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On Rota's conjecture and excluded minors containing large projective geometries

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Abstract

We prove that an excluded minor for the class of GF(q)-representable matroids cannot contain a large projective geometry over GF(q) as a minor.
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1. Introduction

We prove the following theorem.

Theorem 1.1. *For each prime power q, there exists an integer k such that no excluded minor for the class of* GF(q)*-representable matroids contains a* PG(k, q)*-minor.*

We recall that PG(k, q) is the rank-$(k+1)$ projective geometry over GF(q).

Rota's conjecture states that: *for any prime power q, there are only finitely many pairwise nonisomorphic excluded minors for the class of* GF(q)*-representable matroids.* Theorem 1.1 shows that

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excluded minors cannot contain large projective geometries. On the other hand, in [5] we prove that for any integer k there are only finitely many excluded minors that do not contain the cycle matroid of a $k \times k$ grid. While there is still a big gap to bridge between grids and projective geometries, we are encouraged by these complementary results.

We conjecture the following strengthening of Theorem 1.1; however, it is not clear whether this stronger version would provide additional leverage toward resolving Rota's conjecture.

Conjecture 1.2. *For each prime power q, no excluded minor for the class of* $\mathrm{GF}(q)$*-representable matroids contains a* $\mathrm{PG}(2, q)$*-minor.*

Oxley, Vertigan, and Whittle [8] gave examples showing that, for each $q > 5$, there is no bound on the number of inequivalent representations for 3-connected matroids over $\mathrm{GF}(q)$. This is in stark contrast with the following result, which plays a key role in the proof of Theorem 1.1.

Theorem 1.3. *If M is a* 3*-connected* $\mathrm{GF}(q)$*-representable matroid with a* $\mathrm{PG}(q, q)$*-minor, then M is uniquely* $\mathrm{GF}(q)$*-representable.*

We conjecture that this result can be sharpened to:

Conjecture 1.4. *If M is a* 3*-connected* $\mathrm{GF}(q)$*-representable matroid with a* $\mathrm{PG}(2, q)$*-minor, then M is uniquely* $\mathrm{GF}(q)$*-representable.*

We use the notation of Oxley [7], with the exception that the simplification of M is denoted by $\mathrm{si}(M)$ and the cosimplification of M is denoted by $\mathrm{co}(M)$.

2. Connectivity

Let M be a matriod. For any subset A of $E(M)$ we let $\lambda_M(A) = r_M(A) + r_M(E(M) - A) - r_M(E(M))$; λ_M is the *connectivity function* of M. For sets $A, B \subseteq E(M)$, we have

(i) $\lambda_M(A) = \lambda_M(E(M) - A)$,
(ii) $\lambda_M(A) \leqslant \lambda_M(A \cup \{e\}) + 1$ for each $e \in E(M)$, and
(iii) $\lambda_M(A) + \lambda_M(B) \geqslant \lambda_M(A \cup B) + \lambda_M(A \cap B)$.

It can be easily verified that $\lambda_M(X) = r_M(X) + r_{M^*}(X) - |X|$ and, hence, that $\lambda_M(X) = \lambda_{M^*}(X)$.

We let $\kappa_M(X_1, X_2) = \min\,(\lambda_M(A) : X_1 \subseteq A \subseteq E(M) - X_2)$. Note that if M' is a minor of M and $X_1, X_2 \subseteq E(M')$, then $\kappa_{M'}(X_1, X_2) \leqslant \kappa_M(X_1, X_2)$. The following theorem provides a good characterization for $\kappa_M(X_1, X_2)$; this theorem is in fact a generalization of Menger's theorem.

Theorem 2.1 (*Tutte's Linking Theorem [10]*). *Let M be a matroid and let X_1, X_2 be disjoint subsets of $E(M)$. Then there exists a minor M' of M, such that $E(M') = X_1 \cup X_2$ and $\lambda_{M'}(X_1) = \kappa_M(X_1, X_2)$.*

The following result shows that, if we apply Tutte's Linking Theorem when $\lambda_M(X_1) = \kappa_M(X_1, X_2)$, the resulting minor M' satisfies $M|X_1 = M'|X_1$.

Lemma 2.2. *Let M' be a minor of a matroid M and let $X \subseteq E(M)$. If $\lambda_M(X) = \lambda_{M'}(X)$, then $M|X = M'|X$.*

Proof. Note that

$$\begin{aligned}\lambda_M(X) &= r_M(X) + r_{M^*}(X) - |X| \\ &\leqslant r_{M'}(X) + r_{M'^*}(X) - |X| \\ &= \lambda_{M'}(X).\end{aligned}$$

Therefore, if $\lambda_M(X) = \lambda_{M'}(X)$, then $r_M(X) = r_{M'}(X)$ and, hence, $M|X = M'|X$. □

3-*connectivity*: The rest of this section is devoted to the proof of a connectivity result, Lemma 2.8, that is needed in Section 6.

A matroid M is *internally* 3-*connected* if M is connected and for any 2-separation (A, B) of M either $|A| = 2$ or $|B| = 2$. We require the following well-known results on 3-connected matroids.

Theorem 2.3 (*Bixby's Lemma [2]*). *If e is an element of a* 3-*connected matroid, then either $M \setminus e$ or M/e is internally* 3-*connected.*

Theorem 2.4 (*Tutte's Triangle Lemma [11]*). *Let $T = \{a, b, c\}$ be a triangle in a* 3-*connected matroid M with $|E(M)| \geqslant 4$. If neither $M \setminus a$ nor $M \setminus b$ is* 3-*connected, then there is a triad of M that contains a and exactly one of b and c.*

Theorem 2.5 (*Wheels and Whirls Theorem [11]*). *Let M be a* 3-*connected matroid with $E(M) \neq \emptyset$. If M is not a wheel or a whirl, then there exists $e \in E(M)$, such that $M \setminus e$ or M/e is* 3-*connected.*

Corollary 2.6. *If M is a* 3-*connected matroid with $E(M) \neq \emptyset$, then there exists $e \in E(M)$ such that* $\mathrm{si}(M/e)$ *is* 3-*connected.*

Proof. By the Wheels and Whirls Theorem, we can find a sequence of elements $e_1, \ldots, e_k$, such that

- (i) $M \setminus e_1, \ldots, e_i$ is 3-connected for each $i \in \{1, \ldots, k\}$, and
- (ii) either $M \setminus e_1, \ldots, e_k$ is a wheel or a whirl, or there exists an element e of $M \setminus e_1, \ldots, e_k$ such that $(M \setminus e_1, \ldots, e_k)/e$ is 3-connected.

In both cases arising from (ii), there exists an element e of $M \setminus e_1, \ldots, e_k$, such that $\mathrm{si}((M \setminus e_1, \ldots, e_k)/e)$ is 3-connected. But then $\mathrm{si}(M/e)$ is also 3-connected, as required. □

Lemma 2.7. *Let T be a triangle in a* 3-*connected matroid M with $|E(M)| \geqslant 4$. Then there exists $e \in T$ such that $M \setminus e$ is internally* 3-*connected.*

Proof. Suppose otherwise. The result can be readily checked on matroids with at most 6 elements, so we assume that $|E(M)| \geqslant 7$. By Tutte's Triangle Lemma, there exists a triad T^* with $|T \cap T^*| = 2$; let $e \in T - T^*$. Note that, $(T^*, E(M) - T^*)$ is a 2-separation in M/e. Then M/e is not internally 3-connected since $|E(M)| \geqslant 7$. So, by Bixby's Lemma, $M \setminus e$ is internally 3-connected. □

The following lemma is the main result of this section.

Lemma 2.8. *Let M be a 3-connected matroid with $|E(M)|\geqslant 5$. Suppose that no element of M is in both a triangle and a triad. Then there exist $u, v \in E(M)$ such that either*:

(1) *$M \setminus u$ and $M \setminus v$ are 3-connected, and $M \setminus u, v$ is internally 3-connected, or*
(2) *M/u and M/v are 3-connected, and $M/u, v$ is internally 3-connected.*

Proof. Suppose that M is a counterexample. Let $\Lambda(M)$ denote the set of elements $e \in E(M)$ such that $M \setminus e$ is 3-connected, and let $\Lambda^*(M)$ denote $\Lambda(M^*)$. The first three claims are straightforward, we leave the details to the reader.

2.8.1. *$r(M)\geqslant 4$ and $r^*(M)\geqslant 4$.*

2.8.2. *If $e \in \Lambda(M)$, then $\Lambda(M \setminus e) = \emptyset$.*

2.8.3. *If N is a 3-connected matroid, $e \in \Lambda(N)$, and $f \in \Lambda^*(N \setminus e)$, then either $f \in \Lambda^*(N)$ or there is a triangle of N containing both e and f.*

2.8.4. $\Lambda(M) \cup \Lambda^*(M) = E(M)$.

Proof. Suppose not; then there exists $e \in E(M)$ such that neither $M \setminus e$ nor M/e is 3-connected. By Bixby's Lemma and duality, we may assume that M/e is internally 3-connected. But then, since M/e is not 3-connected, e is in a triangle, say $T = \{e, a, b\}$. Now $M \setminus e$ is not 3-connected and neither a nor b is in a triad. Then, by Tutte's Triangle Lemma, both $M \setminus a$ and $M \setminus b$ are 3-connected. (We will obtain a contradiction by proving that $M \setminus a, b$ is internally 3-connected.) Let (A, B) be a 2-separation in $M \setminus e$ with $a \in A$. Note that $b \in B$, since otherwise $(A \cup \{e\}, B)$ would be a 2-separation in M. Since neither a nor b is in a triad, $|A|, |B|\geqslant 3$. Moreover, since $|E(M)|\geqslant 8$, by possibly swapping A and B we may assume that $|A|\geqslant 4$. Note that, $(A, B \cup \{e\})$ is a 3-separation in M, and $a \in \mathrm{cl}_M(B \cup \{e\})$. Thus $(A - \{a\}, B \cup \{e\})$ is a 2-separation in M/a and, hence $(A - \{a\}, (B \cup \{e\}) - \{b\})$ is a 2-separation in $M/a \setminus b$. Thus $(M \setminus b)/a$ is not internally 3-connected. However, $M \setminus b$ is 3-connected, so, by Bixby's Lemma, $M \setminus a, b$ is internally 3-connected. □

It follows from 2.8.4 that, if e is in a triangle, then $M \setminus e$ is 3-connected, and if e is in a triad, then M/e is 3-connected.

2.8.5. *If T is a triangle of M, then $\Lambda(M) \subseteq T$.*

Proof. Suppose, by way of contradiction, that there exists $e \in \Lambda(M) - T$. Thus $M \setminus e$ is 3-connected. Then, by Lemma 2.7, there exists $f \in T$ such that $M \setminus e, f$ is internally 3-connected. Moreover, by 2.8.4, $M \setminus f$ is 3-connected. □

2.8.6. *M contains no triangles and no triads.*

Proof. Suppose otherwise; then, by duality, we may assume that M has a triangle T. By 2.8.4 and 2.8.5, $\Lambda(M) = T$ and $\Lambda^*(M) = E(M) - T$. Thus T is the only triangle of M, and, since $\Lambda^*(M) > 3$, M contains no triads. Let $e \in E(M) - T$. By the duals of 2.8.2 and 2.8.3, $\Lambda^*(M/e) = \emptyset$ and $\Lambda(M/e) \subseteq T$.

Since $r(M) \geqslant 4$, there exists $f \in E(M/e) - \mathrm{cl}_{M/e}(T)$. As $f \notin T$ and $\Lambda(M/e) \subseteq T$, the minor $(M/e) \setminus f$ is not 3-connected. Moreover, since M/e has no triads, $(M/e) \setminus f$ is not internally 3-connected. So, by Bixby's Lemma, $M/e, f$ is internally 3-connected. □

2.8.7. *If* $e \in \Lambda(M)$ *and* $f \in E(M \setminus e)$, *then* $M \setminus e, f$ *is not internally* 3-*connected.*

Proof. Suppose that $M \setminus e, f$ is internally 3-connected. Then $M \setminus f$ is not 3-connected. Let (A, B) be a 2-separation in $M \setminus f$ with $e \in A$. Since M has no triads, $|A|, |B| \geqslant 3$. However, $(A - \{e\}, B)$ is a 2-separation in $M \setminus e, f$ and $M \setminus e, f$ is internally 3-connected, so $|A| = 3$. But, $\lambda_M(A) = 2$ so A is a triangle or a triad, contradicting 2.8.6. □

2.8.8. $\Lambda(M) = E(M)$ and $\Lambda^*(M) = E(M)$.

Proof. By symmetry we may assume that there exists $e \in \Lambda(M)$. By 2.8.7, for each $f \in E(M \setminus e)$, the minor $M \setminus e, f$ is not internally 3-connected. Then, by Bixby's Lemma, $M \setminus e/f$ is internally 3-connected. Moreover, since $M \setminus e$ has no triangles, $M \setminus e/f$ is 3-connected. Thus $\Lambda^*(M \setminus e) = E(M \setminus e)$. So, by 2.8.3 and 2.8.6, $E(M) - \{e\} \subseteq \Lambda^*(M)$. Now, since $|\Lambda^*(M)| \geqslant 2$, we can argue that $\Lambda(M) = E(M)$. Now $|\Lambda(M)| \geqslant 2$, so $\Lambda^*(M) = E(M)$. □

Let $e \in E(M)$. By Corollary 2.6, there exists $f \in E(M/e)$ such that $\mathrm{si}(M/e, f)$ is 3-connected. However, by the dual of 2.8.7, $M/e, f$ is not internally 3-connected. Thus, there is a 4-point line L in M/e that contains f. (That is, the restriction of M/e to L is isomorphic to $U_{2,4}$.) Note that M/e has no triads. Then, by Tutte's Triangle Lemma, there exists $a \in L$ such that $M/e \setminus a$ is 3-connected. Now, by Lemma 2.7, there exists $b \in L - \{a\}$ such that $M/e \setminus a, b$ is internally 3-connected. If $M/e \setminus a, b$ were 3-connected, then $M \setminus a, b$ would be internally 3-connected, contradicting 2.8.7. Thus $M/e \setminus a, b$ has a series-pair $\{c, d\}$. Since M/e has no triads, $\{a, b, c, d\}$ is a cocircuit of M/e. Since a circuit and a cocircuit cannot meet in exactly one element, $|L \cap \{a, b, c, d\}| \geqslant 3$. Moreover, since M/e is 3-connected and has at least 7 elements, $L \neq \{a, b, c, d\}$. By symmetry, we may assume that $d \notin L$. Now $M/e \setminus d$ is not internally 3-connected. So, by Bixby's Lemma, $M/e, d$ is internally 3-connected, contradicting 2.8.7. □

3. Unique representation

In this section we prove Theorem 1.3.

Let $\mathbb{F}$ be a field and let M be a matroid. Two $\mathbb{F}$-representations of M are *algebraically equivalent* if one can be obtained from the other by elementary row operations, column scaling, and field automorphisms. A matroid M is *uniquely* $\mathbb{F}$-representable if it is $\mathbb{F}$-representable and any two $\mathbb{F}$-representations of M are algebraically equivalent. The following result is referred to as the Fundamental Theorem of Projective Geometry (see [1, p. 85]).

Theorem 3.1. *For each prime power* q *and integer* $k \geqslant 2$, *the projective geometry* $\mathrm{PG}(k, q)$ *is uniquely* $\mathrm{GF}(q)$-*representable.*

Two $\mathbb{F}$-representations of M are *projectively equivalent* if one can be obtained from the other by elementary row operations, and column scaling. Two representations that are not projectively equivalent are said to be *projectively inequivalent*. By Theorem 3.1, the number of projectively inequivalent representations of $\mathrm{PG}(k, q)$, for $k \geqslant 2$, is $|\mathrm{Aut}(\mathrm{GF}(q))|$ where $\mathrm{Aut}(\mathrm{GF}(q))$ is the automorphism group of GF(q). Let N be a minor of M. We say that N *stabilizes* M *over* $\mathbb{F}$ if no $\mathbb{F}$-representation of N can be extended to two projectively inequivalent $\mathbb{F}$-representations of M.

Clones: Let e and f be distinct elements of M. We call e and f *clones* if there is an automorphism of M that swaps e and f and that acts as the identity on all other elements of M; that is, e and f are clones if $r_M(X \cup \{e\}) = r_M(X \cup \{f\})$ for each set $X \subseteq E(M) - \{e, f\}$.

Lemma 3.2. *Let e be an element of a matroid M and let* $\mathbb{F}$ *be a field. If* $M \setminus e$ *does not stabilize M over* $\mathbb{F}$, *then there exists an* $\mathbb{F}$*-representable matroid* M' *with* $E(M') = E(M) \cup \{f\}$ *such that* $M = M' \setminus f$, *and e and f are independent clones in* M'.

Proof. If $M \setminus e$ does not stabilize M over $\mathbb{F}$, then there is an $\mathbb{F}$-representation, say A, of $M \setminus e$ that extends to two projectively inequivalent $\mathbb{F}$-representations, say $[A, v_1]$ and $[A, v_2]$, of M. Let M' be the $\mathbb{F}$-representable matroid represented by the matrix $[A, v_1, v_2]$ where the last two columns are indexed by e and f, respectively. Clearly e and f are clones and, since the representations $[A, v_1]$ and $[A, v_2]$ are projectively inequivalent, $\{e, f\}$ is independent in M'. □

Lemma 3.3. *Let M be a* 3*-connected* GF(q)*-representable matroid and let* $L \subseteq E(M)$ *be a line of M. If* $|L| \geqslant q$ *and* $e, f \in E(M) - L$, *then e and f are not clones.*

Proof. Since M is 3-connected, $\kappa_M(L, \{e, f\}) = 2$. Then, by Tutte's Linking Theorem, there exists a minor N of M with $E(N) = L \cup \{e, f\}$ and $\lambda_N(L) = 2$. Since $\lambda_N(L) = 2$, it follows that $r_N(\{e, f\}) = r_N(L) = 2$ and that $e, f \in \mathrm{cl}_N(L)$. Thus $r(N) = 2$. However, N is GF(q)-representable and $|E(N)| \geqslant q + 2$. Thus N contains a parallel pair $\{x, y\}$. Now $\{e, f\}$ is not a parallel pair in N and $N|L = M|L$, so L does not contain a parallel pair. Thus $\{x, y\}$ contains one element of $\{e, f\}$ and one element of L. It follows that e and f are not clones in N, and, hence, they are not clones in M. □

Lemma 3.4. *Let e and f be clones in a matroid M. If* $M \setminus e$ *is* 3*-connected and M is not* 3*-connected, then e and f are parallel.*

Proof. If e and f are clones and $M \setminus e$ is 3-connected, then $M \setminus f$ is also 3-connected and si(M) is 3-connected. Thus, if M is not 3-connected, then e and f are in parallel. □

The following lemma is a key step in the proof of Theorem 1.3.

Lemma 3.5. *Let e and f be elements of a* 3*-connected* GF(q)*-representable matroid M. If* $M/e, f$ *is isomorphic to* PG(q, q), *then e and f are not clones in M.*

Proof. Let $N = M/e, f$ and suppose that e and f are clones. By Lemma 3.5, M has no q-point lines. So, if L is a $(q + 1)$-point line of N, then $r_M(L) \in \{3, 4\}$. Moreover, since M has 2-point lines, $q > 2$.

3.5.1. *There exists a rank-*3 *flat P of N such that* $e, f \in \mathrm{cl}_M(P)$.

Subproof. Suppose not. Then, for each line L of N, we have $r_M(L) = 3$. Consider M as a restriction of PG($q + 2, q$), and let Z be the line in PG($q + 2, q$) spanned by e and f. Each $(q + 1)$-

point line L of N spans a plane in $\mathrm{PG}(q+2,q)$, and this plane intersects Z in a point, say z_L. Suppose that there are two lines L_1 and L_2 of N such that $z_{L_1} \neq z_{L_2}$. If L_1 and L_2 do not meet at a point, then consider a third line L_3 of N that meets both L_1 and L_2. Note that either $z_{L_3} \neq z_{L_1}$ or $z_{L_3} \neq z_{L_2}$. Therefore, by possibly replacing one of L_1 and L_2 with L_3, we may assume that L_1 and L_2 meet at a point. Let $P = \mathrm{cl}_N(L_1 \cup L_2)$. Now e and f are spanned by $\{z_{L_1}, z_{L_2}\}$ and z_{L_1} and z_{L_2} are spanned by $L_1 \cup L_2$ in $\mathrm{PG}(q+2,q)$, so $e, f \in \mathrm{cl}_M(L_1 \cup L_2) \subseteq \mathrm{cl}_M(P)$. Now P is a rank-3 flat of N and $e, f \in \mathrm{cl}_M(P)$, as required.

Thus we may assume that there exists $z \in Z$, such that $z = z_L$ for each $(q+1)$-point line L of N. Let M' be the restriction of $\mathrm{PG}(q+2,q)$ obtained by adding z to M. Now, since $\{e, f, z\}$ is a line, $M'/e, z \setminus f = M'/e, f \setminus z = N$. Since M is 3-connected, $M'/z \setminus f$ is connected. Thus e is in the closure of $E(N)$ in $M'/z \setminus f$. So there is a circuit C of N such that C is independent in M'/z; among all such circuits we choose C as small as possible. Note that, each line of N is also a line of M'/z; thus $|C| > 3$. Let (I_1, I_2) be a partition of C into two sets with $|I_1|, |I_2| \geqslant 2$. Since C is a circuit of N and since N is a projective geometry, there exists a unique element a in $\mathrm{cl}_N(I_1) \cap \mathrm{cl}_N(I_2)$. Now $I_1 \cup \{a\}$ and $I_2 \cup \{a\}$ are both circuits of N and are both smaller than C. Thus, by our choice of C, $I_1 \cup \{a\}$ and $I_2 \cup \{a\}$ are both circuits in M'/z. However, this implies that $C = I_1 \cup I_2$ is dependent in M'/z. This contradiction completes the proof. □

3.5.2. *If P is a rank-3 flat of N, then there exists a restriction K of N such that $E(K) = P \cup L'$ where L' is a q-point line in K^*.*

Subproof. Let H be a matroid with $E(H) = L \cup \{a, b, c\}$, where L is a q-point line of H and a, b, and c are placed in parallel with distinct elements of L (recall that $q > 2$). Note that, H is $\mathrm{GF}(q)$-representable, H is cosimple, and $r^*(H) = q+1$. Thus there is a spanning restriction H' of N that is isomorphic to H^*. Now let $E(H') = L' \cup \{a', b', c'\}$ where a', b', c' are the elements corresponding to a, b, c. By the symmetry of N, we may assume that $a', b', c' \in P$. Finally, let $K = N|(L' \cup P)$; it is straightforward to check that K has the desired properties. □

Let P be the rank-3 flat of N given by 3.5.1, let K be the restriction of N given by 3.5.2, and let K' be the restriction of M to $E(K) \cup \{e, f\}$. Thus $K'/e, f = K$. Since $e, f \in \mathrm{cl}_{K'}(P)$, the elements e and f are not in series. Then, by the dual of Lemma 3.4, K' is 3-connected. Moreover, since L' is a q-point coline of K, it is also a coline in K'. Thus, by applying the dual of Lemma 3.3 to K' we obtain a final contradiction. □

Stabilizers for a class of matroids: We say that N *stabilizes* a class $\mathcal{M}$ of matroids *over* $\mathbb{F}$ if N stabilizes each 3-connected matroid in $\mathcal{M}$ that contains N as a minor. For brevity, when N stabilizes the class of $\mathbb{F}$-representable matroids over $\mathbb{F}$, we simply say that N is a *stabilizer* for $\mathbb{F}$.

Lemma 3.6. *Let q be a prime power and let N be a uniquely* $\mathrm{GF}(q)$*-representable stabilizer for* $\mathrm{GF}(q)$. Then N has $|\mathrm{Aut}(\mathrm{GF}(q))|$ *projectively inequivalent representations.*

Proof. This follows easily from Theorem 3.1 and the fact that N is a stabilizer for all projective geometries of sufficiently large rank. □

The following result shows that to test whether N stabilizes $\mathcal{M}$ we need only check matroids $M \in \mathcal{M}$ with $r(M) \leqslant r(N)+1$ and $r^*(M) \leqslant r^*(N)+1$.

Theorem 3.7 (*Whittle [12]*). *Let $\mathcal{M}$ be a class of matroids that is closed with respect to taking minors, duality, and isomorphism. A* 3*-connected matroid $N \in \mathcal{M}$ stabilizes $\mathcal{M}$ with respect to a field $\mathbb{F}$ if and only if N stabilizes each* 3*-connected matroid $M \in \mathcal{M}$ satisfying one of the following conditions*:

(i) $N = M \setminus e$ *for some* $e \in E(M)$,
(ii) $N = M/e$ *for some* $e \in E(M)$, or
(iii) $N = M \setminus e/f$ *for some* $e, f \in E(M)$ *where* $M \setminus e$ *and* M/f *are both* 3*-connected.*

We can now prove one of the main results of the paper.

Theorem 3.8. *For each prime power q,* $\mathrm{PG}(q, q)$ *is a stabilizer for* $\mathrm{GF}(q)$.

Proof. Let M be a 3-connected $\mathrm{GF}(q)$-representable matroid with a minor N isomorphic to $\mathrm{PG}(q, q)$. Since there are no 3-connected $\mathrm{GF}(q)$-representable extensions of $\mathrm{PG}(q, q)$, then, by Theorem 3.7, it suffices to consider the case that $N = M/e$ for some $e \in E(M)$.

Suppose that M is not stabilized by N. Then, by applying the dual of Lemma 3.2, we see that there exists a matroid M' with $E(M') = E(M) \cup \{f\}$ such that $M'/f = M$, the elements e and f are clones in M', and $\{e, f\}$ is coindependent in M'. Since $\{e, f\}$ is coindependent in M', e and f are not in series in M'. Then, by the dual of Lemma 3.4, M' is 3-connected. This contradicts Lemma 3.5. □

Theorem 1.3 is an immediate consequence of Theorems 3.8 and 3.1.

4. Path-width

Let M be a matroid on E. The *path-width* of M is the least integer k, such that there exists an ordering $(e_1, \ldots, e_n)$ of E, such that $\lambda_M(\{e_1, \ldots, e_i\}) \leqslant k$ for all $i \in \{1, \ldots, n\}$. In the remainder of the paper we shift our attention from Theorem 1.1 to the following result.

Theorem 4.1. *For any prime power q, there exists an integer k such that, each excluded minor for the class of* $\mathrm{GF}(q)$*-representable matroids that contains a* $\mathrm{PG}(q+6, q)$*-minor has path-width at most k.*

Theorem 4.1 implies Theorem 1.1. Indeed, it is straightforward to show that $\mathrm{PG}(k+1, q)$ has path-width $k+2$, and that path-width is non-increasing with respect to taking minors. Then, by Theorem 4.1, there is no excluded minor for the class of $\mathrm{GF}(q)$-representable matroids that contains a $\mathrm{PG}(k+1, q)$-minor, proving Theorem 1.1.

Let $\mathcal{A} = (A_1, \ldots, A_l)$ be an ordered partition of E. We let $\rho_M(\mathcal{A}) = \max(\lambda_M(A_1 \cup \cdots \cup A_i) : i \in \{1, \ldots, l\})$. We use the following two lemmas to obtain bounds on the path-width.

Lemma 4.2. *Let M be a matroid, $\mathcal{A} = (A_1, \ldots, A_l)$ and $\mathcal{B} = (B_1, \ldots, B_m)$ be two ordered partitions of $E(M)$, and let $\mathcal{C} = (A_1 \cap B_1, A_1 \cap B_2, \ldots, A_1 \cap B_m, \ldots, A_l \cap B_1, A_l \cap B_2, \ldots, A_l \cap B_m)$. Then $\rho_M(\mathcal{C}) \leqslant 2\rho_M(\mathcal{A}) + \rho_M(\mathcal{B})$.*

Proof. For each $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, m\}$, we let

$$\widehat{A}_i = A_1 \cup \cdots \cup A_i,$$

$$\begin{aligned}\widehat{B}_j &= B_1 \cup \cdots \cup B_j, \text{ and}\\ S_{ij} &= ((A_1 \cap B_1) \cup \cdots \cup (A_1 \cap B_m)) \cup \cdots\\ &\quad \cup ((A_{i-1} \cap B_1) \cup \cdots \cup (A_{i-1} \cap B_m)) \cup \cdots\\ &\quad \cup ((A_i \cap B_1) \cup \cdots \cup (A_i \cap B_j))\\ &= \widehat{A}_{i-1} \cup (\widehat{A}_i \cap \widehat{B}_j).\end{aligned}$$

Now there exists $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, m\}$, such that $\rho_M(\mathcal{C}) = \lambda_M(S_{ij})$. By submodularity,

$$\begin{aligned}\lambda_M(\widehat{A}_{i-1} \cup (\widehat{A}_i \cap \widehat{B}_j)) &\leqslant \lambda_M(\widehat{A}_{i-1}) + \lambda_M(\widehat{A}_i) + \lambda(\widehat{B}_j)\\ &\leqslant 2\rho_M(\mathcal{A}) + \rho_M(\mathcal{B}).\end{aligned}$$

Therefore $\rho_M(\mathcal{C}) = \lambda_M(S_{ij}) = \lambda_M(\widehat{A}_{i-1} \cup (\widehat{A}_i \cap \widehat{B}_j)) \leqslant 2\rho_M(\mathcal{A}) + \rho_M(\mathcal{B})$, as required. □

Lemma 4.3. *Let A, B, and X be disjoint sets of elements in a matroid M such that, for each $e \in X$, either $\kappa_{M\backslash e}(A, B) < \kappa_M(A, B)$ or $\kappa_{M/e}(A, B) < \kappa_M(A, B)$. Then there exists an ordering $(e_1, \ldots, e_m)$ of X and a partition $(Y_0, \ldots, Y_m)$ of $E(M) - X$ such that $A \subseteq Y_0$, $B \subseteq Y_m$, and $\rho_M(Y_0, \{e_1\}, Y_1, \ldots, \{e_m\}, Y_m) = \kappa_M(A, B)$.*

Proof. Let $k = \kappa_M(A, B)$. The result is vacuous when $X = \emptyset$. Suppose then that X is non-empty and let $e \in X$. Now, inductively, we can find an ordering $(e_1, \ldots, e_m)$ of $X - \{e\}$ and a partition $(Y_0, \ldots, Y_m)$ of $E(M) - (X - \{e\})$ such that $A \subseteq Y_0$, $B \subseteq Y_m$, and $\rho_M(Y_0, \{e_1\}, Y_1, \ldots, \{e_m\}, Y_m) = \kappa_M(A, B)$. Now $e \in Y_i$ for some $i \in \{0, \ldots, m\}$. Define

$$A' = \begin{cases} A & \text{if } i = 0,\\ (Y_0 \cup \cdots \cup Y_{i-1}) \cup \{e_1, \ldots, e_i\} & \text{if } i > 1\end{cases}$$

and

$$B' = \begin{cases} B & \text{if } i = m,\\ (Y_{i+1} \cup \cdots \cup Y_m) \cup \{e_{i+1}, \ldots, e_m\} & \text{if } i < m.\end{cases}$$

By duality we may assume that $\kappa_{M/e}(A, B) < k$. Thus there exists a partition (X_1, X_2) of $E(M/e)$ with $A \subseteq X_1$, $B \subseteq X_2$, and $\lambda_{M/e}(X_1) = k - 1$. It follows that $\lambda_M(X_1) = \lambda_M(X_1 \cup \{e\}) = k$ and that $e \in \mathrm{cl}_M(X_1) \cap \mathrm{cl}_M(X_2)$. If $A' = A$, then $A' \subseteq X_1$. On the other hand, if $A' \neq A$, then $\lambda_M(A') = k$. Then, by submodularity, $\lambda_M(A' \cap X_1) = k$ and $\lambda_M(A' \cup X_1) = k$. So, by replacing X_1 by $A' \cup X_1$, we get $A' \subseteq X_1$. Thus, in either case, we may assume that $A' \subseteq X_1$. Similarly, we may assume that $B' \subseteq X_2$. Finally, we get $\rho_M(Y_0, \{e_1\}, \ldots, Y_{i-1}, \{e_{i-1}\}, Y_i \cap X_1, \{e\}, Y_i \cap X_2, \{e_{i+1}\}, Y_{i+1}, \ldots, \{e_m\}, Y_m) = k$, as required. □

5. Final preparations

The following lemma is well-known; we prove it here for the sake of completeness.

Lemma 5.1. *Let $\mathbb{F}$ be a field and let M be an excluded minor for the class of $\mathbb{F}$-representable matroids. If $|E(M)| \geqslant 5$ then no element of M is in both a triangle and a triad.*

Proof. Suppose, by way of contradiction that $e \in E(M)$ is in both a triangle T and a triad T^*. Note that $|T \cap T^*| \geqslant 2$. Since M is 3-connected and $|E(M)| \geqslant 5$, we cannot have $T = T^*$. Thus $|T \cap T^*| = 2$; suppose that $T = \{e_1, e_2, e_3\}$ and $T^* = \{e_2, e_3, e_4\}$. Let N be a matroid isomorphic to $M(K_4)$, where one of the triangles in N is labelled by $\{e_1, e_2, e_3\}$. Now let M' be obtained by taking the generalized parallel connection of M/e_4 and N across the triangle $\{e_1, e_2, e_3\}$. Since M/e_4 is $\mathbb{F}$-representable, so is M'. However, $M' \setminus e_2, e_3$ is isomorphic to M. This contradiction completes the proof. □

Lemma 5.2. *Let M be a* GF(q)*-representable matroid and let N be a minor of M isomorphic to* PG$(k+2, q)$*. Then for each $e \in E(M)$ there exists a restriction N' of N isomorphic to* PG(k, q) *such that N' is a minor of both $M \setminus e$ and M/e.*

Proof. By deleting or contracting the other elements in a way that keeps N as a minor, we may assume that $E(M) = E(N) \cup \{e\}$. The result is straightforward if $e \in E(N)$; so assume that $e \notin E(N)$. We may also assume that e is neither a loop nor a coloop.

First consider the case that $N = M \setminus e$. Since M is GF(q)-representable, e is in parallel with some element $e' \in E(N)$. Since $e' \in E(N)$, there is a restriction N' of N isomorphic to PG(k, q) such that N' is a minor of both $M \setminus e'$ and M/e'. Thus, since e and e' are in parallel, N' is a minor of both $M \setminus e$ and M/e.

Now consider the case that $N = M/e$. Since e is not a coloop of M, there exists some triangle T of N such that $e \in \mathrm{cl}_M(T)$. Choose a restriction N' of N isomorphic to PG(k, q) such that $r_N(T \cup E(N')) = r(N') + 2$. Thus N' is a minor of N/T and hence also of M/T. However, e is a loop in M/T. So N' is a minor of both M/e and $M \setminus e$. □

A matroid M is called *stable* if it is connected and it cannot be written as the 2-sum of two non-binary matroids. This differs from the original definition in [4] since we require that M is connected. Suppose that $\eta_q(M)$ denotes the number of GF(q)-representations of M up to projective equivalence. It is easy to see that if M is the 2-sum of M_1 and M_2, then $\eta_q(M) = \eta_q(M_1)\eta_q(M_2)$. Moreover, if M is a binary matroid, then $\eta_q(M) = 1$. It follows that if M is a stable GF(q)-representable matroid, then by repeatedly decomposing across 2-separations we will obtain a 3-connected matroid M' such that $\eta_q(M) = \eta_q(M')$. It follows that if N is a stabilizer for GF(q), and if M is a stable matroid that contains N as a minor, then N stabilizes M over GF(q).

The following two lemmas can be derived from results in [12]; we include direct proofs for completeness.

Lemma 5.3. *Let M be a 3-connected matroid, let $u, v \in E(M)$ be such that $M \setminus u, v$ is stable, and suppose that $M \setminus u, v$ has a minor N that is uniquely* GF(q)*-representable and is a stabilizer for* GF(q)*. If $M \setminus u$ and $M \setminus v$ are both* GF(q)*-representable, then there exists a* GF(q)*-representable matroid M', such that $M' \setminus u = M \setminus u$ and $M' \setminus v = M \setminus v$.*

Proof. Let B be a basis of M containing neither u nor v. Consider GF(q)-representations A_1 and A_2 of $M \setminus u$ and $M \setminus v$, respectively. By applying row operations we may assume that:

$$A_1 = \begin{pmatrix} \overset{B}{I} & C_1 & \overset{v}{y} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \overset{B}{I} & C_2 & \overset{u}{x} \end{pmatrix}.$$

Thus (I, C_1) and (I, C_2) are both GF(q)-representations of $M \setminus u, v$. However, $M \setminus u, v$ is uniquely GF(q)-representable since N is a minor of $M \setminus u, v$. Therefore, by possibly applying a field automorphism and rescaling, we may assume that $C_1 = C_2$. Now let M' be the matroid represented over GF(q) by

$$\begin{array}{cccc} B & & u & v \\ \big(I & C_1 & x & y\big). \end{array}$$

Clearly $M' \setminus u = M \setminus u$ and $M' \setminus v = M \setminus v$, as required. □

Lemma 5.4. *Let M_1 and M_2 be* GF(q)*-representable matroids on the same ground set and let $u, v \in E(M_1)$ be such that $M_1 \setminus u = M_2 \setminus u$* and *$M_1 \setminus v = M_2 \setminus v$. If $M_1 \setminus u$ and $M_2 \setminus v$ are both stable, $M_1 \setminus u, v$ is connected, and $M_1 \setminus u, v$ has a minor N that is uniquely* GF(q)*-representable and is a stabilizer for the class of* GF(q)*-representable matroids, then $M_1 = M_2$.*

Proof. Since $M_1 \setminus u$ and $M_1 \setminus v$ are connected, $\{u, v\}$ is co-independent. Thus there exists a basis B of M_1 disjoint from u and v. For each $i \in \{1, 2\}$, consider a GF(q)-representation A_i of M_i where:

$$\begin{array}{ccccc} & B & & u & v \\ A_i = & \big(I & C_i & x_i & y_i\big). \end{array}$$

Now (I, C_1, x_1) and (I, C_2, x_2) are both representations of $M_1 \setminus v$. However, $M_1 \setminus v$ is uniquely GF(q)-representable since it is stable and contains N as a minor. Therefore, by possibly applying a field automorphism and rescaling, we may assume that $C_2 = C_1$ and $x_2 = x_1$. So we may assume that $A_2 = (I, C_1, x_1, y_2)$. Now we have two representations, (I, C_1, y_1) and (I, C_1, y_2), of $M_1 \setminus u$ and, since $M_1 \setminus u$ is stable and contains N as a minor, these representations are algebraically equivalent. Consider the operations required to transform (I, C_1, y_1) into (I, C_1, y_2); we have at our disposal row operations, column scaling, and field automorphisms. The common identity matrix limits the row operations to row scaling. Since $M_1 \setminus u, v$ contains N as a minor and since, by Theorem 3.1, N has $|\mathrm{Aut}(\mathrm{GF}(q))|$ weakly inequivalent representations, we cannot apply field automorphisms (while keeping (I, C_1) and (I, C_2) projectively equivalent). Moreover, since $M_1 \setminus u, v$ is connected, the only scalings that we may apply to (I, C_1) without changing it are trivial (that is, multiply every row by a constant α and divide all columns by α). Therefore y_2 is obtained from y_1 by scaling, and, hence, $M_2 = M_1$. □

The next result is considerably harder to prove; we defer the proof to Sections 8–10. Before stating the result we need some definitions. If M_1 and M_2 are two matroids on a common ground set, then a set B is said to *distinguish* M_1 from M_2 if B is a basis of exactly one of M_1 and M_2. Let X be a set of elements in a matroid M. We say that X is *connected* in M if X is contained in a single component of M. We say that X is *3-connected* in M if X is connected and for any partition (X_1, X_2) of X with $|X_1|, |X_2| \geqslant 2$ we have $\kappa_M(X_1, X_2) \geqslant 2$.

Lemma 5.5. *Let M, M', and N be matroids, let B be a basis of M, let $u, v \in E(M) - B$, and let $a, b \in B$ be such that*

(1) *M' is a* GF(q)*-representable matroid on the same ground set as M, $M' \setminus u = M \setminus u$, $M' \setminus v = M \setminus v$, and $(B - \{a, b\}) \cup \{u, v\}$ distinguishes M from M';*
(2) *N is a uniquely* GF(q)*-representable stabilizer for* GF(q) *and N is a minor of $M \setminus u, v$;* and
(3) *$E(N) \cup \{a, b, u\}$ is* 3*-connected in $M \setminus v$ and $E(N) \cup \{a, b, v\}$ is* 3*-connected in $M \setminus u$.*

Then M is not GF(q)*-representable.*

6. Proof of Theorem 4.1

Let s denote the number of elements of $\mathrm{PG}(q,q)$, and let t be the number of $\mathrm{PG}(q,q)$ restrictions of $\mathrm{PG}(q+2,q)$. In this section we prove Theorem 4.1 with $k = 24t2^{s+3} + 4$.

Let M be an excluded minor for the class of $\mathrm{GF}(q)$-representable matroids. Suppose by way of contradiction that M contains a $\mathrm{PG}(q+6,q)$- or a $\mathrm{PG}(q+6,q)^*$-minor and that the path-width of M is greater than k. By Lemma 5.1, no element of M is in both a triangle and a triad. Therefore, by Lemma 2.8 and by possibly replacing M with M^*, we may assume that there exist elements $u, v \in E(M)$ such that $M \setminus u$ and $M \setminus v$ are 3-connected and $M \setminus u, v$ is internally 3-connected. By Lemma 5.2, $M \setminus u, v$ has a $\mathrm{PG}(q+2,q)$- or a $\mathrm{PG}(q+2,q)^*$-minor N. Therefore, by Lemma 5.3 and Theorem 3.8, there exists a $\mathrm{GF}(q)$-representable matroid M' on the same ground set as M such that $M' \setminus u = M \setminus u$ and $M' \setminus v = M \setminus v$. Moreover, by Lemma 5.4, M' is unique.

6.1. *There exists a basis B of M and elements $a, b \in B$ such that $u, v \notin B$ and $(B - \{a,b\}) \cup \{u,v\}$ distinguishes M from M'.*

Proof. Suppose that B' distinguishes M from M'. Since M is 3-connected, there exists a basis B of M that is disjoint from $\{u,v\}$; we choose such B minimizing $|B' - B|$. Note that $|B| = |B'|$ and that $u, v \in B' - B$; thus, if $|B' - B| = 2$, then 6.1 holds (take a and b to be the two elements in $B - B'$). Hence, we may assume that $|B' - B| > 2$; let $x \in (B' - B) - \{u,v\}$. By one of the standard basis exchange axioms, there exists $y \in B - B'$ such that $(B \cup \{x\}) - \{y\}$ is a basis of at least one of M and M'; let $B'' = (B \cup \{x\}) - \{y\}$. Since $u, v \notin B''$, B'' does not distinguish M from M'. Thus B'' is a basis of M that contains neither u nor v. However, $|B' - B''| < |B' - B|$, contradicting our choice of B. □

Let $N' \in \{N, N^*\}$ be isomorphic to $\mathrm{PG}(q+2,q)$, and let $N'_1, \ldots, N'_t$ be the $\mathrm{PG}(q,q)$-restrictions of N'. Now, for each $i \in \{1,\ldots,t\}$, let $N'_i = N_i$ if $N' = N$ and let $N'_i = N^*_i$ if $N' = N^*$. Let $Z = E(M) - \{a,b,u,v\}$. Now, for each $i \in \{1,\ldots,t\}$, let Z_i denote the set of all elements $e \in Z$ such that $(M \setminus u, v) \setminus e$ and $(M \setminus u, v)/e$ both contain N_i as a minor. By Lemma 5.2, each element in Z is contained in at least one of $Z_1, \ldots, Z_t$.

For each $i \in \{1,\ldots,t\}$, let $\Pi_i(u)$ denote the set of all partitions (A_1, A_2) of $E(N_i) \cup \{a,b,v\}$ such that $\kappa_{M\setminus u}(A_1,A_2) = 2$, and let $\Pi_i(v)$ denote the set of all partitions (A_1,A_2) of $E(N_i) \cup \{a,b,u\}$ such that $\kappa_{M\setminus v}(A_1,A_2) = 2$. Recall that $|E(N_i)| = s$, so we trivially get $|\Pi_i(u)|, |\Pi_i(v)| \leqslant 2^{s+3}$.

6.2. *For each $e \in Z_i$ either*

(a) *there exists $(A_1,A_2) \in \Pi_i(u)$ such that either $\kappa_{(M\setminus u)\setminus e}(A_1,A_2) < 2$ or $\kappa_{(M\setminus u)/e}(A_1,A_2) < 2$; or*
(b) *there exists $(A_1,A_2) \in \Pi_i(v)$ such that either $\kappa_{(M\setminus v)\setminus e}(A_1,A_2) < 2$ or $\kappa_{(M\setminus v)/e}(A_1,A_2) < 2$.*

Proof. If $e \notin B$, then let

$$M_1 = M \setminus e,\ M'_1 = M' \setminus e,\ \text{and } B_1 = B.$$

If $e \in B$, then let

$$M_1 = M/e,\ M'_1 = M'/e,\ \text{and } B_1 = B - \{e\}.$$

Note that, B_1 is a basis of M_1. Moreover

(1) M_1 and M_1' are GF(q)-representable matroids on the same ground set, $M_1' \setminus u = M_1 \setminus u$, $M_1' \setminus v = M_1 \setminus v$, and $(B_1 - \{a, b\}) \cup \{u, v\}$ distinguishes M_1 from M_1'; and
(2) N_i is a uniquely GF(q)-representable stabilizer for GF(q) and N_i is a minor of $M_1 \setminus u, v$.

Then, by Lemma 5.5, either

(i) $E(N_i) \cup \{a, b, u\}$ is not 3-connected in $M_1 \setminus v$, or
(ii) $E(N_i) \cup \{a, b, v\}$ is not 3-connected in $M_1 \setminus u$.

However, $E(N_i) \cup \{a, b, u\}$ is 3-connected in $M \setminus v$ and $E(N_i) \cup \{a, b, v\}$ is 3-connected in $M \setminus u$. It follows that one of (a) and (b) hold. □

The result is now relatively straightforward, we just apply Lemmas 4.3 and 4.2 to bound the path-width of M.

For each $i \in \{1, \ldots, t\}$, $w \in \{u, v\}$, and $\pi = (A_1, A_2) \in \Pi_i(w)$, let $Z_i(w, \pi)$ denote the set of all elements $e \in Z_i$ for which either $\kappa_{(M\setminus w)\setminus e}(A_1, A_2) < 2$ or $\kappa_{(M\setminus w)/e}(A_1, A_2) < 2$.

6.3. *For each $i \in \{1, \ldots, t\}$, $w \in \{u, v\}$, and $\pi = (A_1, A_2) \in \Pi_i(w)$ there exists an ordering $(e_1, \ldots, e_m)$ of $Z_i(w, \pi)$ and a partition $(Y_0, \ldots, Y_m)$ of $E(M) - Z_i(w, \pi)$, such that $\rho_M(Y_0, \{e_1\}, Y_1, \ldots, \{e_m\}, Y_m) \leqslant 3$.*

Proof. By Lemma 4.3, there exists an ordering $(e_1, \ldots, e_m)$ of $Z_i(w, \pi)$ and a partition $(Y_0, \ldots, Y_m)$ of $(E(M) - Z_i(w, \pi)) - \{w\}$ such that $\rho_{M\setminus w}(Y_0, \{e_1\}, Y_1, \ldots, \{e_m\}, Y_m) \leqslant 2$. Adding w to Y_0 gives the result. □

Now let $Z_i(w)$ denote the union of the sets $Z_i(w, \pi)$ over all $\pi \in \Pi_i(w)$. By 6.3 and Lemma 4.2, we get

6.4. *For each $i \in \{1, \ldots, t\}$ and $w \in \{u, v\}$, there exists an ordering $(e_1, \ldots, e_m)$ of $Z_i(w)$ and a partition $(Y_0, \ldots, Y_m)$ of $E(M) - Z_i(w)$, such that $\rho_M(Y_0, \{e_1\}, Y_1, \ldots, \{e_m\}, Y_m) \leqslant 6|\Pi_i(w)| \leqslant 6\left(2^{s+3}\right)$.*

Now, for each $e \in Z$, there exists $i \in \{1, \ldots, t\}$ such that $e \in Z_i(u)$ or $e \in Z_i(v)$. Then, by 6.4 and Lemma 4.2, we get

6.5. *There exists an ordering $(e_1, \ldots, e_m)$ of Z and a partition $(Y_0, \ldots, Y_m)$ of $E(M) - Z$ such that $\rho_M(Y_0, \{e_1\}, Y_1, \ldots, \{e_m\}, Y_m) \leqslant 24t2^{s+3}$.*

Now $E(M) - Z = \{a, b, u, v\}$ so, by 6.5, $M \setminus \{u, v, a, b\}$ has path-width at most $24t2^{s+3}$. Hence, M has path-width at most $24t2^{s+3} + 4 = k$. This contradiction completes the proof. □

7. Fixing a basis

In the proof of Lemma 5.5, we work with a pair (M, B) where B is a fixed basis of the matroid M. In this section we formalize the notion of a matroid viewed with respect to a fixed basis. The results given here were introduced in [4]; we use different notation in the hope of keeping a closer connection to more familiar matroid notions.

We denote the symmetric difference of sets X and Y by $X\Delta Y$; that is, $X\Delta Y = (X-Y)\cup(Y-X)$.

Let B be a basis of a matroid M. A set $X \subseteq E(M)$ is a *feasible set* of (M, B) if $X\Delta B$ is a basis of M. Duality is quite transparent in this setting, since (M, B) and $(M^*, E(M) - B)$ have the same feasible sets.

Representations: An $\mathbb{F}$-*representation* of (M, B) is a $B \times (E(M) - B)$ matrix A over $\mathbb{F}$, such that

$$\begin{pmatrix} \overset{B}{I} & A \end{pmatrix}$$

is an $\mathbb{F}$-representation of M. (Elsewhere, A is often called a *standard representation*.) Note that, $X \subseteq E(M)$ is a feasible set of (M, B) if and only if $|X \cap B| = |X - B|$ and the submatrix $A[X \cap B, X - B]$ is non-singular. (Many of the results given below are straightforward for representable matroids.)

Fundamental graphs: The *fundamental graph* of (M, B), denoted by $G_{(M,B)}$ or by G_B, is the graph whose vertex set is $E(M)$ and whose edge set is given by the 2-element feasible sets of (M, B). Note that G_B is bipartite with bipartition $(B, E(M) - B)$. For $X \subseteq E(M)$, we denote by $G_B[X]$ the subgraph of G_B induced by the vertex set X. The following results relate feasible sets to the fundamental graph.

Lemma 7.1 (*Brualdi [3]*). *If X is a feasible set of* (M, B), *then* $G_B[X]$ *has a perfect matching.*

Lemma 7.2 (*Krogdahl [6]*). *If* $G_B[X]$ *has a unique perfect matching, then X is a feasible set of* (M, B).

Minors: For any $X \subseteq E(M)$, we let

$$M[X, B] = M \setminus (E(M) - (X \cup B))/(B - X);$$

such minors are said to be *visible* with respect to B. It is straightforward to show that, for any minor N of M, there exists a basis B' of M such that $N = M[E(N), B']$. Note that $B \cap X$ is a basis of $M[X, B]$ and the fundamental graph of $(M[X, B], B \cap X)$ is $G_B[X]$. Moreover, if A is a representation of (M, B) then $A[B \cap X, X - B]$ is a representation of $(M[X, B], B \cap X)$.

Pivoting: We will need to change bases; for example, to make some minor visible. Suppose that X is a feasible set of (M, B). Then $B\Delta X$ is a basis of M. Now Y is a feasible set of $(M, B\Delta X)$ if and only if $X\Delta Y$ is a feasible set of (M, B). Typically we will shift from (M, B) to $(M, B\Delta\{x, y\})$ for some edge $\{x, y\}$ of G_B; such a change is referred to as a *pivot on xy*. Let $B' = B\Delta\{x, y\}$. We can determine much of the structure of $G_{B'}$ from G_B. Note that uv is an edge of $G_{B'}$ if and only if $\{u, v\}\Delta\{x, y\}$ is feasible in (M, B). Thus

(i) $\{x, y\}$ is an edge of $G_{B'}$.
(ii) If $v \in E(M) - \{x, y\}$, then xv is an edge of $G_{B'}$ if and only if yv is an edge of G_B. Similarly, yv is an edge of $G_{B'}$ if and only if xv is an edge of G_B.
(iii) If $u, v \in E(M) - \{x, y\}$ and v is adjacent to neither x nor y in G_B, then uv is an edge of $G_{B'}$ if and only if uv is an edge of G_B.
(iv) If $u, v \in E(M) - \{x, y\}$ where ux and vy are edges of G_B but uv is not, then uv is an edge of $G_{B'}$.

This leaves only one problematic case: if $G_B[\{x, y, u, v\}]$ is a circuit, then we cannot determine whether uv is an edge of $G_{B'}$ using only information from G_B. All we can say in this case is that, uv is an edge of $G_{B'}$ if and only if $\{x, y, u, v\}$ is a feasible set of (M, B).

A set $X \subseteq E(M)$ is a *twirl* of (M, B) if $G_B[X]$ is an induced circuit and X is feasible; it is easy to check that if X is a twirl, then $M[X, B]$ is a whirl. We are only interested in 4-element twirls; these are precisely visible $U_{2,4}$-minors.

Connectivity and fundamental graphs: The following results help us identify 1- and 2-separations using fundamental graphs. In each of the these results, B is a basis of a matroid M.

Lemma 7.3. *Let $Y \subseteq E(M)$. Then, $\lambda_M(Y) > 0$ if and only if there exists an edge uv of G_B with $u \in Y$ and $v \in V - Y$.*

Corollary 7.4. *M is connected if and only if G_B is connected.*

A partition (X_1, X_2) of $E(M)$ is called a *split* of G_B if $|X_1|, |X_2| \geqslant 2$ and the edges of G_B connecting X_1 to X_2 induce a complete bipartite graph; that is, there exist $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ such that each vertex in Y_1 is adjacent to each vertex in Y_2, and these are the only edges between X_1 and X_2.

Lemma 7.5. *If (X_1, X_2) is a 2-separation in M, then (X_1, X_2) is a split of G_B.*

A partial converse is given by the following result.

Lemma 7.6 (*See [4, Proposition 4.12]*). *Let (X_1, X_2) be a split in G_B and let $x_1 \in X_1$ and $x_2 \in X_2$ where x_1 and x_2 are adjacent in G_B. Then, (X_1, X_2) is a 2-separation in M if and only if there is no twirl $\{x_1, x_2, y_1, y_2\}$ in (M, B) with $y_1 \in X_1$ and $y_2 \in X_2$.*

Series and parallel elements: Suppose that x and y are parallel in M. We may assume that $y \notin B$. If $x \in B$, then y is *pendant* to x in G_B; that is, x is the only neighbour of y. On the other hand, if $x \notin B$, then x and y are *twins* in G_B; that is, x and y have the same neighbours. Similarly, if x and y are in series in M and $y \in B$, then either x is pendant to y in G_B or x and y are twins. The converse need not be true. If x and y are twins in G_B, then x and y need not be in series or in parallel. However, by 7.6, if x is pendant to y in G_B, then either x and y are in series (when $x \in B$) or x and y are in parallel (when $x \notin B$).

8. 3-Connected sets and fundamental graphs

In this section we prove various connectivity results, most of which concern 3-connected sets in a matroid with a fixed basis. Let X be a 3-connected set in a connected matroid M. Now let $\mathcal{F}_M(X) = \{Z \subseteq E(M) : \lambda_M(Z) \leqslant 1 \text{ and } |X \cap Z| \leqslant 1\}$ and let $\Pi_M(X)$ be the collection of maximal sets in $\mathcal{F}_M(X)$.

Lemma 8.1. *If X is a 3-connected set in a connected matroid M and $|X| \geqslant 4$, then $\Pi_M(X)$ is a partition of $E(M)$.*

Proof. Note that, for each $v \in E(M)$, we have $\{v\} \in \mathcal{F}_M(X)$. Thus it suffices to prove that, if $Z_1, Z_2 \in \mathcal{F}_M(X)$ and $Z_1 \cap Z_2 \neq \emptyset$, then $Z_1 \cup Z_2 \in \mathcal{F}_M(X)$. By submodularity, $\lambda_M(Z_1) + \lambda_M(Z_2) \geqslant \lambda_M(Z_1 \cap Z_2) + \lambda_M(Z_1 \cup Z_2)$. Since $Z_1, Z_2 \in \mathcal{F}_M(X)$, we have $\lambda_M(Z_1), \lambda_M(Z_2) \leqslant 1$. Moreover, since $Z_1 \cap Z_2 \neq \emptyset$ and since M is connected, we have $\lambda_M(Z_1 \cap Z_2) \geqslant 1$. Therefore $\lambda_M(Z_1 \cup Z_2) \leqslant 1$. Now $|(Z_1 \cup Z_2) \cap X| \leqslant 2$ so $|X - (Z_1 \cup Z_2)| \geqslant 2$. Hence, since X is a 3-connected set, we must have $|(Z_1 \cup Z_2) \cap X| \leqslant 1$ and, so, $Z_1 \cup Z_2 \in \mathcal{F}_M(X)$, as required. □

For any $\pi \subseteq E(M)$, we let $\partial_{(M,B)}(\pi)$ be the elements of π that have a neighbour in $E(M) - \pi$ in G_B. For a partition Π of $E(M)$, we let $\partial_{(M,B)}(\Pi)$ denote $(\partial_{(M,B)}(\pi) : \pi \in \Pi)$. Where there is no fear of ambiguity we denote $\partial_{(M,B)}$ by ∂_B. Now suppose that B is a basis of M and that (X_1, X_2) is a 2-separation of M. Then, as noted in the previous section, (X_1, X_2) is a split of G_B. Now let $x_1 \in \partial_B(X_1)$ and $x_2 \in \partial_B(X_2)$. It is straightforward to prove that M is the 2-sum of $M[X_1 \cup \{x_2\}, B]$ and $M[\{x_1\} \cup X_2, B]$ (identifying x_1 with x_2) and that, up to isomorphism, these matroids do not depend on the particular choice of x_1 and x_2. Decomposing across each of the 2-separations given by the parts of $\Pi_M(X)$, we obtain the following lemma.

Lemma 8.2. *Let B be a basis of a connected matroid M and let X be a 3-connected set of M with $|X| \geqslant 4$. If T is a transversal of $\partial_B(\Pi_M(X))$, then $M[T, B]$ is 3-connected. Moreover, if N is a 3-connected minor of M with $X \subseteq E(N)$, then $M[T, B]$ has a minor isomorphic to N.*

Lemma 8.2 provides a way of recognizing that certain minors are 3-connected; we also need to recognize that certain minors are stable.

Lemma 8.3. *Let B be a basis in a connected matroid M and let $X \subseteq E(M)$ be a 3-connected set in M with $|X| \geqslant 4$. If $S \subseteq E(M)$ where $S \cap \pi \neq \emptyset$ for each $\pi \in \Pi_M(X)$ and each component of $G_B[S \cap \pi]$ is a tree containing exactly one element of $\partial_B(\pi)$, then $M[S, B]$ is stable.*

Proof. Note that, there is a transversal $T \subseteq S$ of $\partial_B(\Pi_M(X))$. By Lemma 8.2, $M[T, B]$ is 3-connected. Moreover, we can obtain $M[T, B]$ from $M[S, B]$ by repeated simplification and cosimplification. Thus $M[S, B]$ is stable. □

We need the following elementary fact about bipartite graphs; the easy proof is left to the reader.

Lemma 8.4. *If $G = (V, E)$ is a connected bipartite graph and $u, v, w \in V$, then there exists $A \subseteq V$, such that $u, v, w \in A$ and $G[A]$ is a tree.*

Lemma 8.5. *Let B be a basis in a connected matroid M and let $X \subseteq E(M)$ be a 3-connected set in M with $|X| \geqslant 4$. If $\pi \in \Pi_M(X)$ and $Z \subseteq \pi$ with $|Z| \leqslant 2$, then there exists $S \subseteq \pi$, such that $Z \subseteq S$ and each component of $G_B[S]$ is a tree with exactly one vertex in $\partial_B(\pi)$.*

Proof. Let $v \in E(M) - \pi$ be a vertex of G_B that has a neighbour in π. By Lemma 8.4, there exists $S \subseteq \pi$, such that $Z \subseteq S$ and $G_B[S \cup \{v\}]$ is a tree. Since v is adjacent to every vertex in $\partial_B(\pi)$, each component of $G_B[S]$ is a tree with exactly one vertex in $\partial_B(\pi)$. □

Lemma 8.6. *Let e be an element of a connected matroid M and let N be a 3-connected non-binary minor of $M \setminus e$. If $M \setminus e$ is stable but M is not stable, then there exists $\pi \in \Pi_{M \setminus e}(E(N))$ such that $\lambda_M(\pi \cup \{e\}) = 1$.*

Proof. If M is not stable, then M can be expressed as the 2-sum of two non-binary matroids M_1 and M_2 on ground sets $X_1 \cup \{z\}$ and $X_2 \cup \{z\}$ respectively. By symmetry, we may assume that $e \in X_1$. Moreover, since $M \setminus e$ is stable, $M_1 \setminus e$ is binary. It follows that $|X_1 \cap E(N)| \leqslant 1$. Thus there exists $\pi \in \Pi_{M \setminus e}(E(N))$ such that $X_1 - \{e\} \subseteq \pi$. Now, since $\lambda_M(X_1) = \lambda_{M \setminus e}(X_1 - \{e\})$, we have $e \in \text{cl}_M(X_1 - \{e\})$. Then $e \in \text{cl}_M(\pi)$ and, hence, $\lambda_M(\pi \cup \{e\}) = 1$. □

We conclude this section with two easy connectivity results.

Lemma 8.7. *Let (X, D, Y) be a partition of the ground set of a matroid M where D is co-independent in M. Then, $\lambda_M(X) = \lambda_{M \setminus D}(X)$ if and only if $D \subseteq \text{cl}_M(Y)$.*

Proof. Note that,

$$\begin{aligned}
\lambda_M(X) - \lambda_{M\setminus D}(X) &= (r_M(X) + r_M(D \cup Y) - r(M)) \\
&\quad - (r_M(X) + r_M(Y) - r_M(X \cup Y)) \\
&= (r_M(X) + r_M(D \cup Y) - r(M)) \\
&\quad - (r_M(X) + r_M(Y) - r(M)) \\
&= r_M(D \cup Y) - r_M(Y).
\end{aligned}$$

Thus, $\lambda_M(X) = \lambda_{M \setminus D}(X)$ if and only if $D \subseteq \text{cl}_M(Y)$. □

Lemma 8.8. *Let X and Y be disjoint sets of elements of a matroid M and let B be a basis of M. If $\lambda_M(X) > \lambda_{M[X\cup Y, B]}(X)$, then there exists $e \in E(M) - (X \cup Y)$, such that $\lambda_{M[X\cup Y\cup\{e\}, B]}(X) > \lambda_{M[X\cup Y, B]}(X)$.*

Proof. Let $C = (E(M) - (X \cup Y)) \cap B$ and let $D = E(M) - (X \cup Y \cup C)$. By using duality, we may assume that D is not empty. Now let $N = M/C$; thus $N \setminus D = M[X \cup Y, B]$. Suppose that $\lambda_N(X) > \lambda_{N\setminus D}(X)$. Then, by Lemma 8.7, there exists $e \in D$ such that $e \notin \text{cl}_N(Y)$. Then, again by Lemma 8.7, $\lambda_{M[X\cup Y\cup\{e\}, B]}(X) = \lambda_{N\setminus(D-\{e\})}(X) > \lambda_{N\setminus D}(X) = \lambda_{M[X\cup Y, B]}(X)$, as required. Therefore we may assume that $\lambda_N(X) = \lambda_{N\setminus D}(X)$. Then, by Lemma 8.7, $D \subseteq \text{cl}_N(Y)$. However, since $N = M/C$, we have $D \subseteq \text{cl}_M(Y \cup C)$. So, by Lemma 8.7, $\lambda_{M\setminus D}(X) = \lambda_M(X) > \lambda_{(M\setminus D)/C}(X)$. But $D \neq \emptyset$, so by replacing M with $M \setminus D$ the result follows inductively. □

9. Proof of Lemma 5.5

Recall that M, M', and N are matroids, B is a basis of M, $u, v \in E(M) - B$, and $a, b \in B$ sayisfying

(1) M' is a GF(q)-representable matroid on the same ground set as M, $M' \setminus u = M \setminus u$, $M' \setminus v = M \setminus v$, and $(B - \{a, b\}) \cup \{u, v\}$ distinguishes M from M';
(2) N is a uniquely GF(q)-representable stabilizer for GF(q) and N is a minor of $M \setminus u, v$; and
(3) $E(N) \cup \{a, b, u\}$ is 3-connected in $M \setminus v$ and $E(N) \cup \{a, b, v\}$ is 3-connected in $M \setminus u$.

We will need that N is non-binary. It is straightforward to show that a binary matroid can only be a stabilizer over GF(2) or GF(3). On the other hand, Lemma 5.5 is straightforward when $q \in \{2, 3\}$. Therefore we may assume that N is non-binary.

Note that $G_{(M,B)}$ and $G_{(M',B)}$ are the same; we denote this graph by G_B. Since $E(N)\cup\{u,a,b\}$ is 3-connected in $M\setminus v$, the set $E(N)\cup\{a,b\}$ is connected in $M\setminus u,v$. Thus $E(N)\cup\{a,b\}$ is contained in a component, say H, of G_B-u-v. Now it is easy to check that the hypotheses of Lemma 5.5 are satisfied when we replace M and M' by $M[V(H)\cup\{u,v\},B]$ and $M'[V(H)\cup\{u,v\},B]$, respectively. Thus we may assume that $M\setminus u,v$ is connected.

A set $F\subseteq E(M)$ *distinguishes* (M,B) from (M,B') if F is a feasible set of exactly one of (M,B) and (M,B'). Thus $\{a,b,u,v\}$ distinguishes (M,B) from (M,B'). Since $M\setminus u=M'\setminus u$ and $M\setminus v=M'\setminus v$, both u and v are contained in any set that distinguishes (M,B) from (M,B'). During the proof we change our choice of a, b, and B; however, we are careful that a, b, and B are chosen such that they satisfy the following four conditions:

9.1. *B is a basis of M with $u,v\notin B$ and $a,b\in B$;*

9.2. *$\{a,b,u,v\}$ distinguishes (M,B) from (M',B);*

9.3. *no two of a, b, and u are in the same part of $\Pi_{M\setminus v}(E(N))$; and*

9.4. *no two of a, b, and v are in the same part of $\Pi_{M\setminus u}(E(N))$.*

Conditions 9.1 and 9.2 are trivially satisfied by our initial a, b, and B. Moreover, since $E(N)\cup\{a,b,u\}$ is 3-connected in $M\setminus v$ and, $E(N)\cup\{a,b,v\}$ is 3-connected in $M\setminus u$, conditions 9.3 and 9.4 are also satisfied.

Let $\Pi=\Pi_{M\setminus u,v}(E(N))$. For each $e\in E(M)-\{u,v\}$, we let π_e denote the set in Π that contains e. In this section we abbreviate $\partial_{(M\setminus u,v,B)}$ to ∂.

9.5. *If X is a transversal of $\partial(\Pi)$, then $M[X,B]$ is 3-connected, uniquely* GF(q)*-representable, and is a stabilizer for* GF(q).

Proof. By Lemma 8.2, $M[X,B]$ is 3-connected and contains an N-minor. Then, since N is uniquely GF(q)-representable and is a stabilizer for GF(q), $M[X,B]$ is uniquely GF(q)-representable and is a stabilizer for GF(q). □

Since $\{a,b,u,v\}$ distinguishes (M,B) from (M,B'), we see, by Lemmas 7.1 and 7.2, that:

9.6. *$G_B[\{u,v,a,b\}]$ is a circuit.*

9.7. *If x is adjacent to both a and b in G_B, then $\{x,a,b,u\}$ and $\{x,a,b,v\}$ are both twirls of (M,B).*

Proof. Suppose that $\{x,a,b,v\}$ is not a twirl of (M,B). Then x and v are in parallel in $M[\{x,a,b,u,v\},B]$ and, hence, also in $M'[\{x,a,b,u,v\},B]$. Thus, $\{a,b,u,v\}$ is feasible in (M,B) if and only if $\{x,a,b,u\}$ is feasible in (M,B). Similarly, $\{a,b,u,v\}$ is feasible in (M',B) if and only if $\{x,a,b,u\}$ is feasible in (M',B). Then, since $\{a,b,u,v\}$ distinguishes (M,B) and (M',B), the set $\{a,b,u,v'\}$ also distinguishes (M,B) and (M',B). This contradicts the fact that $M\setminus v=M'\setminus v$. □

We rely on the following result to prove that M is not GF(q)-representable.

9.8. *Let X be a transversal of $\partial(\Pi)$ and let $S \subseteq E(M) - \{u, v\}$ with $X \cup \{a, b\} \subseteq S$. If $M[S \cup \{u\}, B]$ and $M[S \cup \{v\}, B]$ are stable and $M[S, B]$ is connected, then M is not* GF(q)-*representable.*

Proof. Let $M_1 = M[S \cup \{u, v\}, B]$ and $M_2 = M'[S \cup \{u, v\}, B]$. Note that $M_1 \setminus u = M_2 \setminus u$ and $M_1 \setminus v = M_2 \setminus v$. However, $M_1 \neq M_2$ since $\{a, b, u, v\}$ distinguishes (M, B) from (M, B'). Moreover, $M_1 \setminus u$ and $M_1 \setminus v$ are stable and $M_1 \setminus u, v$ is connected. Then, by Lemma 5.4, M_1 is not GF(q)-representable. □

Henceforth, we assume that M is GF(q)-representable, and, hence, there does not exist a set S satisfying the hypotheses of 9.8. By 9.8 we can exclude an easy case.

9.9. *No transversal of $\partial(\Pi)$ contains both a and b.*

Proof. Suppose that there is a transversal X of $\partial(\Pi)$ with $a, b \in X$ and let $S = X \cup \{u, v\}$. By 9.5, $M[S - \{u, v\}, B]$ is 3-connected. Thus $M[S - \{u\}, B]$ and $M[S - \{v\}, B]$ are both internally 3-connected, and, hence, stable. Thus we have a contradiction to 9.8. □

Currently a and b play interchangeable roles in the proof. By possibly swapping a and b we may assume that:

9.10. *If $b \in \partial(\pi_b)$, then $a \in \partial(\pi_b)$.*

Proof. Suppose that $b \in \partial(\pi_b)$. By the symmetry between a and b we may also suppose that $a \in \partial(\pi_a)$. If $\pi_a = \pi_b$, then the assumption holds. On the other hand, if $\pi_a \neq \pi_b$, then there is a transversal X of $\partial(\Pi)$ that contains both a and b, contradicting 9.9. □

9.11. *Suppose that $b' \in \partial(\pi_b)$ such that if $a \in \partial(\pi_b)$ then $a = b'$. Now let $v' \in E(M) - (\{u, v\} \cup \pi_b)$ be a neighbour of b'. Then $\lambda_{M[\{b,b',v,v'\},B]}(\{b, b'\}) > 1$.*

Proof. By 9.10, $b' \neq b$. Suppose to the contrary that $\lambda_{M[\{b,b',v,v'\},B]}(\{b, b'\}) = 1$. Thus $(\{b, b'\}, \{v, v'\})$ is a split in $G_B[\{b, b', v, v'\}]$. However, note that b is adjacent to v and b' is adjacent to v'. It follows that b and b' are both adjacent to v and v'. Moreover, $\{b, b', v, v'\}$ is not a twirl in (M, B). Since b is adjacent to v', we have $b \in \partial(\pi_b)$. Then, by 9.10, $a \in \partial(\pi_b)$. Hence, by our definition of b', we have $b' = a$. Now v' is adjacent to both a and b but $\{v', a, b, v\}$ is not a twirl in (M, B), contradicting 9.7. □

9.12. *Let $S \subseteq E(M) - \{u, v\}$ where $a, b \in S$, $M[S, B]$ is stable, and $S \cap \pi \neq \emptyset$ for each $\pi \in \Pi$. If $M[S \cup \{v\}, B]$ is not stable, then $\lambda_{M[\pi_b \cup \{v\} \cup S, B]}(\pi_b \cup \{v\}) = 1$.*

Proof. Let $\widehat{M} = M[S \cup \{v\}, B]$ and let $X \subseteq S$ be a transversal of $\partial(\Pi)$. By 9.5, X is a 3-connected set in $\widehat{M} \setminus v$, so $\Pi_{\widehat{M} \setminus v}(X) = (S \cap \pi : \pi \in \Pi)$. If $M[S \cup \{v\}, B]$ is not stable, then, by Lemma 8.6, there exists $\pi \in \Pi_{\widehat{M} \setminus v}(X)$ such that $\lambda_{\widehat{M}}(\pi \cup \{v\}) = 1$. It follows that $v \in \mathrm{cl}_{\widehat{M}}(\pi)$. Therefore, for any $\pi' \in \Pi_{\widehat{M} \setminus v}(X)$ where $\pi \neq \pi'$, we have $\lambda_{\widehat{M}}(\pi') = 1$. However, by 9.11, $\lambda_{\widehat{M}}(\pi_b \cap S) > 1$. Thus $\pi = S \cap \pi_b$. Suppose that $\lambda_{M[\pi_b \cup \{v\} \cup S, B]}(\pi_b \cup \{v\}) > 1$. We know that $\lambda_{M[S,B]}(\pi) = 1$. So, by Lemma 8.8, there exists $e \in (\pi_b \cup \{v\}) - \pi$ such that $\lambda_{M[S \cup \{e\},B]}(\pi \cup \{e\}) > 1$. Since $\lambda_{M[\pi_b \cup S,B]}(\pi_b) = 1$, it follows that $e = v$. But this contradicts the fact that $\lambda_{\widehat{M}}(\pi \cup \{v\}) = 1$. □

Note that there is still symmetry between u and v. Thus, an analogous result holds with the roles of u and v swapped in 9.12.

Case 1: $\pi_a = \pi_b$.

By Lemma 8.5, there exists $S_b \subseteq \pi_b$ such that $a, b \in S_b$ and each component of $G_B[S_b]$ is a tree containing exactly one element of $\partial(\pi_b)$. Now let $b' \in \partial(\pi_b) \cap S_b$ and let X be a transversal of $\partial(\Pi)$ that contains b'. Finally, let x be a neighbour of b' in $G_B[X]$. By 9.4, $\lambda_{M\backslash u}(\pi_b \cup \{v\}) > 1 = \lambda_{M[\pi_b \cup \{v,x\},B]}(\pi_b \cup \{v\})$. Then, by Lemma 8.8, there exists $e_v \in E(M) - (\pi_b \cup \{u, v, x\})$ such that $\lambda_{M[\pi_b \cup \{e_v,v,x\},B]}(\pi_b \cup \{v\}) > 1$. Similarly, there exists $e_u \in E(M) - (\pi_b \cup \{u, v, x\})$ such that $\lambda_{M[\pi_b \cup \{e_u,u,x\},B]}(\pi_b \cup \{u\}) > 1$.

Case 1.1: *e_u and e_v are not both contained in π_x.*

By Lemmas 8.3 and 8.5, there exists $S \subseteq E(M) - \{u, v\}$ such that $M[S, B]$ is stable, $e_u, e_v, x \in S$, $S \cap \pi_b = S_b$, and $S \cap \pi \neq \emptyset$ for each $\pi \in \Pi$. Since $b', x, e_u, e_v \in S$, we have $\lambda_{M[\pi_b \cup \{u\} \cup S,B]}(\pi_b \cup \{u\}) > 1$ and $\lambda_{M[\pi_b \cup \{v\} \cup S,B]}(\pi_b \cup \{v\}) > 1$. Therefore, by 9.12, $M[S \cup \{u\}, B]$ and $M[S \cup \{v\}, B]$ are both stable, contradicting 9.8.

Case 1.2: $e_u, e_v \in \pi_x$.

Since X is a transversal of $\partial(\Pi)$, the minor $M[X, B]$ is 3-connected. Hence, $G_B[X]$ has no vertices of degree one. Therefore b' has a neighbour x' in $G_B[X - \{x\}]$. Note that, $\lambda_{M[\pi_b \cup \{x,x',u,e_u\},B]}(\pi_b \cup \{u\}) > 1 = \lambda_{M[\pi_b \cup \{u,x'\},B]}(\pi_b \cup \{u\})$. Then, by Lemma 8.8, there exists $e'_u \in \{x, e_u\}$ such that $\lambda_{M[\pi_b \cup \{u,x',e'_u\},B]}(\pi_b \cup \{u\}) > 1$. Similarly, there exists $e'_v \in \{x, e_v\}$ such that $\lambda_{M[\pi_b \cup \{v,x',e'_v\},B]}(\pi_b \cup \{v\}) > 1$. Note that, $e'_u, e'_v \in \pi_x$ and that $\pi_x \neq \pi_{x'}$. Therefore replacing x, e_u, and e_v with x', e'_u, and e'_v returns us to Case 1.1.

Case 2: $\pi_a \neq \pi_b$.

We choose $S_a \subseteq \pi_a$ such that $G_B[S_a]$ is a path connecting a to some element $a' \in \partial(S_a)$. Now we choose $S_b \subseteq \pi_b$ such that $G_B[S_b]$ is a path connecting b to some element $b' \in \partial(S_b)$. Now let X be a transversal of $\partial(\Pi)$ containing both a' and b', and let $S = S_a \cup S_b \cup X$. By Lemma 8.3, $M[S, B]$ is stable. By 9.8 and by possibly swapping u and v, we may assume that $M[S \cup \{u\}, B]$ is not stable. Then, by 9.12, $\lambda_{M[\pi_b \cup \{u\} \cup S,B]}(\pi_b \cup \{u\}) = 1$. Thus $(\pi_b \cup \{u\}, S - \pi_b)$ is a split in $G_B[\pi_b \cup \{u\} \cup S]$. Recall that u is adjacent to a in G_B. It follows that $a \in \partial(\pi_a)$ and that a is adjacent to b' in G_B.

Now let $\widehat{a} = b'$, $\widehat{b} = b$, and $\widehat{B} = B\Delta\{a, b'\}$. Observe that $\widehat{a}$ and $\widehat{b}$ are in the same part of Π. We will show that $\widehat{a}$, $\widehat{b}$, and $\widehat{B}$ satisfy 9.1, 9.2, 9.3, 9.4, and 9.9; thus reducing Case 2 to Case 1. Note that, $\widehat{a}$, $\widehat{b}$, and $\widehat{B}$ trivially satisfy 9.1. Moreover, as $\{\widehat{a}, \widehat{b}, u, v\} = \{a, b, u, v\}\Delta\{a, b'\}$ and $\{a, b, u, v\}$ distinguishes (M, B) from (M', B), the set $\{\widehat{a}, \widehat{b}, u, v\}$ distinguishes $(M, \widehat{B})$ from $(M', \widehat{B})$. Thus $\widehat{a}$, $\widehat{b}$, and $\widehat{B}$ also satisfy 9.2. Note that, a and b' remain adjacent in $G_{\widehat{B}}$, so $\widehat{a} \in \partial_{(M\backslash u,v,\widehat{B})}(\pi_b)$. Hence, $\widehat{a}$, $\widehat{b}$, and $\widehat{B}$ satisfy 9.9.

It remains to prove that $\widehat{a}$, $\widehat{b}$, and $\widehat{B}$ satisfy 9.3 and 9.4; suppose otherwise. By the symmetry between u and v, we may assume that there exists $\pi \in \Pi_{M\backslash v}(E(N))$, such that $|\pi \cap \{\widehat{a}, \widehat{b}, u\}| \geqslant 2$. However, by 9.3, π cannot contain both of $\widehat{b} = b$ and u. Thus $\widehat{a} = b' \in \pi$. Again using 9.3, since π contains one of u and b, we have $a \notin \pi$. Now $(\pi, E(M) - (\{v\} \cup \pi))$ is a split in $G_B - v$ and both of the edges ub and ab' cross this split. It follows that $u, b' \in \pi$, $a, b \notin \pi$, and that u and b' are both adjacent to a and b. By 9.7, $\{b', a, b, u\}$ is a twirl of (M, B); this contradicts the fact that $\lambda_{M\backslash v}(\pi) = 1$. This final contradiction completes the proof of Lemma 5.5. □

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