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Groups Generated by Transvections over the Field of Two Elements

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In this paper we give a complete description of the linear groups over $\mathbf{F}_2$ generated by transvections and free of non-trivial unipotent normal subgroups. Recall that for a *transvection* T, $\operatorname{Ker}(T-1)$ is a hyperplane H and $\operatorname{Im}(T-1)$ is a line $\langle x \rangle$; we call H the *axis* of T and $\langle x \rangle$ the *center* of T. If $T \in G$, we say H is an axis for G, $\langle x \rangle$ is a center for G, $\langle x \rangle$ is a center for H, and H is an axis for $\langle x \rangle$. Recall also that a transformation T is *unipotent* if $T-1$ is nilpotent, and that a group of transformations is unipotent if each of its elements is. Suppose V decomposes into $V_1 \oplus V_2$ with respect to a group G generated by transvections on V. Then if G_i is the subgroup of G generated by transvections centered in V_i, we see easily that $G = G_1 \times G_2$, $G_i \mid V_j = 1$ for $j \neq i$, $G_i \cong G \mid V_i$ and G_i is generated by transvections. Hence in the remainder of this paper we will assume that V is indecomposable with respect to G, unless it is explicitly designated otherwise.

Our point of departure is the following theorem of J. E. McLaughlin [5].

THEOREM 1. *Let V be a vector space of dimension $n \geqslant 2$ over a finite field K, and let $G \leqslant SL(V)$ be indecomposable on V, generated by transvections, and free of unipotent normal subgroups $\neq \{1\}$. Then there is a decomposition $A \oplus W_1 \oplus \cdots \oplus W_m \oplus X$ of V such that with respect to this decomposition, $T \in G$ has the form*

$$T(a) = a \qquad \text{for all} \quad a \in A$$
$$T(w_i) = \delta_i(T)(w_i) + \theta_i(T)(w_i) \qquad \text{for all} \quad w_i \in W_i$$
$$T(x) = \alpha(T)(x) + \textstyle\sum_{i=1}^{m} \epsilon_i(T)(x) + x \qquad \text{for all} \quad x \in X,$$

where the $\delta_i(T) \in \operatorname{Hom}_K(W_i, A)$, the $\theta_i(T) \in GL(W_i)$, $\alpha(T) \in \operatorname{Hom}_K(X, A)$, the $\epsilon_i(T) \in \operatorname{Hom}_K(X, W_i)$ and the θ_i are irreducible representations generated by transvections.

We indicate briefly the proof of this theorem. Let W be the subspace of V spanned by the centers for G and let A be the intersection of the axes for G.

Then it may be shown that $A \leqslant W$. Write $V = A \oplus W' \oplus X$ where X is a complement for W in V and W' is a complement for A in W. Then if $T \in G$, $T(a) = a$ for $a \in A$, $T(w) = \delta(T)w + \bar{T}w$ for $w \in W'$ and $T(x) = \alpha(T)x + \epsilon(T)x + x$ for $x \in X$, where $\bar{T} \in GL(W')$, $\delta(T) \in \mathrm{Hom}_K(W', A)$, $\alpha(T) \in \mathrm{Hom}_K(X, A)$ and $\epsilon(T) \in \mathrm{Hom}_K(X, W')$. The map from $T \in G$ to its associated $\bar{T} \in GL(W')$ is a homomorphism, and its kernel is clearly unipotent, and so is trivial. Thus we may write $\delta(T) = \delta(\bar{T})$, $\alpha(T) = \alpha(\bar{T})$, and $\epsilon(T) = \epsilon(\bar{T})$.

Now consider the transvections $\bar{T}$ on W' ($W' \cong W/A$) induced by G. For $\langle x \rangle$ a center for G, let $d(x)$ be the dimension of the intersection of all axes for $\langle x \rangle$. There is a natural action of G on its centers and axes, for if T has center $\langle x \rangle$ and axis H, STS^{-1} has center $\langle Sx \rangle$ and axis SH. Clearly, if $\langle x \rangle$ and $\langle y \rangle$ are centers in the same G-orbit, then $d(x) = d(y)$. Let Γ_1 be a G-orbit of centers such that for $\langle x \rangle \in \Gamma_1$, $d(x)$ is minimal. Let G_1 be the group on W' generated by the transvections centered in Γ_1, and let G_1^* be the group on W' generated by the transvections centered outside Γ_1. Then $G = G_1 \times G_1^*$. Let W_1 be the subspace spanned by the centers for G_1 and let A_1 be the intersection of the axes for G_1. Define W_1^* and A_1^* similarly for G_1^*. Then $W_1 = A_1^*$, $A_1 = W_1^*$ and $W' = W_1 \oplus A_1 = W_1^* \oplus A_1^*$. Moreover G_1 acts faithfully on W_1, $G_1 \mid A_1 = 1$, and G_1 is a group generated by transvections having no unipotent normal subgroups $\neq \{1\}$. G_1 is now transitive on its centers and so is irreducible on W_1. Iterating this construction we obtain the decomposition of the theorem.

We observe further that if G_i is the subgroup of G consisting of those $T \in G$ for which $\theta_j(T) = 1$ for $j \neq i$, then $G = G_1 \times \cdots \times G_m$ and G_i induces an irreducible group $\bar{G}_i$ on W_i generated by transvections and free of unipotent normal subgroups $\neq \{1\}$. Moreover $G_i \cong \bar{G}_i$ and $G_i \mid W_j = 1$ for $j \neq i$. So $\theta_i \mid G_i$ is an isomorphism, and for $T \in G_i$, $\theta_i(T)$ determines $\alpha(T)$ and $\delta_j(T)$, $\epsilon_j(T)$ for all j.

McLaughlin has determined all the irreducible subgroups of $SL_n(\mathbf{F}_2)$ generated by transvections [4]. $SL_n(\mathbf{F}_2)$ is itself irreducible and generated by transvections. For $n \geqslant 4$ and even we also have the sympletic group $Sp_n(\mathbf{F}_2)$, the orthogonal group of maximal index $\mathbf{O}_n(1, \mathbf{F}_2)$ for $n \neq 4$, the orthogonal group of non-maximal index $\mathbf{O}_n(-1, \mathbf{F}_2)$, and the symmetric groups S_{n+1} and S_{n+2}. We know [2, Lemma 4, p. 441; 6, sections 13 and 14; 7, sections 4 and 5] that for each of these groups G, the $\mathbf{F}_2$-dimension of the first cohomology group $H^1(G, V)$ is at most one, where V is an n-dimensional $\mathbf{F}_2$-space. (Here, as in [3, p. 130-131], $H^1(G, V)$ is represented as $\mathrm{Der}(G, V)/\mathrm{Inn}(G, V)$, where $\mathrm{Der}(G, V) = \{\delta : G \to V \mid \delta(TS) = T(\delta(S)) + \delta(T) \text{ for all } T, S \in G\}$ is a vector space under pointwise addition and scalar-multiplication, and $\mathrm{Inn}(G, V) = \{\delta \in \mathrm{Der}(G, V) \mid \text{there exists } v \in V \text{ with } \delta(T) = (T - 1)v \text{ for all } T \in G\}$ is a subspace of $\mathrm{Der}(G, V)$. The elements of $\mathrm{Der}(G, V)$ are *derivations*, and the elements of $\mathrm{Inn}(G, V)$ are *inner derivations*.) From this point we will always assume the ground field is $\mathbf{F}_2$.

Continuing the notation of Theorem 1, let $b_1, \ldots, b_r$ be a basis for A. For $T \in G_i$ and $w \in W_i$, $Tw = \bar{T}w + \sum_{j=1}^{r} \delta_{ji}(\bar{T})(w)b_j$, where $\bar{T} \in \bar{G}_i$ and $\delta_{ji}(\bar{T}) \in W_i^*$. We check easily that $\delta_{ji} \in \mathrm{Der}(\bar{G}_i, W_i^*)$. Suppose $H^1(\bar{G}_i, W_i^*) = \langle \delta^{(i)} \rangle$. Then for $j = 1, \ldots, r$, $\delta_{ji}(\bar{T}) = \lambda_{ji}\delta^{(i)}(\bar{T}) + \psi_{ji}(\bar{T} - 1)$ for each $\bar{T} \in \bar{G}_i$, where $\lambda_{ji} \in \mathbf{F}_2$ and $\psi_{ji} \in W_i^*$. Let $W_i' = \langle w + \sum_j \psi_{ji}(w)b_j \mid w \in W_i \rangle$. Then W_i' is a complement for $A \oplus \sum_{j \neq i} \oplus W_i \oplus X$ in V, and $W_i' \equiv W_i$ modulo A. Let $w' \in W_i'$, say $w' = w + \sum \psi_{ji}(w)\, b_j$ for $w \in W_i$. Then for $T \in G_i$, $T(w') = \bar{T}w + \sum_j \psi_{ji}(\bar{T}w)b_j + \delta^{(i)}(\bar{T})(w) \sum_j \lambda_{ji}b_j$. For $T \in G_i$, define $\bar{T}' \in GL(W_i')$ by $\bar{T}'(w + \sum \psi_{ji}(w)\, b_j) = \bar{T}w + \sum \psi_{ji}(\bar{T}w)b_j$ for $w \in W_i$, and define $\delta^{(i)\prime} \in \mathrm{Der}(\bar{G}_i, W_i'^*)$ by $\delta^{(i)\prime}(\bar{T})(w + \sum \psi_{ji}(w)b_j) = \delta^{(i)}(\bar{T})(w)$ for $\bar{T} \in \bar{G}_i$ and $w \in W_i$. Let $a_i = \sum \lambda_{ji}b_j$. Then $T(w') = \bar{T}'(w') + \delta^{(i)\prime}(\bar{T})(w')a_i$ for $T \in G_i$ and $w' \in W_i'$. (Note that if the δ_{ji} are inner for all j, then $a_i = 0$.)

Change notation by omitting the "primes". Then $\langle a_1, \ldots, a_m \rangle \oplus W_1 \oplus \cdots \oplus W_m \oplus X$ is stable for G and has a stable complement in V, namely the complement for $\langle a_1, \ldots, a_m \rangle$ in A. Since V is indecomposable, $A = \langle a_1, \ldots, a_m \rangle$ and $\dim A \leqslant m$.

Now, for $x \in X$ and $T \in G_i$, $Tx \equiv x + \epsilon_i(\bar{T})(x)$ modulo A, where $\epsilon_i(\bar{T}) \in \mathrm{Hom}(X, W_i)$. Let $\epsilon_{i,x}(\bar{T}) = \epsilon_i(\bar{T})(x)$. Then $\epsilon_{i,x}$ induces a derivation from $\bar{G}_i$ to $\bar{W}_i = (W_i + A)/A$. Suppose $H^1(\bar{G}_i, \bar{W}_i) = \langle \epsilon^{(i)} \rangle$, and suppose that for $T_i \in G_i$, $x \in X$ and $i = 1, \ldots, m$, $\epsilon_{i,x}(T_i) \equiv \phi_i(x)\epsilon^{(i)}(\bar{T}_i) + (\bar{T}_i - 1)(w_i(x))$ modulo A, with $\phi_i \in X^*$ and $w_i \in \mathrm{Hom}(X, W_i)$. Let

$$X' = \left\langle x - \sum_{j=1}^{m} w_j(x) \,\middle|\, x \in X \right\rangle.$$

Then for $x' \in X'$, say $x' = x - \sum w_j(x)$ for $x \in X$, and for $T_i \in G_i$, $T_i(x') \equiv x' + \phi_i(x)\, \epsilon^{(i)}(T_i)$ modulo A. Define $\phi_i' \in X'^*$ by $\phi_i'(x + \sum w_j(x)) = \phi_i(x)$ for $x \in X$. Then $T_i(x') \equiv x' + \phi_i'(x')\, \epsilon^{(i)}(\bar{T}_i)$ modulo A. (Note that if the $\epsilon_{i,x}$ are inner for all $x \in X$, $\phi_i = 0$.)

Again change notation by omitting the "primes". Suppose $x \in \bigcap_{i=1}^{m} \mathrm{Ker}\, \phi_i$. Then $(T - 1)(x) \in A$ for every $T \in G$. If A contains the center of a transvection R, then $(R - 1)(v) \in A$ and $(R - 1)^2 = 0$, so R is unipotent. Since $G \mid A = 1$, G centralizes every transvection centered in A, and $\langle R \rangle$ is a unipotent normal subgroup. Hence $R = 1$ and A contains no centers. Thus $(T - 1)(x) = 0$ for all $T \in G$ and $x \in X \cap A = \{0\}$. Therefore $X^* = \langle \phi_1, \ldots, \phi_m \rangle$ and $\dim V/W = \dim X \leqslant m$. Thus we see that if we know the G_i, $i = 1, \ldots, m$, we have an upper bound on the degree of the representation of G. To summarize,

THEOREM 2. *Under the hypotheses of Theorem* 1 *and modifying the choices of the* W_i *and* X *as above, there exist spanning sets* $a_1, \ldots, a_m$ *and* $\phi_1, \ldots, \phi_m$

for A and X^* rsspectively such that $T_i(w_i) = \bar{T}_i(w_i) + \delta^{(i)}(\bar{T}_i)(w_i)a_i$ and $T_i(x) \equiv x + \phi_i(x)\,\epsilon^{(i)}(\bar{T}_i)$ modulo A, where $H^1(\bar{G}_i, W_i^*) = \langle\delta^{(i)}\rangle$ and $H^1(\bar{G}_i, W_i) = \langle\epsilon^{(i)}\rangle$, for $w_i \in W_i$, $x \in X$ and $T_i \in G_i$.

Suppose that G satisfies the hypotheses of Theorem 1. Continuing our earlier notation, a transvection $T \in G$ induces a transvection $\bar{T}$ on W/A. Since, by our construction, T is centered in $W_i + A$ for some i, $\bar{T}$ must be in $\bar{G}_i$ for some i. Each of the groups $\bar{G}_i$ has a single conjugacy class of transvections. so every transvection in $\bar{G}_i$ comes from a transvection in G, and so from a transvection in G_i. Hence if $\bar{T}_i \in \bar{G}_i$ is a transvection, then $T_i \in G_i$ is a transvection.

Consider the group G_i acting on the subspace $V_i = A \oplus W_i \oplus X$. Dropping the subscripts, with respect to this decomposition of V, $T \in G$ has the matrix

$$\begin{vmatrix} 1 & \delta(\bar{T}) & \alpha(\bar{T}) \\ 0 & \bar{T} & \epsilon(\bar{T}) \\ 0 & 0 & 1 \end{vmatrix}.$$

Modifying the choices of W and X as above and choosing appropriate bases for A and X, we may assume that $\delta(\bar{T})$ has at most one nonzero row and $\epsilon(\bar{T})$ has at most one non-zero column for all $T \in G$. If dim $H^1(\bar{G}, W) = 0$, we can choose $\epsilon = 0$ and $\delta = 0$, so V decomposes with respect to G. Since we have assumed that V is indecomposable, $\bar{G}$ must be one of the irreducible groups over $\mathbf{F}_2$ having non-inner derivations. By [2, Lemma 4, p. 441; 6, sections 13 and 14; 7, sections 4 and 5], $\bar{G}$ must be one of $SL_3(\mathbf{F}_2)$, $Sp_{2n}(\mathbf{F}_2)$ for $n \geqslant 3$, or S_n for $n \geqslant 6$, even.

Suppose one of δ, ϵ is identically zero, say ϵ is. Then every $T \in G$ has the form

$$\begin{vmatrix} 1 & \delta(\bar{T}) & \alpha(\bar{T}) \\ 0 & \bar{T} & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

with $\delta \in \mathrm{Der}(\bar{G}, W^*)$ non-inner. Since whenever $\bar{T}$ is a transvection, T is, we must have $\alpha(\bar{T}) = 0$ for every transvection $\bar{T}$. But since $\bar{G}$ is generated by transvections and since $\alpha(\bar{T}\bar{S}) = \alpha(\bar{T}) + \alpha(\bar{S})$ for $\bar{T}, \bar{S} \in \bar{G}$, $\alpha = 0$.

Suppose $\bar{G} \cong SL_3(\mathbf{F}_2)$. In [2, Lemma 4, p. 441], a derivation from $SL_3(\mathbf{F}_2)$ to its standard module is found to have one of two forms. One of these is shown to be inner. The other, call it δ, is not shown to be inner (or non-inner); but it is described so explicitly that it is clear that there are transvections $\bar{T} \in SL_3(\mathbf{F}_2)$ for which $\left|\begin{smallmatrix} 1 & \delta(\bar{T}) \\ 0 & \bar{T} \end{smallmatrix}\right|$ is not a transvection. Hence $\bar{G} \cong SL_3(\mathbf{F}_2)$ cannot occur.

Suppose $\bar{G} \cong Sp_{2n}(\mathbf{F}_2)$ for $n \geqslant 3$, so G is isomorphic as a linear group to the group G' of transformations on the $(2n+1)$-dimensional $\mathbf{F}_2$-space $\langle a\rangle \oplus W$ having matrix representations of the form $\left|\begin{smallmatrix}1 & \delta(\bar{T}) \\ 0 & \bar{T}\end{smallmatrix}\right|$ for $\bar{T} \in \bar{G} = Sp(W)$ and $\delta \in \mathrm{Der}(Sp(W), W^*)$, non-inner. By [6, Theorem 10.4, p. 43; 7, Corollary to Theorem 1.10], we may assume $\delta(\bar{T})(w) = \sqrt{(Q(\bar{T}(w)) + Q(w))}$ for $w \in W$, where Q is a quadratic form associated with the bilinear form B on W defining $Sp(W)$. If $\bar{T}$ is a transvection with center $v \in W$, choose a basis $v = v_1, \ldots, v_{2n}$ of W such that $B(v_i, v_j) = \delta_{j,2n-i+1}$. Then, by [6, Theorem 4.8, p. 13; 7, Theorem 1.6], for $w \in W$, $\delta(\bar{T})(w) = \sqrt{(1 + Q(v))}\, B(v, w)$, so with respect to the $\{v_i\}$,

$$T = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & \sqrt{(1+Q(v))} \\ & 1 & 0 & \cdots & 0 & 1 \\ & & 1 & \cdots & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{vmatrix}.$$

Clearly the rank of $T - 1$ is one, so T is a transvection whenever $\bar{T}$ is. Hence G' is a group generated by transvections. In fact, $G \cong G' \cong \mathbf{O}_{2n+1}(\mathbf{F}_2)$ [6, Theorem 10.1, p. 41].

Now suppose $\bar{G} \cong S_{n-2}$ for $n \geqslant 8$, even. S_{n-2} is regarded as a linear group in the following way [1]. Viewed as a permutation group on the letters $\{3, \ldots, n\}$, S_{n-2} faithfully induces a linear group on an $(n-2)$-dimensional $\mathbf{F}_2$-space $\langle x_3, \ldots, x_n\rangle$ by $\pi(x_k) = x_{\pi(k)}$. If η is a linear functional defined by $\eta(\sum_{k=3}^{n} \lambda_k x_k) = \sum_{k=3}^{n} \lambda_k$, and if $x_0 = \sum_{k=3}^{n} x_k$, then $x_0 \in \operatorname{Ker} \eta$ and S_{n-2} acts faithfully on $\operatorname{Ker} \eta/\langle x_0\rangle$. We take W to be $\operatorname{Ker} \eta/\langle x_0\rangle$ and $\bar{G}$ to be the group on W induced by S_{n-2}. (Note that if $n = 8$, $\bar{G} = Sp(W)$.) Then we may suppose G is isomorphic as a linear group to G', the group of transformations on the $(2n+1)$-dimensional $\mathbf{F}_2$-space $W \oplus \langle x\rangle$ having the matrix representations $\left|\begin{smallmatrix}\bar{T} & \epsilon(\bar{T}) \\ 0 & 1\end{smallmatrix}\right|$ for $\bar{T} \in \bar{G}$ and $\epsilon \in \mathrm{Der}(\bar{G}, W)$, non-inner. By [6, p. 81; 7, Theorem 5.2], we may assume that $\epsilon(\bar{T}) = (\bar{T} - 1)\bar{x}_3$, where $\bar{x}_3$ is the coset of x_3 in $\operatorname{Ker} \eta/\langle x_0\rangle$ and $\bar{T} \in \bar{G}$. Write $x_{ij} = x_i + x_j$, and write $\bar{x}_{ij}$ for the coset of x_{ij} in $\operatorname{Ker} \eta/\langle x_0\rangle$. Then $\bar{x}_{34}, \bar{x}_{45}, \ldots, \bar{x}_{n-2,n-1}$ is a basis for W. S_{n-2} is generated by the transpositions $(i, i+1)$, $i = 3, \ldots, n$. If $\bar{T}$ is induced by $(i, i+1)$, $i > 3$, $\epsilon(\bar{T}) = 0$ and T is clearly a transvection. If $\bar{T}$ is induced by (34), then $\bar{T}(\bar{x}_{34}) = \bar{x}_{34}$, $\bar{T}(\bar{x}_{45}) = \bar{x}_{34} + \bar{x}_{45}$, and $\epsilon(\bar{T}) = \bar{x}_{34}$, so T has the matrix

$$\begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 1 \\ & 1 & 0 & \cdots & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{vmatrix}.$$

Clearly the rank of $T - 1$ is one, so T is a transvection. Thus G' is generated by transvections. In fact, if we let S_n act on $\operatorname{Ker} \eta'/\langle x_0'\rangle$, where η' is a linear functional on the $\mathbf{F}_2$-space $\langle x_1, \ldots, x_n \rangle$ defined by $\eta'(\sum_{k=1}^{n} \lambda_k x_k) = \sum_{k=1}^{n} \lambda_k$ and $x_0' = \sum_{k=1}^{n} x_k$, then S_n preserves the bilinear form B defined on $\operatorname{Ker} \eta'/\langle x_0' \rangle$ by $B(\sum_{j=1}^{n} \lambda_j x_j, \sum_{k=1}^{n} \mu_k x_k) = \sum_{k \neq j} \lambda_j \mu_k$. By [6, p. 79; 7, Theorem 5.2], $\langle \bar{x}_{12}, \bar{x}_{23} \rangle^{\perp}$ and W are isomorphic as S_{n-2}-modules and we see that $G \cong G' \cong (S_n)_{\bar{x}_{12}} \mid \langle \bar{x}_{12} \rangle^{\perp}$ as linear groups.

Suppose now that neither ϵ nor δ is zero, so every element of G has the form

$$\begin{vmatrix} 1 & \delta(\bar{T}) & \alpha(\bar{T}) \\ 0 & \bar{T} & \epsilon(\bar{T}) \\ 0 & 0 & 1 \end{vmatrix},$$

with $\bar{T} \in \bar{G}$, an irreducible group generated by transvections, $\delta \in \operatorname{Der}(\bar{G}, W^*)$ non-inner, and $\epsilon \in \operatorname{Der}(\bar{G}, W)$ non-inner.

Suppose first that $\bar{G} \cong SL_3(\mathbf{F}_2)$. If δ is the derivation from $SL_3(V)$ to V^* of [2, Lemma 4, p. 441] referred to earlier, and $\epsilon \in \operatorname{Der}(SL_3(V), V)$ is its dual, then we see again that there are transvections $\bar{T} \in SL_3(\mathbf{F}_2)$ for which

$$\begin{vmatrix} 1 & \delta(\bar{T}) & \alpha(\bar{T}) \\ 0 & \bar{T} & \epsilon(\bar{T}) \\ 0 & 0 & 1 \end{vmatrix}$$

cannot be a transvection. Hence $\bar{G} \cong SL_3(\mathbf{F}_2)$ cannot occur in this situation either.

Now suppose $\bar{G} \cong Sp_{2n}(\mathbf{F}_2)$ with $n \geqslant 3$. Then G is isomorphic as a linear group to the group G' of transformations on the $(2n + 2)$-dimensional space $\langle a \rangle \oplus W \oplus \langle x \rangle$ having the form

$$\begin{vmatrix} 1 & \delta(\bar{T}) & \alpha(\bar{T}) \\ 0 & \bar{T} & \epsilon(\bar{T}) \\ 0 & 0 & 1 \end{vmatrix}$$

where $\bar{T} \in Sp(W)$ and we may assume by [6, Theorem 10.4, p. 43; 7, Corollary to Theorem 1.10] that $\delta(\bar{T})(w) = \sqrt{(Q(\bar{T}(w)) + Q(w))} = B(u(\bar{T}), \bar{T}(w))$ and $\epsilon(\bar{T}) = u(\bar{T}) \in W$. As before, Q is a quadratic form associated with the bilinear form B on W defining $Sp(W)$. We see that $\alpha(\bar{T}\bar{S}) = \alpha(\bar{T}) + \alpha(\bar{S}) + B(u(\bar{T}), \bar{T}u(\bar{S}))$ for $\bar{T}, \bar{S} \in \bar{G}$. If d is the extension of the Dickson Invariant on $0(Q)$ to $Sp(B)$ defined in [6, Theorem 6.1, p. 28; 7, Theorem 1.11], then $L = d + \alpha$ is a homomorphism from $Sp(W)$ to the additive group of $\mathbf{F}_2$. Since $Sp(W)$ is simple, $L = 0$ and $\alpha = d$. Hence $G' \cong \mathbf{O}^{+}_{2n+2}(\mathbf{F}_2)_{\langle v \rangle}$ for v non-singular in V, and G' is not generated by transvections. We should note that $0_{2n+2}(\mathbf{F}_2)_{\langle v \rangle}$ is generated by transvections, but it contains the unipotent normal subgroup generated by the orthogonal transvection with center $\langle v \rangle$.

Now suppose $\bar{G} \simeq S_{n-2}$ for $n \geqslant 8$, even. Then again, $G \simeq G'$, a group on $\langle a\rangle \oplus W \oplus \langle x\rangle$ whose elements have the form

$$\begin{vmatrix} 1 & \delta(\bar{T}) & \alpha(\bar{T}) \\ 0 & \bar{T} & \epsilon(\bar{T}) \\ 0 & 0 & 1 \end{vmatrix}$$

with $\bar{T} \in \bar{G}$, the group induced by S_{n-2} on $W = \operatorname{Ker} \eta/\langle x_0\rangle$, $\delta \in \operatorname{Der}(\bar{G}, W^*)$ non-inner, and $\epsilon \in \operatorname{Der}(\bar{G}, W)$ non-inner. As noted before, W is isomorphic as an S_{n-2}-module to $U = \langle \bar{x}_{12}, \bar{x}_{23}\rangle^{\perp}$ ($\leqslant \operatorname{Ker} \eta'/\langle x_0'\rangle$). To be more explicit, suppose $\phi : W \to U$ is defined by $\phi(\bar{x}_{i,i+1}) = z_{i,i+1}$, $i = 3,\ldots, n-2$, where $z_{34} = \bar{x}_{12} + \bar{x}_{34}$ and $z_{i,i+1} = \bar{x}_{i,i+1}$, $i = 4,\ldots, n-2$. The $z_{i,i+1}$ form a basis for U. U is not stable for $S_{n-2} = (S_n)_{x_1,x_2}$ but $\langle \bar{x}_{12}\rangle^{\perp} = \langle \bar{x}_{12}\rangle \oplus U$ is. However U may be regarded as an S_{n-2}-module via $\bar{T} : u \to \bar{T}^*(u)$, where $\bar{T}(u) = \delta_0(\bar{T})(u)\bar{x}_{12} + \bar{T}^*(u)$, and $\bar{T}^* \in GL(U)$ for $\bar{T} \in S_{n-2}$, $u \in U$. Then $\bar{T}(\bar{x}_{i,i+1}) = \bar{T}^*(z_{i,i+1})$. By [6, Theorem 14.2, p. 78; 7, Theorem 5.2], we may assume $\delta(\bar{T}) = \delta_0(\bar{T})$ and $\epsilon(\bar{T}) = \epsilon_0(\bar{T})$, where ϵ_0 is defined by $(\bar{T} + 1)(\bar{x}_3) = \alpha_0(\bar{T})\bar{x}_{12} + \epsilon_0(\bar{T})$. Then we see that $L = \alpha + \alpha_0$ is a homomorphism from S_{n-2} to the additive group of $\mathbf{F}_2$, so either $L = 0$ or $\operatorname{Ker} L = A_{n-2}$. If $L = 0$, then $\alpha = \alpha_0$ and $G \simeq G' \simeq (S_n)_{\bar{x}_{12}}$, where we identify $a \in A$ with $\bar{x}_{12}$ and $x \in X$ with $\bar{x}_{23}$. If $\bar{T} \in \bar{G}$ is induced by a transposition in S_{n-2} fixing 3, $\epsilon(\bar{T}) = 0$, $\delta(\bar{T}) = 0$ and $\alpha(\bar{T}) = 0$, so T is clearly a transvection. If $\bar{T}$ is induced by (34), then $\bar{T}(z_{34}) = z_{34}$, $\bar{T}(z_{45}) = \bar{x}_{12} + z_{34} + z_{45}$, and $(\bar{T} + 1)(\bar{x}_3) = \bar{x}_{12} + z_{34}$, so the matrix of T is

$$\begin{vmatrix} 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\ & 1 & 1 & 0 & \cdots & 0 & 1 \\ & & 1 & 0 & \cdots & 0 & 0 \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{vmatrix}$$

with respect to the decomposition $\langle \bar{x}_{12}\rangle \oplus U \oplus \langle \bar{x}_{23}\rangle$ and the basis $z_{i,i+1}$, $i = 3,\ldots, n-2$, for U. Clearly the rank of $T - 1$ is one, so T is a transvection and G is generated by transvections. If $\operatorname{Ker} L = A_{n-2}$, then for every transvection in $\bar{G}$ (induced by a transposition in S_{n-2}), $L(\bar{T}) = 1$. Referring to the above discussion we see that T cannot then be a transvection, so G is not generated by transvections.

To summarize, we have

THEOREM 3. *If $G = G_1 \times \cdots \times G_m$ is a linear group on the $\mathbf{F}_2$-space $V = A \oplus W_1 \oplus \cdots \oplus W_m \oplus X$ satisfying the hypotheses of Theorem* 1, *then either $\bar{G}_i \simeq Sp_{2n}(\mathbf{F}_2)$ for $n \geqslant 3$ and $G_i \simeq \mathbf{O}_{2n+1}(\mathbf{F}_2)$; or $G_i \simeq S_{n-2}(\mathbf{F}_2)$ for $n \geqslant 8$, even, and $G_i \simeq (S_n)_{\bar{x}_{12}} \mid \langle \bar{x}_{12}\rangle^{\perp}$ or $G_i \simeq (S_n)_{\bar{x}_{12}}$. Equivalently, if for*

$T_i \in G_i$, $w_i \in W_i$, and $x \in X$, $T_i(w_i) = \bar{T}_i(w_i) + \delta^{(i)}(\bar{T}_i)(w_i)a_i$ and $T_i(x) \equiv x + \phi_i(x)\epsilon^{(i)}(\bar{T}_i)$ modulo A, with $H^1(\bar{G}_i, W_i^*) = \langle \delta^{(i)} \rangle$, $H^1(\bar{G}_i, W_i) = \langle \epsilon^{(i)} \rangle$, $a_i \in A$ and $\phi_i \in X^*$ as in Theorem 2, then either $\bar{G}_i \simeq Sp_{2n}(\mathbf{F}_2)$ $(n \geqslant 3)$ and a_i and ϕ_i are neither both zero nor both non-zero; or $\bar{G}_i \simeq S_n$ $(n \geqslant 6$, even) and a_i and ϕ_i are not both zero.

Clearly a group $G = G_1 \times \cdots \times G_m$, with the G_i as in Theorem 3, is generated by transvections. However, one may still ask, when is such a group indecomposable?

THEOREM 4. *Using the notation of Theorem 3, V decomposes with respect to G if and only if there is a partition I, J of $\{1,..., m\}$ such that*

$$A = \langle a_i \mid i \in I \rangle \oplus \langle a_j \mid j \in J \rangle$$

and

$$X^* = \langle \phi_i \mid i \in I \rangle \oplus \langle \phi_j \mid j \in J \rangle.$$

Proof. First we show that if V decomposes into $U \oplus W$ with respect to G, then for each i, $i = 1,..., m$, $W_i \oplus \langle a_i \rangle \leqslant U$ or $W_i \oplus \langle a_i \rangle \leqslant W$. Suppose $T \in G_i$ is a transvection $\neq 1$. Then since U and W are stable for T, the center of T must lie in U or in W; say the center is $u \in U$. Then for all $S \in G$, STS^{-1} is centered in U. Since the transvections in G_i form a single conjugacy class in G_i, U must contain all centers for G_i. Since T stabilizes $W_i \oplus \langle a_i \rangle$ and $T \neq 1$, $u \in W_i \oplus \langle a_i \rangle$, $u \notin A$. Suppose first that $u \in W_i$. Then since W_i is spanned by the centers for $\bar{G}_i$, $W_i \leqslant U$ and so $\langle a_i \rangle \oplus W_i \leqslant U$. Now suppose that $u = w_i + a_i$, $w_i \in W_i$. For $S \in G_i$, STS^{-1} has center $Sw_i + a_i$, so the space W_i' spanned by the centers for G_i is $\langle w + a_i \mid w$ a center for $\bar{G}_i \rangle$. Clearly if w and v are centers for $\bar{G}_i$, $w + v \in W_i'$. By Theorem 3 we need consider only two cases.

(a) If $\bar{G}_i \simeq Sp(W_i)$ (dim $W_i \geqslant 6$), then every non-zero line (vector) of W_i is a center, so $W_i \leqslant W_i'$, and $W_i' = W_i \oplus \langle a_i \rangle \leqslant U$.

(b) If $\bar{G}_i \simeq S_n$ $(n \geqslant 6$, even), the centers are the $\bar{x}_{ij}$, $i \neq j$. Clearly $\bar{x}_{ij} = \bar{x}_{ik} + \bar{x}_{kj}$, $k \neq i, j$. So again $W_i' = W_i \oplus \langle a_i \rangle \leqslant U$.

Thus if V decomposes into $U \oplus W$ with respect to G, there is a partition I, J of $\{1,..., m\}$ such that $\sum_{i \in I}(W_i \oplus \langle a_i \rangle) \leqslant U$ and $\sum_{j \in J}(W_j \oplus \langle a_j \rangle) \leqslant W$. Clearly $A = (A \cap U) \oplus (A \cap W) = \langle a_i \mid i \in I \rangle \oplus \langle a_j \mid j \in J \rangle$. Likewise, $X = (X \cap U) \oplus (X \cap W)$. For $T \in G$, $T - 1$ maps $X \cap U$ into U and maps $X \cap W$ into W. But if T is centered in U, $\mathrm{Im}(T - 1) \leqslant U$, so for $x \in X \cap W$, $(T - 1)x \in U \cap W = \{0\}$ and $x \in \bigcap_{i \in I} \mathrm{Ker}\, \phi_i$. Hence $X \cap W \leqslant \bigcap_{i \in I} \mathrm{Ker}\, \phi_i$. Similarly, $X \cap U \leqslant \bigcap_{j \in J} \mathrm{Ker}\, \phi_j$. Since $\bigcap_{k=1}^{m} \mathrm{Ker}\, \phi_k = \{0\}$, $X = \bigcap_{i \in I} \mathrm{Ker}\, \phi_i \oplus \bigcap_{j \in J} \mathrm{Ker}\, \phi_j$, and so $X^* = \langle \phi_j \mid j \in J \rangle \oplus \langle \phi_i \mid i \in I \rangle$.

For the converse, suppose the partition I, J exists and let $U = \sum_{i\in I}(W_i \oplus \langle a_i\rangle) \oplus \bigcap_{j\in J} \operatorname{Ker} \phi_j$, $W = \sum_{j\in J}(W_j \oplus \langle a_j\rangle) \oplus \bigcap_{i\in I} \operatorname{Ker} \phi_i$.

COROLLARY. *If $G = G_1 \times \cdots \times G_m$ satisfies the hypotheses of Theorem* 1, *then the following values of $r = \dim A$ and $s = \dim X$ cannot occur*: (i) $r = s = m$, (ii) $r = m$, $s = 0$, (iii) $r = 0$, $s = m$, (iv) $r = s = 0$.

Finally, we have the question: suppose G and G' are linear groups on V satisfying the hypotheses of Theorem 1; under what conditions are they isomorphic as linear groups?

THEOREM 5. *Suppose $G = G_1 \times \cdots \times G_m$ and $G' = G_1' \times \cdots \times G_m'$ are indecomposable groups on V and V' respectively, generated by transvections and having no unipotent normal subgroups $\neq \{1\}$, and suppose* $\dim V = \dim V'$ *and $\bar{G}_i \cong \bar{G}_i'$ as linear groups, $i = 1,\ldots, m$. Suppose further that the spanning sets $a_1,\ldots, a_m$ and $\phi_1,\ldots,\phi_m$ (resp. $a_1',\ldots, a_m'$ and $\phi_1',\ldots,\phi_m'$) for A and X^* (resp. A' and X'^*) are chosen as in Theorem* 2. *Then G and G' are isomorphic as linear groups if and only if*

(i) $\sum_{i=1}^{m} \lambda_i a_{\pi(i)} = 0$ *if and only if* $\sum_{i=1}^{m} \lambda_i a_i' = 0$, *and*
(ii) $\sum_{i=1}^{m} \lambda_i \phi_{\pi(i)} = 0$ *if and only if* $\sum_{i=1}^{m} \lambda_i \phi_i' = 0$, *for all $\lambda_i \in \mathbf{F}_2$, where π is a permutation of $1,\ldots, m$ such that $\pi(i) = j$ only if $\bar{G}_i \cong \bar{G}_j'$.*

The proof of Theorem 5 requires several lemmas.

LEMMA 1. *Let V be an $\mathbf{F}_2$-space of dimension at least* 6 *with a nondegenerate alternate bilinear form B, and let $\delta \in \operatorname{Der}(Sp(V), V)$, δ non-inner. Then $d = u_Q$ for some quadratic form Q associated with B, where u_Q is defined by*

$$B(u_Q(T), T(v)) = \sqrt{(Q(T(v)) + Q(v))}$$

for all $v \in V$. In particular, there is an element $T \in Sp(V)$ with $T + 1$ nonsingular and $\delta(T) = 0$.

Proof. Choose $0(\epsilon, Q) \leqslant Sp(V)$, where $\epsilon = 1$ if the index of Q is maximal and $\epsilon = -1$ otherwise. By [6, Section 13; 7, Section 4] $\dim H^1(Sp(V), V)$ is one and u_Q is non-inner, so there is $v_0 \in V$ such that $\delta(T) = u_Q(T) + (T+1)(v_0)$ for all $T \in Sp(V)$. Suppose first that v_0 is singular. Let S be the symplectic transvection centered at v_0 (i.e.: $S(v) = v + B(v_0, v)v_0$), so $u_Q(S) = \sqrt{(Q(v_0) + 1)}v_0 = v_0$. Since $u_Q \mid \mathbf{O}(Q) = 0$, $T \in 0(Q)$ implies $u_Q(STS^{-1}) = (STS^{-1} + 1)\, u_Q(S)$, and so $\delta \mid O(Q)^S = 0$ ($O^S = SOS^{-1}$). Let $u' = u_{QS^{-1}}$. u' is also non-inner, so there exists $w_0 \in V$ such that $\delta(T) = u'(T) + (T+1)(w_0)$ for all $T \in Sp(V)$. Then $\delta \mid O(Q)^S = 0$ and $u' \mid O(Q)^S = 0$ imply that w_0 is a fixed point of $O(Q)^S$. But $O(Q)^S$ is irreducible, so $w_0 = 0$ and $\delta = u'$. QS^{-1} is then the quadratic form appearing in the statement of the lemma.

Now suppose that v_0 is non-singular. There are two cases to consider.

(a) $\epsilon = +1$. Choose $w_0 \in V$ such that $\langle v_0, w_0 \rangle$ is a hyperbolic plane. Since $\epsilon = 1$, the index of $Q \mid \langle v_0, w_0 \rangle$ is one, so we may assume $v_0 = u_0 + w_0$, where u_0, w_0 is a hyperbolic pair of singular vectors. Choose a symplectic basis of singular vectors $x_1 = u_0, \ldots, x_n$, $y_1 = w_0, \ldots, y_n$ for V, with $B(x_i, y_j) = \delta_{ij}$. Define a quadratic form Q', associated with B on V, by $Q'(x_i) = Q'(y_i) = 0$, $i = 2, \ldots, n$ and $Q'(x_1) = Q'(y_1) = 1$. Let $u' = u_{Q'}$. Then there exists $v \in V$ such that $u_Q(T) + u'(T) = (T+1)(v)$ for all $T \in Sp(V)$. If T is a symplectic transvection with center x_i (resp. y_i), $i = 2, \ldots, n$, then $u_Q(T) = u'(T)$, so $(T+1)(v) = 0$. That is, $B(v, x_i) = B(v, y_i) = 0$, $i = 2, \ldots, n$. If T is a symplectic transvection with center x_1 (resp. y_1), then $u'(T) = 0$, $u_Q(T) = x_1$ (resp. y_1). Thus $B(x_1, v) = B(y_1, v) = 1$. So we see $v = x_1 + y_1 = u_0 + w_0 = v_0$, and so $\delta = u'$ and Q' is the quadratic form specified by the lemma.

(b) $\epsilon = -1$. Again form the hyperbolic pair v_0, w_0 and let $u_0 = v_0 + w_0$. Since $O(Q)$ is irreducible we may choose w_0 to be non-singular, so $Q \mid \langle v_0, w_0 \rangle^{\perp}$ is of maximal index. Form the symplectic basis $x_1 = u_0, \ldots, x_n, y_1 = w_0, \ldots, y_n$ with $B(x_i, y_j) = \delta_{ij}$ and x_i, y_i singular for $i = 2, \ldots, n$. Define Q' associated with B on V by $Q'(x_i) = Q'(y_i) = 0$ for $i = 1, \ldots, n$, and let $u' = u_{Q'}$. As in (a) we find that $u'(T) + u_Q(T) = (T+1)(v_0)$, so $\delta = u'$.

Now, by [6, Theorem 10.3, p. 43; 7, Theorem 1.10], if Q is the form specified in the lemma, there is $T \in O(Q)$ such that $T + 1$ is non-singular.

Lemma 2. *Under the hypotheses of Lemma* 1, *if* $\delta \in \operatorname{Der}(Sp(V), V)$ *is non-inner, then* $\langle \delta(T) \mid T \in Sp(V) \rangle = V$.

Proof. By Lemma 1 we may assume $\delta = u_Q$ for a suitable Q. Thus if T is a symplectic transvection whose center v is singular with respect to Q, $\delta(T) = V$. Therefore $\langle \delta(T) \mid T \in Sp(V) \rangle$ contains all singular vectors. Since $O(Q)$ is irreducible, Lemma 2 follows.

Lemma 3. *In our earlier notation, if* $\delta \in \operatorname{Der}(S_n, H/\langle x_0 \rangle)$ *is non-inner* ($n \geq 6$, *even*), *then there is* $T \in S_n$ *with* $T + 1$ *non-singular on* $H/\langle x_0 \rangle$ *and* $\delta(T) = 0$.

Proof. Recall that S_n acts on $\langle x_1, \ldots, x_n \rangle$ by $T(x_i) = x_{T(i)}$ for $T \in S_n$. If η is the linear functional on $\langle x_1, \ldots, x_n \rangle$ defined by $\eta(\sum \lambda_i x_i) = \sum \lambda_i$, then $x_0 = \sum x_i \in H = \operatorname{Ker} \eta$. S_n acts faithfully on $H/\langle x_0 \rangle$. Let $\delta \in \operatorname{Der}(S_n, H/\langle x_0 \rangle)$ be non-inner. As in the proof of Theorem 3, we may assume that there is $\bar{v}_0 \in H/\langle x_0 \rangle$ ($v_0 \in H$) such that $\delta(T) = \delta_0(T) + (T+1)(v_0)$ for all $T \in S_n$, where $\delta_0(T)$ is the coset of $(T+1)(x_1)$ in $H/\langle x_0 \rangle$. Thus $\delta(T)$ is the coset of $(T+1)(x)$ in $H/\langle x_0 \rangle$, where $x \notin H$. Clearly $\delta \mid (S_n)_x = 0$. Suppose $x = \sum_{i \in I} x_i$. Write $I = \{i_1, \ldots, i_s\}$ and let $J = \{j_1, \ldots, j_t\}$ be the complement I^C of I in

$\{1,\ldots,n\}$. Let $T = (i_1 \cdots i_s)(j_1 \cdots j_t) \in (S_n)_x$, so $\delta(T) = 0$. Since $x \notin H$, s is odd, and so $s + t = n$ implies t is odd. Suppose $y \in H$ with $T(y) = y$ modulo $\langle x_0 \rangle$; say $y = \sum_{k \in K} x_k$, with $\#K$ even. Let $T_I = (i_1 \cdots i_s)$, $T_J = (j_1 \cdots j_t)$. $T(y) \equiv y$ modulo $\langle x_0 \rangle$ implies $T_I(y) = y$ or $y + x_0$, $T_J(y) = y$ or $y + x_0$. If $T_I(y) = y + x_0 = \sum_{k \notin K} x_k$, then $T_I(K) = K^C$, so $K^C \leqslant I$ and $K \leqslant I$. But then $I = \{1,\ldots,n\}$ and $x = x_0 \in H$, which is impossible. Similarly $T_J(y) = y + x_0$ implies $J = \{1,\ldots,n\}$ and $x = 0 \in H$. So we must have $T_I(y) = T_J(y) = y$, and thus either $I \cap K = \phi$ or $I \leqslant K$, and either $J \cap K = \phi$ or $J \leqslant K$. Since s and t are odd and $\#K$ is even, either $K = I \cup J$ and $y = x_0$, or $K = \phi$ and $y = 0$. In any case $y \equiv 0$ modulo $\langle x_0 \rangle$ and $T + 1$ is non-singular on $H/\langle x_0 \rangle$.

LEMMA 4. *Under the hypotheses of Lemma* 3, *if* $\delta \in \operatorname{Der}(S_n, H/\langle x_0 \rangle)$ *is non-inner, then* $\langle \delta(T) \mid T \in S_n \rangle = H/\langle x_0 \rangle$.

Proof. Let $\delta \in \operatorname{Der}(S_n, H/\langle x_0 \rangle)$ be non-inner, and let $W = \langle \delta(T) \mid T \in S_n \rangle$. As before, we may suppose $\delta(T)$ is the coset of $(T + 1)(x)$ in $H/\langle x_0 \rangle$, with $x = \sum \alpha_i x_i \notin H$. Since $x \neq x_0$, there is an i with $\alpha_i = 0$. If $\alpha_j \neq 0$, $j \neq i$, then $((ij) + 1)(x) = x_{ij}$, so $\bar{x}_{ij} \in W$. If $\alpha_j = 0$, $\alpha_k \neq 0$, $((jk) + 1)(x) = x_{jk}$ and $\bar{x}_{jk} \in W$, $((ik) + 1)(x) = x_{ik}$ and $\bar{x}_{ik} \in W$, and so $\bar{x}_{ij} \in W$. Thus $\bar{x}_{ij} \in W$ for all j and so $W = H/\langle x_0 \rangle$.

Now we return to the proof of Theorem 5. Let G and G' be as in the statement of the theorem. Referring to the construction of the spanning sets $a_1,\ldots,a_m$ and $\phi_1,\ldots,\phi_m$ for A and X^* in the proof of Theorem 2, we note that if A has basis $b_1,\ldots,b_r$ then $a_i = \sum_{j=1}^{r} \lambda_{ji} b_j$ where $\lambda_{ji} \neq 0$ if and only if $\delta_{ji} \in \operatorname{Der}(\bar{G}_i, W_i^*)$ is non-inner. Also $\phi_j(x) \neq 0$ if and only if $\epsilon_{j,x} \in \operatorname{Der}(\bar{G}_j, W_j)$ is non-inner. Thus if $x_1,\ldots,x_s$ is a basis of X and $\chi_1,\ldots,\chi_s$ is the dual basis of X^*, then $\phi_j = \sum_{k=1}^{s} \mu_{kj} \chi_k$ where $\mu_{kj} \neq 0$ if and only if $\epsilon_{j,k} = \epsilon_{j,x_k} \in \operatorname{Der}(\bar{G}_j, W_j)$ is non-inner.

With respect to the decompositions $V = A \oplus W_1 \oplus \cdots \oplus W_m \oplus X$ and $V' = A' \oplus W_1' \oplus \cdots \oplus W_m' \oplus X'$ of V and V' respectively, choose bases $\mathscr{B}$ and $\mathscr{B}'$ of V and V' respectively such that if $\bar{G}_i \simeq \bar{G}_j'$ as linear groups, then $\bar{G}_i = \bar{G}_j'$ as matrix groups with respect to the bases of W_i and W_j' respectively. Suppose a non-singular $C \in \operatorname{Hom}_{\mathbf{F}_2}(V, V')$, written as a matrix with respect to the bases $\mathscr{B}$ and $\mathscr{B}'$, intertwines the elements of G and G'. That is, suppose $CT = SC$ for $T \in G$, $S \in G'$. As matrices, $C = (C_{ij})_{i,j=0,\ldots,m+1}$,

$$T = \begin{vmatrix} 1 & \delta_1(T) \cdots \delta_m(T) & & \alpha(T) \\ & T_1 & & \epsilon_1(T) \\ & & \ddots & \vdots \\ & & T_m & \epsilon_m(T) \\ & & & 1 \end{vmatrix} \qquad S = \begin{vmatrix} 1 & \delta_1'(S) \cdots \delta_m'(S) & & \alpha'(S) \\ & S_1 & & \epsilon_1'(S) \\ & & \ddots & \vdots \\ & & S_m & \epsilon_m'(S) \\ & & & 1 \end{vmatrix}.$$

Then $CT = SC$ gives the matrix relations

$$\sum_{k=1}^{m} \delta_k'(S)\, C_{k0} + \alpha'(S)\, C_{m+1,0} = 0 \tag{1}$$

$$C_{00}\delta_i(T) + \sum_{k=1}^{m} \delta_k'(S)\, C_{ki} = C_{0i}(T_i + 1), \qquad i = 1,..., m \tag{2}$$

$$C_{00}\alpha(T) + \sum_{k=1}^{m} C_{0k}\epsilon_k(T) = \sum_{k=1}^{m} \delta_k'(S)\, C_{k,m+1} + \alpha'(S)\, C_{m+1,m+1} \tag{3}$$

$$(S_j + 1)\, C_{j,0} = \epsilon_j'(S)\, C_{m+1,0}\,, \qquad j = 1,..., m \tag{4}$$

$$C_{j0}\delta_i(T) + C_{ji}T_i = S_jC_{ji} + \epsilon_j'(S)\, C_{m+1,i}\,, \qquad i, j = 1,..., m \tag{5}$$

$$(S_j + 1)\, C_{j,m+1} = \epsilon_j'(S)\, C_{m+1,m+1} + \sum_{k=1}^{m} C_{jk}\epsilon_k(T) + C_{j0}\alpha(T),\ j = 1,..., m \tag{6}$$

$$C_{m+1,0}\delta_i(T) = C_{m+1,i}(T_i + 1), \qquad i = 1,..., m \tag{7}$$

$$C_{m+1,0}\alpha(T) + \sum_{k=1}^{m} C_{m+1,k}\epsilon_k(T) = 0 \tag{8}$$

Relation (4) and Lemmas 1 and 3 imply $C_{j0} = 0$, $j = 1,..., m$. Likewise, (7) implies $C_{m+1,i} = 0$, $i = 1,..., m$. Lemmas 2 and 4 and relation (4) (or (7)) imply $C_{m+1,0} = 0$. If C_{ji} is of rank r, then relation (5) implies $\bar{G}_j'$ has a stable subspace of dimension r. Since the $\bar{G}_j'$ are irreducible, C_{ji} must be non-singular or zero. Again using (5), C_{ji} is non-singular only if $\bar{G}_i \simeq \bar{G}_j'$. Since we've assumed that for $\bar{G}_i \simeq \bar{G}_j'$, $\bar{G}_i = \bar{G}_j'$ as matrix groups, C_{ji} non-singular implies $C_{ji} = 1$. Suppose $C_{ji} = 1$, $C_{jk} = 1$, $k \neq i$. Then (5) implies $T_k = S_jC_{jk} = S_j$ and $T_i = S_jC_{ji} = S_j$, so $T_i = T_k$. But for $T \in G_i$, $T \neq 1$, $T_i \neq 1$ and $T_k = 1$ when $k \neq i$. So $C_{ji} = 1$ implies $C_{jk} = 0$ for $k \neq i$. Similarly $C_{ji} = 1$ implies $C_{ki} = 0$ for $k \neq j$. Thus $(C_{u,v})_{u,v=1,...,m}$ is a "permutation matrix" with $C_{ji} = 1$ only if $\bar{G}_i \simeq \bar{G}_j'$.

If we modify the choice of the basis $\mathscr{B}$ by changing the bases for A and X and permuting the W_i, we may assume $C_{ii} = 1$ for $i = 0,..., m + 1$ and $C_{ij} = 0$ for $i \neq j$, $i, j = 1,..., m$. So finally we have the relations

$$(\delta_i + \delta_i')(T) = C_{0i}(T_i + 1), \qquad i = 1,..., m \tag{2'}$$

$$\alpha(T) + \alpha'(T) = \sum_{k=1}^{m} C_{0k}\epsilon_k(T) + \sum_{k=1}^{m} \delta_k'(S)\, C_{k,m+1} \tag{3'}$$

$$(\epsilon_j' + \epsilon_j)(T) = (T_j + 1)\, C_{j,m+1}\,, \qquad j = 1,..., m \tag{6'}$$

Write

$$(\delta_i + \delta_i')(T) = \begin{vmatrix} \delta_{1i} + \delta'_{1i} \\ \vdots \\ \delta_{ri} + \delta'_{ri} \end{vmatrix} (T)$$

and

$$C_{0i} = \begin{vmatrix} c_{1i} \\ \vdots \\ c_{ri} \end{vmatrix} \qquad \text{for} \qquad i = 1,\dots, m;$$

and $(\epsilon'_j + \epsilon_j)(T) = ((\epsilon'_{j1} + \epsilon_{j1}),\dots, (\epsilon'_{js} + \epsilon_{js}))(T)$ and $C_{j,m+1} = (c_{j1},\dots, c_{js})$ for $j = 1,\dots, m$, where $r = \dim A$ and $s = \dim X$. Then we have the relations $(\delta_{ki} + \delta'_{ki})(T_i) = c_{ki}(T_i + 1)$ for $k = 1,\dots, r$, $i = 1,\dots, m$, and $(\epsilon'_{jk} + \epsilon_{jk})(T_j) = (T_j + 1)c_{jk}$ for $k = 1,\dots, s$, $j = 1,\dots, m$, where δ_{ki}, $\delta'_{ki} \in \mathrm{Der}(\bar{G}_i, W_i^*)$, $c_{ki} \in W_i^*$, and ϵ'_{jk}, $\epsilon_{jk} \in \mathrm{Der}(\bar{G}_j, W_j)$, $c_{jk} \in W_j$. That is, $\delta_{ki} \equiv \delta'_{ki}$ modulo $\mathrm{Inn}(\bar{G}_i, W_i^*)$ and $\epsilon'_{jk} \equiv \epsilon_{jk}$ modulo $\mathrm{Inn}(\bar{G}_i, W_i)$. Hence if $\{a_i\}$, $\{\phi_i\}$ and $\{a_i'\}$, $\{\phi_i'\}$ are the spanning sets for A, X^* and A', X'^* chosen as in Theorem 2, then up to a change of bases for A and X and a permutation of the indices $1,\dots, m$, $a_i = a_i'$ and $\phi_i = \phi_i'$, $i = 1,\dots, m$. Thus we have proved Theorem 5.

COROLLARY. *Under the hypotheses of Theorem* 5, *if* $\dim A = \dim A' = m - 1, 1,$ *or* 0 *and* $\dim X = \dim X' = 1$ *or* 0, *then* $G_{\pi(i)} \cong G_i'$, $i = 1,\dots, m$, *as linear groups for some permutation* π *of* $1,\dots, m$ *implies* $G \cong G'$ *as linear groups. Dually, if* $\dim A = \dim A' = 1$ *or* 0 *and* $\dim X = \dim X' = m - 1$, 1, *or* 0, *then* $G_{\pi(i)} \cong G_i'$, $i = 1,\dots, m$, *as linear groups implies* $G \cong G'$ *as linear groups.*

Note, however, that the proposition (stated as a conjecture in [6, p. 94]), "$G_i \cong G_i'$ as linear groups for $i = 1,\dots, m$, $\dim A = \dim A'$ and $\dim X = \dim X'$ implies $G \cong G'$ as linear groups" is false. For example, suppose $m = 4$, $\dim A = \dim A' = 2$, $\dim X = \dim X' = 0$, $G_i \cong G_i'$ for $i = 1,\dots, 4$ and $G_i \ncong G_j$ for $i \neq j$. Choose the a_i satisfying $A = \langle a_1, a_2\rangle$, $a_1 = a_3$, $a_4 = a_1 + a_2$ and the a_i' satisfying $A' = \langle a_1', a_3'\rangle$, $a_1' = a_2'$, $a_4' = a_1' + a_3' \neq 0$. Then Theorem 5 implies G and G' cannot be isomorphic as linear groups since $a_1 + a_3 = 0$, $a_1' + a_3' \neq 0$.

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