of gravity. As an illustration of convergence of the results, we show in Fig. 2 the results for the case of $l/a = 10$ by taking 20, 30, and 40 terms of the series for computation. Dis-

Table 1
Dimensionless eigenvalues $\mu_i a$ for cylinders of isotropic and transversely isotropic materials ($c_{12}/c_{11} = c_{13}/c_{11} = 1/3$).

Mode	$c_{11}/c_{33} = 1$ (isotropic)	$c_{11}/c_{33} = 1/5$	$c_{11}/c_{33} = 1/10$	$c_{11}/c_{33} = 1/20$
1	0	0	0	0
2	$2.698 \pm 1.367i$	1.061	0.723	0.503
3	$6.051 \pm 1.638i$	1.955	1.324	0.920
4	$9.261 \pm 1.829i$	2.921	1.924	1.335
5	$12.438 \pm 1.967i$	$3.456 \pm 0.363i$	2.532	1.749
6	$15.602 \pm 2.076i$	4.378	3.204	2.164
7	$18.760 \pm 2.166i$	5.282	$3.535 \pm 0.241i$	2.583
8	$21.912 \pm 2.242i$	6.160	4.194	3.011
9	$25.062 \pm 2.308i$	7.036	4.812	$3.514 \pm 0.149i$
10	$28.210 \pm 2.367i$	7.933	5.412	3.678
11	$31.358 \pm 2.419i$	$8.851 \pm 0.311i$	6.008	4.171
12	$34.504 \pm 2.466i$	9.442	6.602	4.595
13	$37.649 \pm 2.510i$	10.382	7.197	5.012
14	$40.794 \pm 2.549i$	11.263	7.796	5.426
15	$43.939 \pm 2.586i$	12.138	8.407	5.840
16	$47.083 \pm 2.621i$	13.027	$9.082 \pm 0.195i$	6.253
17	$50.227 \pm 2.653i$	$14.075 \pm 0.236i$	9.429	6.665
18	$53.370 \pm 2.683i$	14.489	10.089	7.078
19	$56.514 \pm 2.712i$	15.482	10.693	7.491
20	$59.657 \pm 2.738i$	16.365	11.290	7.906
21	$62.800 \pm 2.764i$	17.240	11.884	8.323
22	$65.943 \pm 2.788i$	18.125	12.479	8.750
23	$69.086 \pm 2.812i$	19.199	13.076	$9.208 \pm 0.141i$
24	$72.228 \pm 2.834i$	$19.444 \pm 0.146i$	13.683	9.466
25	$75.371 \pm 2.855i$	20.582	$14.471 \pm 0.094i$	9.922

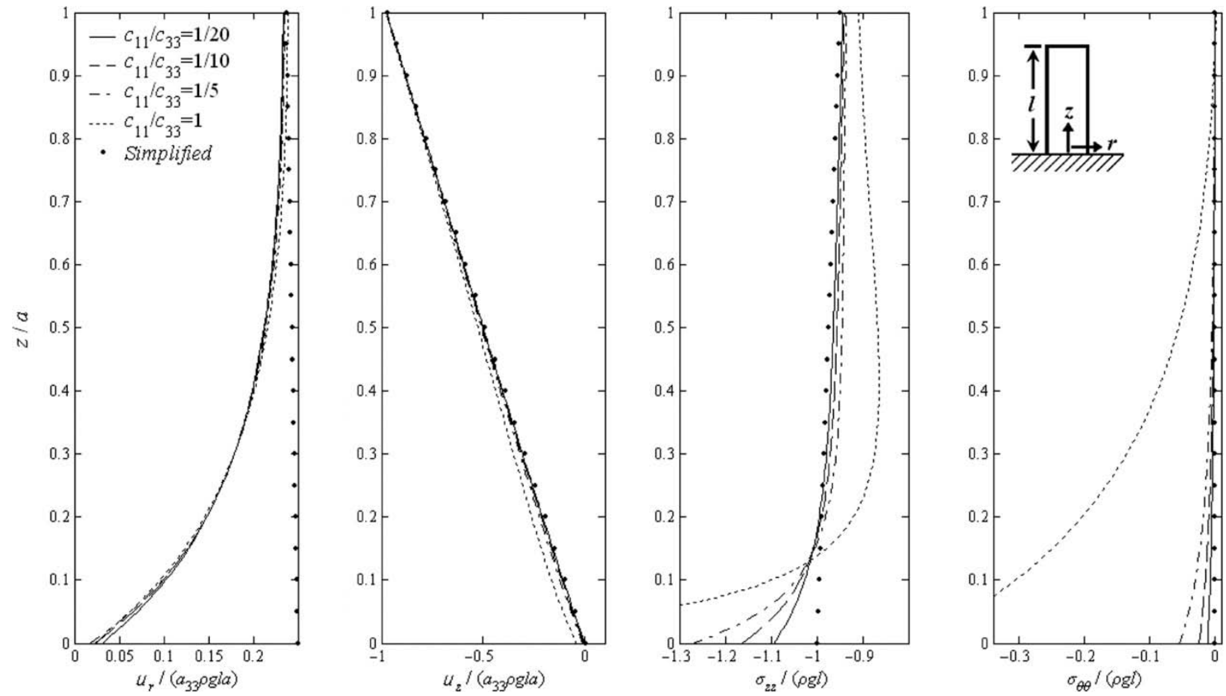

Fig. 1. Variations of displacements and stresses in the axial direction at the rim ($r = a$) of the cylinder with a fixed end.

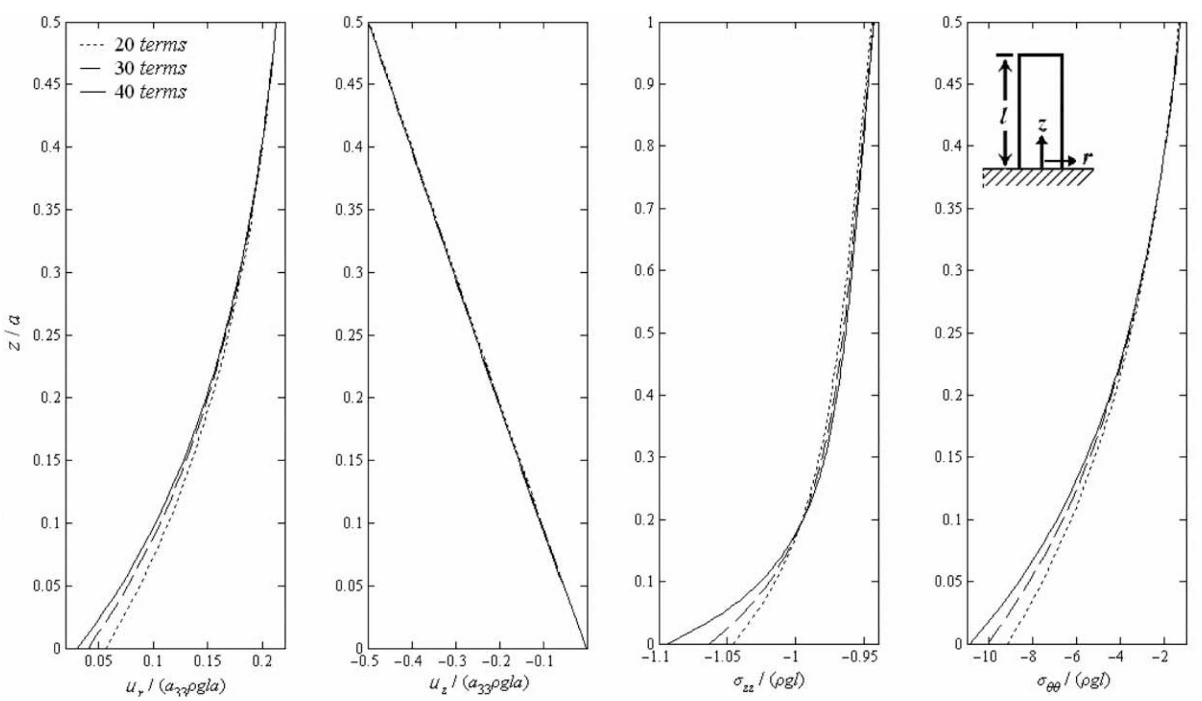

Fig. 2. Convergence of displacements and stresses in the axial direction at the rim ($r = a$) of the cylinder with a fixed end ($c_{11}/c_{33} = 1/20$).

placements and stresses converge rapidly (taking less than 20 terms) in region away from the fixed end, but relatively slow (taking up to 40 terms) in region where the end effect is significant.

Figs. 3 and 4 depict variations of displacements and stresses in the radial direction at $z = 0.1a$ and at the fixed end $z = 0$ by taking 40 terms of the series. The lateral BC $\sigma_{rr} = \sigma_{rz} = 0$ at $r = a$ are satisfied as required. The stresses σ_{rr}, $\sigma_{\theta\theta}$ and σ_{rz} are small but non-zero, whereas the simplified solution gives $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{rz} = 0$. Remarkable differences in the axial stress σ_{zz} from the one given by the simplified solution exist around the rim of the cylinder. This is expected because the fixed end conditions $u_r = u_z = 0$ at $z = 0$ are not satisfied by the simplified solution. Fig. 4 show that the lateral BC and end conditions are satisfied in the present analysis. The stresses σ_{zz} and σ_{rz} overshoot near the boundary points $r = a$, indicating stress concentration appears at the rim on the base plane at the fixed end due to geometric discontinuity. The values of σ_{zz} and σ_{rz} near $r = a$ reach $1.3\rho gl$ and $0.1\rho gl$, respectively, comparing with $\sigma_{zz} = \rho gl$ and $\sigma_{rz} = 0$ according to the simplified solution. As a con-

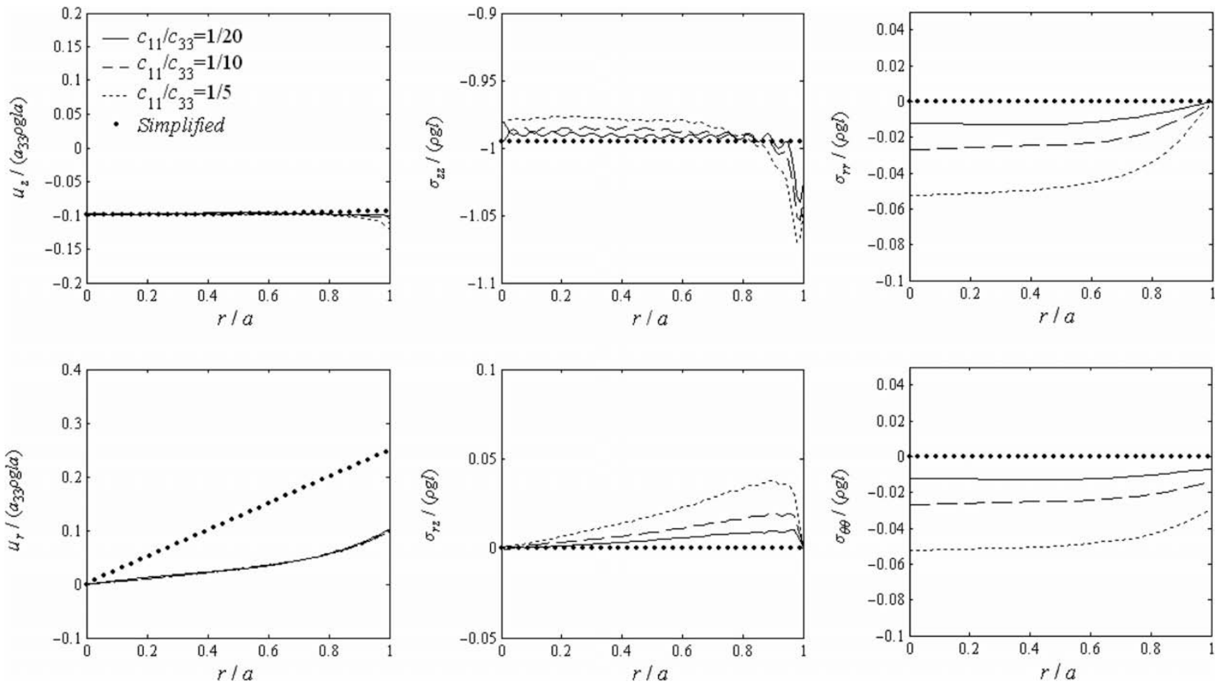

Fig. 3. Variations of stresses in the radial direction at $z = 0.1a$ of the cylinder with a fixed end.

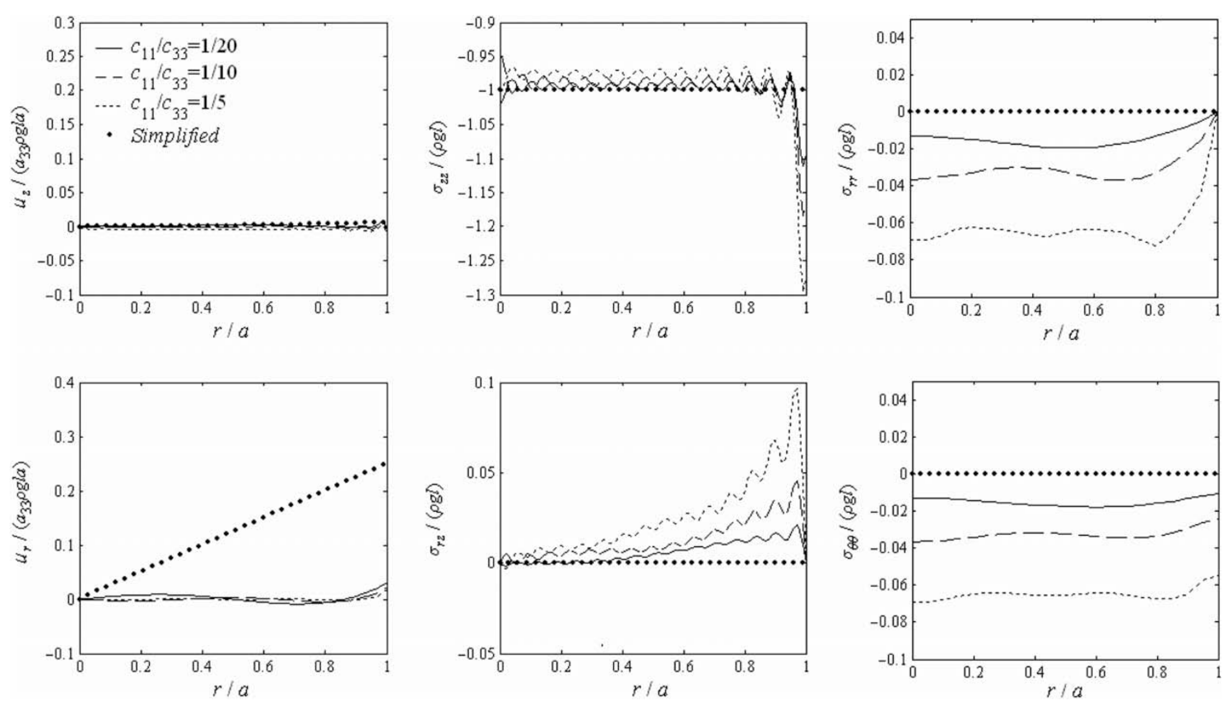

Fig. 4. Variations of stresses in the radial direction at the base ($z = 0$) of the cylinder with a fixed end.

sequence of the series solution using eigenfunction expansion, the axial stress σ_{zz} and the shear stress σ_{rz} at the base plane are oscillating with respect to mean values. The oscillation has been magnified as a result of using small scales to illustrate the variations. It reduces rapidly at a distance $z = 0.1a$ away from the fixed end.

In general, it is expected that the end effect is more significant in the material with stronger anisotropy. Therefore, as c_{33} increases, the curves of the rigorous results should deviate from the one based on the simplified solution. On the contrary, the analysis shows that, as c_{33} increases, or equivalently, as the value of c_{11}/c_{33} decreases, the rigorous results turn toward those of the simplified solution, not turn aside. This could be explained by the fact that the base plane of a fiber-reinforced cylinder under the force of gravity experiences smaller axial deformation than the one

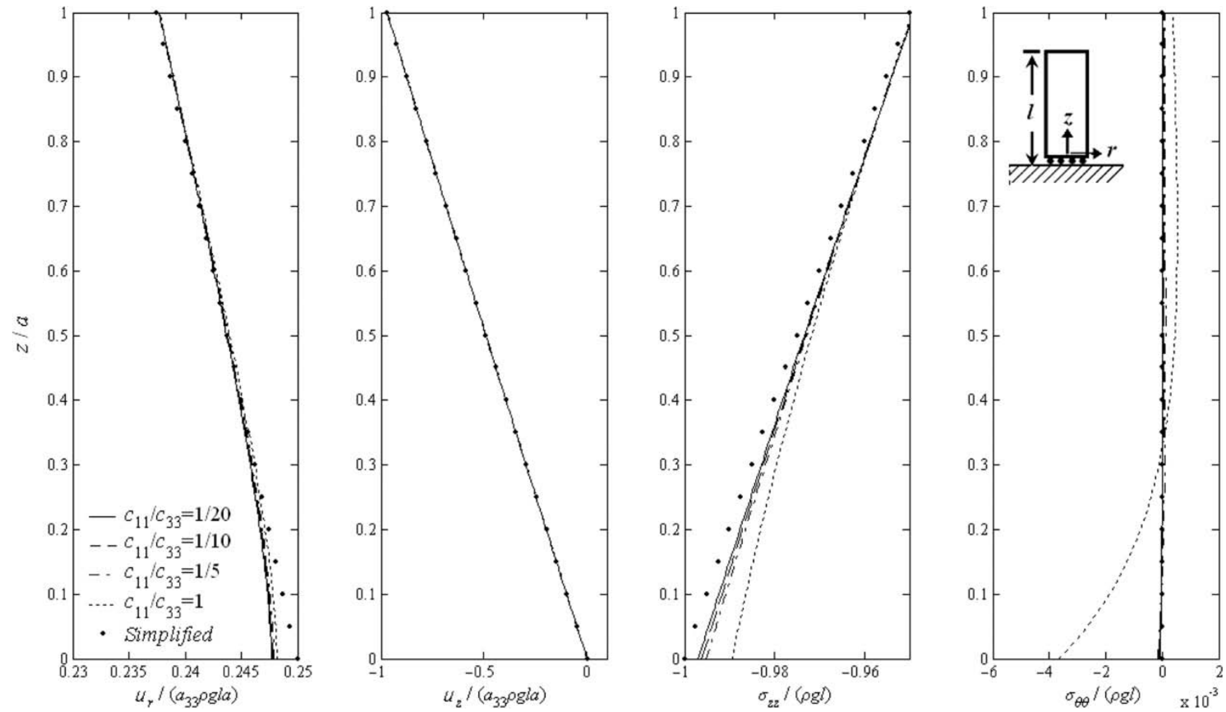

Fig. 5. Variations of displacements and stresses in the axial direction at the rim ($r = a$) of the cylinder with a sliding-contact end.

without fiber reinforcements. Consequently, as the axial stiffness increases, the end effect on the stress field decreases and the stress state in the cylinder under its own weight tends to be uniaxial.

To show the effect of the sliding contact, Fig. 5 depicts the displacements and stresses in the axial direction around the rim ($r = a$) of the cylinder with a sliding-contact end. The deformed shape of the cylinder can be observed from Figs. 1 and 5 in which the variations of u_r at $r = a$ in the axial direction are depicted. It is interesting to note that the deformed shape of the fixed-end cylinder and that of the cylinder with a sliding-contact end are remarkably different in the region where the end effect is significant. The results for the case of the sliding-contact end are close to the simplified ones except for appreciable differences in u_r and σ_{zz} near the bottom plane. This is in accord with the physical situation in that the sliding-contact allows the base plane free to move radially so that $\sigma_{rz} = \sigma_{\theta z} = 0$ at $z = 0$, which are satisfied identically by the uniaxial stress field. Hence the only end condition which is not satisfied by the simplified solution is $u_z = 0$ at $z = 0$. Yet, under the force of gravity, u_z at $z = 0$ is small, so the condition $u_z = 0$ plays a minor role as the axial stiffness increases. Variations of the displacements and stresses in the radial direction at the bottom plane of the cylinder are nearly identical to the simplified ones. The calculations show that all the stress components are almost zero except for σ_{zz} varying linearly in z and u_r linearly in r. This suggests that, when the bottom plane of the cylinder is in smooth contact with a rigid base, the simplified solution is a good approximation for the circular cylinder under its own weight.

10. Conclusions

The problem of a circular elastic cylinder under its own weight is revisited. On the basis of Hamiltonian state space formulation, an exact analysis of the deformations and stress distributions in finite cylinders of transversely isotropic and isotropic elastic materials is conducted. The results show that the displacement and stress fields in the region near the base plane of the cylinder are significantly different from those according to the simplified solution that gives a uniaxial stress state. As the axial stiffness increases, the end effect on the stress state in the cylinder under its own weight decreases. The end effect is more pronounced in the cylinder with the bottom plane being perfectly bonded than in smooth contact with a rigid base.

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