Some functions reversing the order of positive operators

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Abstract

As a complement to our previous results about the function preserving the operator order, we shall show the following reversing version: Let A and B be positive operators on a Hilbert space H satisfying $MI \geq B \geq mI > 0$. Let $f(t)$ be a continuous convex function on $[m, M]$. If $g(t)$ is a continuous decreasing convex function on $[m, M] \cup \text{Sp}(A)$, then for a given $\alpha > 0$

$A \geq B \geq 0$ implies $\alpha g(B) + \beta I \geq f(A)$,

where $\beta = \max_{m \leq t \leq M} \{f(m) + (f(M) - f(m))(t - m)/(M - m) - \alpha g(t)\}$. Our main result is to classify complementary inequalities on power means of positive operators. As a matter of fact, we determine real constants $\alpha_1$ and $\alpha_1$ such that

$\alpha_2 M^{[s]}(A; \omega) \leq M^{[r]}(A; \omega) \leq \alpha_1 M^{[s]}(A; \omega)$ if $r \leq s$, where $M^{[r]}(A; \omega) := (\sum_{j=1}^{k} \omega_j A_j^r)^{1/r} (r \in \mathbb{R} \{0\})$ is weighted power mean of positive operators $A_j$, $\text{Sp}(A_j) \subseteq [m, M]$ for some scalars $0 < m < M$ and $\omega_j \in \mathbb{R}_+$ such that $\sum_{j=1}^{k} \omega_j = 1 (j = 1, \ldots, k)$.

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1. Introduction

The Löwner–Heinz theorem asserts that the function \( f(t) = t^p \) is operator monotone only for \( 1 \geq p \geq 0 \) though it is monotone increasing for \( p > 0 \). Then

\[
A \geq B > 0 \quad \text{implies} \quad A^p \geq B^p > 0 \quad \text{for all} \quad 0 < p \leq 1,
\]

and consequently

\[
A \geq B > 0 \quad \text{implies} \quad B^p \geq A^p > 0 \quad \text{for all} \quad -1 \leq p < 0.
\]

For convenience we denote by \( C(J) \) the set of real valued continuous functions on an interval \( J \) and by \( \text{Sp}(A) \) the spectrum of an operator \( A \) on a Hilbert space \( H \).

Furuta [4] showed several extensions of the Kantorovich inequality and applied them to show the following order preserving operator inequalities.

**Theorem A.** Let \( A \) and \( B \) be positive operators on a Hilbert space \( H \) satisfying \( \text{Sp}(A) \subseteq [m, M] \) for some scalars \( 0 < m < M \) (resp. \( \text{Sp}(B) \subseteq [n, N] \) for some scalars \( 0 < n < N \)). If \( A \geq B > 0 \), then for each \( p > 1 \)

\[
\left( \frac{M}{m} \right)^{p-1} A^p \geq K(m, M, p) A^p \geq B^p
\]

(resp. \( \left( \frac{N}{n} \right)^{p-1} A^p \geq K(n, N, p) A^p \geq B^p \)),

where a generalized Kantorovich constant \( K(m, M, p)[1,4,6] \) is defined as

\[
K(m, M, p) := \frac{mM^p - Mm^p}{(p-1)(M-m)} \left( \frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \right)^p
\]

for all \( p \in \mathbb{R} \). (★)

Moreover, these extensions are discussed by many authors [2,3,5,7–9,13] and a distinction between the usual order and the chaotic one is clarified in the framework of Kantorovich type inequalities.

In our previous result [9] we showed the function order preserving operator inequalities under a general setting.

**Theorem B.** Let \( A \) and \( B \) be positive operators on a Hilbert space \( H \) satisfying \( \text{Sp}(B) \subseteq [m, M] \) for some scalars \( m > M > 0 \). Let \( f \in \mathcal{C}([m, M]) \) be a convex function and \( g \in \mathcal{C}(J) \), where \( J \supseteq [m, M] \cup \text{Sp}(A) \). Suppose that either of the following conditions holds (a) \( g \) is increasing convex on \( J \), or (b) \( g \) is decreasing concave on \( J \). If \( A \geq B > 0 \), then for a given \( \alpha > 0 \) in the case (a) or \( \alpha < 0 \) in the case (b)

\[
\alpha g(A) + \beta I \geq f(B)
\]

holds for \( \beta = \max_{m \leq t \leq M} \{ f(m) + \mu(t - m) - \alpha g(t) \} \), where \( \mu = \frac{f(M) - f(m)}{M-m} \).
Next we consider the weighted power means of positive operators as follows. Let $A_j$ be positive operators on a Hilbert space $H$ satisfying $\text{Sp}(A_j) \subseteq [m, M]$, for some scalars $0 < m < M$ and $\omega_j \in \mathbb{R}_+$ such that $\sum_{j=1}^k \omega_j = 1$ ($j = 1, \ldots, k$). We define

$$M_k^{[r]}(A; \omega) := \left( \sum_{j=1}^k \omega_j A_j^r \right)^{1/r} \quad \text{if } r \in \mathbb{R} \setminus \{0\}.$$  

We proved in [12–Theorem 1] that

$$A^{-1} M_k^{[s]}(A; \omega) \leq M_k^{[r]}(A; \omega) \leq A M_k^{[s]}(A; \omega)$$

holds if $r \leq s$, $s \notin (-1, 1)$, or $1/2 \leq r \leq 1 \leq s$ or $r \leq -1 \leq s \leq -1/2$ and

$$A^{-1} M_k^{[s]}(A; \omega) \leq M_k^{[r]}(A; \omega) \leq A M_k^{[s]}(A; \omega)$$

holds if $s \geq 1$, $-1 < r < 1/2$, $r \neq 0$ or $r \leq -1$, $-1/2 < s < 1$, $s \neq 0$, where

$$A = \left\{ \frac{r(k^r - k^s)}{(s - r)(k^r - 1)} \right\}^{1/7} \left\{ \frac{s(k^r - k^s)}{(r - s)(k^s - 1)} \right\}^{-1/7}, \quad \kappa = \frac{M}{m}.$$  

The object of this paper is to pursue further the study of reversing Kantorovich type operator inequalities under a general setting. As our main result, we determine real constants $\alpha_2$ and $\alpha_1$ such that

$$\alpha_2 M_k^{[r]}(A; \omega) \leq M_k^{[r]}(A; \omega) \leq \alpha_1 M_k^{[r]}(A; \omega)$$

holds if $r \leq s$, $r, s \neq 0$.

2. Functions reversing the operator order

First we show the function order reversing operator inequalities under the operator order.

The following theorem is similar to Theorem B but for reversing order.

**Theorem 2.1.** Let $A$ and $B$ be positive operators on a Hilbert space $H$ satisfying $\text{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. Let $f \in \mathcal{C}'([m, M])$ be a convex function and $g \in \mathcal{C}(J)$, where $J$ be any interval $J \supseteq \{m, M\} \cup \text{Sp}(B)$. Suppose that either of the following conditions holds: (a) $g$ is decreasing convex on $J$, or (b) $g$ is increasing concave on $J$. If $A \geq B > 0$, then for a given $\alpha > 0$ in the case (a) or $\alpha < 0$ in the case (b)

$$\alpha g(B) + \beta I \geq f(A)$$  

holds for $\beta = \max_{m \leq t \leq M} \left\{ f(m) + \mu(t - m) - \alpha g(t) \right\}$, where $\mu = \frac{f(M) - f(m)}{M - m}$.

**Proof.** Though the proof is quite similar to the proof of Theorem B in [9], we give proof for the sake of convenience. Let $x \in H$ be any unit vector. By the convexity of $\alpha g$, it follows from Jensen’s inequality that
\[ \alpha(g(B)x, x) \geq \alpha g((Bx, x)). \]

By the decrease of \( \alpha g \), we have
\[ \alpha g((Bx, x)) \geq \alpha g((Ax, x)). \]

Next, it follows from complementary inequality to Jensen’s inequality [10—Theorem 4] that for any real number \( \alpha \in \mathbb{R} \), a function \( g \in \mathcal{C}([m, M]) \) and a convex function \( f \in \mathcal{C}([m, M]) \) the following inequality
\[ \alpha g((Ax, x)) + \beta \geq (f(A)x, x) \]
holds, where \( \beta = \max_{m \leq t \leq M} \{ f(t) + \mu(t - m) - \alpha g(t) \} \). Therefore, combining the three inequalities above we have
\[ \alpha(g(B)x, x) + \beta \geq \alpha g((Bx, x)) + \beta \geq \alpha g((Ax, x)) + \beta \geq (f(A)x, x). \]
\[ \square \]

The following theorem is a complementary result to Theorem 2.1:

**Theorem 2.2.** Let \( A \) and \( B \) be positive operators on a Hilbert space \( H \) satisfying \( \text{Sp}(B) \subseteq [m, M] \) for some scalars \( M > m > 0 \). Let \( f \in \mathcal{C}([m, M]) \) be a convex function and \( g \in \mathcal{C}(J) \), where \( J \supseteq [m, M] \cup \text{Sp}(A) \). Suppose that either of the following conditions holds: (a) \( g \) is decreasing concave on \( J \), or (b) \( g \) is increasing convex on \( J \). If \( A \geq B > 0 \), then for a given \( \alpha > 0 \) in the case (a) or \( \alpha < 0 \) in the case (b)
\[ f(B) \geq \alpha g(A) + \beta I \quad (2) \]
holds for \( \beta = \min_{m \leq t \leq M} \{ f(t) + \mu(t - m) - \alpha g(t) \} \), where \( \mu = \frac{f(M) - f(m)}{M - m} \).

**Remark 2.3.** If we put \( \alpha = 1 \) in Theorems 2.1 and 2.2, then we have the following: Let \( A \) and \( B \) be positive operators on a Hilbert space \( H \) satisfying \( \text{Sp}(A) \subseteq [m, M] \) (resp. \( \text{Sp}(B) \subseteq [m, M] \)) for some scalars \( M > m > 0 \). Let \( f \in \mathcal{C}(J) \) be a convex (resp. concave) function and \( g \in \mathcal{C}(J) \) an decreasing convex (resp. concave) function, where \( J \supseteq [m, M] \cup \text{Sp}(A) \cup \text{Sp}(B) \).

If \( A \geq B > 0 \), then
\[ g(B) + \beta I \geq f(A) \quad (\text{resp. } f(B) \geq g(A) + \beta I) \]
holds for \( \beta = \max_{m \leq t \leq M} \{ (f(t) + \mu(t - m)) - g(t) \} \) (resp. \( \beta = \min_{m \leq t \leq M} \{ (f(t) + \mu(t - m)) - g(t) \} \)), where \( \mu = \frac{f(M) - f(m)}{M - m} \).

If we choose \( \alpha \) such that \( \beta = 0 \) in Theorems 2.1 and 2.2, then we have the following corollary:

**Corollary 2.4.** Let \( A \) and \( B \) be positive operators on a Hilbert space \( H \) satisfying \( \text{Sp}(A) \subseteq [m, M] \) (resp. \( \text{Sp}(B) \subseteq [m, M] \)) for some scalars \( M > m > 0 \). Let
$f \in \mathcal{C}(J)$ be a convex (resp. concave) function and $g \in \mathcal{C}(J)$, where $J \supseteq [m, M] \cup \text{Sp}(A) \cup \text{Sp}(B)$. Suppose that either of the following conditions holds:

(i) $g$ is decreasing convex (resp. concave) on $J$, $g > 0$ on $[m, M]$ and $f(m) > 0$, $f(M) > 0$,
(ii) $g$ is decreasing convex (resp. concave) on $J$, $g < 0$ on $[m, M]$ and $f(m) < 0$, $f(M) < 0$,
(iii) $g$ is increasing concave (resp. convex) on $J$, $g > 0$ on $[m, M]$ and $f(m) < 0$, $f(M) < 0$,
(iv) $g$ is increasing concave (resp. convex) on $J$, $g < 0$ on $[m, M]$ and $f(m) > 0$, $f(M) > 0$.

If $A \succeq B > 0$, then

$$\alpha_+ g(B) \geq f(A) \quad \text{(resp. } f(B) \geq \alpha_- g(A))$$

holds for

$$\alpha_+ = \max_{m \leq t \leq M} \left\{ \frac{f(m) + \mu(t - m)}{g(t)} \right\}$$

(resp. $\alpha_- = \min_{m \leq t \leq M} \left\{ \frac{f(m) + \mu(t - m)}{g(t)} \right\}$),

in case (i) and (iii), or

$$\alpha_+ = \min_{m \leq t \leq M} \left\{ \frac{f(m) + \mu(t - m)}{g(t)} \right\}$$

(resp. $\alpha_- = \max_{m \leq t \leq M} \left\{ \frac{f(m) + \mu(t - m)}{g(t)} \right\}$),

in case (ii) and (iv), where $\mu = \frac{f(M) - f(m)}{M - m}$.

As applications of Theorems 2.1, 2.2 and Corollary 2.4 for $f \equiv g$ we can obtain the function order reversing operator inequalities under operator order similarly to [9–Section 6] for the function order preserving operator ones. In particular, if we put $f(t) = g(t) = t^p$ for $p < -1$ in Theorem 2.1 and Corollary 2.4, then we have the following corollaries which we need in next section.

**Corollary 2.5.** Let $A$ and $B$ be positive operators on a Hilbert space $H$ satisfying $\text{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. If $A \succeq B > 0$, then for a given $\alpha > 0$

$$\alpha B^p + \beta I \succeq A^p \quad \text{for all } p < -1,$$

where
\[ \beta = \begin{cases} 
\alpha(p - 1) \left( \frac{1}{\alpha p} \frac{M^p - m^p}{M - m} \right)^{\frac{1}{p-1}} + \frac{Mm^p - M^p - m^p}{M - m} & \text{if } \frac{pm^p - m^p}{\alpha (M - m)} < pM^p - m^p, \\
\max \{M^p - \alpha M^p, m^p - \alpha m^p\} & \text{otherwise.} 
\end{cases} \]

The following theorem is similar to Theorem A but for reversing order.

**Corollary 2.6.** Let \( A \) and \( B \) be positive operators on a Hilbert space \( H \) satisfying \( \text{Sp}(A) \subseteq [m, M] \) for some scalars \( 0 < m < M \) (resp. \( \text{Sp}(B) \subseteq [n, N] \) for some scalars \( 0 < n < N \)). If \( A \succeq B > 0 \), then for each \( p < -1 \)
\[ K(m, M, p)B^p \succeq A^p \quad \text{(resp. } K(n, N, p)B^p \succeq A^p), \]
where a generalized Kantorovich constant \( K(m, M, p) \) is defined as (\( \star \)).

### 3. Weighted power mean

In this section we discuss the usual operator order among power means. First we have the following result:

**Theorem 3.1.** Let \( A_j \) be positive operators on a Hilbert space \( H \) satisfying \( \text{Sp}(A_j) \subseteq [m, M] \) for some scalars \( 0 < m < M \) and \( \omega_j \in \mathbb{R}_+ \) such that \( \sum_{j=1}^k \omega_j = 1 (j = 1, 2, \ldots, k) \).

(I) If \( 0 < p \leq 1 \), then
\[ K(m, M, p) \left( \sum_{j=1}^k \omega_j A_j \right)^p \leq \sum_{j=1}^k \omega_j A_j^p \leq K(m, M, p) \left( \sum_{j=1}^k \omega_j A_j \right)^p. \]

(II) If \( -1 \leq p < 0 \) or \( 1 \leq p \leq 2 \), then
\[ \left( \sum_{j=1}^k \omega_j A_j \right)^p \leq \sum_{j=1}^k \omega_j A_j^p \leq K(m, M, p) \left( \sum_{j=1}^k \omega_j A_j \right)^p. \]

(III) If \( p < -1 \) or \( p > 2 \), then
\[ \frac{1}{K(m, M, p)} \left( \sum_{j=1}^k \omega_j A_j \right)^p \leq \sum_{j=1}^k \omega_j A_j^p \leq K(m, M, p) \left( \sum_{j=1}^k \omega_j A_j \right)^p, \]
where a generalized Kantorovich constant \( K(m, M, p) \) is defined as (\( \star \)).

We need the following three theorems to prove Theorem 3.1.
Theorem C [10–Corollary 4]. Let $A_j$ be positive operators on a Hilbert space $H$ satisfying $\text{Sp}(A_j) \subseteq [m, M]$ for some scalars $0 < m < M$ $(j = 1, 2, \ldots, k)$. Let $f \in C([m, M])$ be a convex function and let $x_1, x_2, \ldots, x_k$ be any finite number of vectors in $H$ such that $\sum_{j=1}^k \|x_j\|^2 = 1$. If $f$ satisfies either (a) $f > 0$ or (b) $f < 0$ on $[m, M]$, then

$$\sum_{j=1}^k (f(A_j)x_j, x_j) \leq \lambda f \left( \sum_{j=1}^k (A_jx_j, x_j) \right)$$

(3)

holds for $\lambda > 1$ in case (a) or $0 < \lambda < 1$ in case (b).

More precisely, a value of $\lambda = \lambda(m, M, f)$ for (3) may be determined as follows:

Let $\mu = \frac{f(M) - f(m)}{M - m}$. If $\mu = 0$, let $t = \bar{t}$ be the unique solution of the equation $f'(t) = 0$ ($m < \bar{t} < M$); then $\lambda = f(m)/f(\bar{t})$ suffices for (3). If $\mu \neq 0$, let $t = \bar{t}$ be the unique solution in $(m, M)$ of the equation $\mu f(t) - f'(t)(f(m) + \mu(t_m)) = 0$; then $\lambda = \mu/f(\bar{t})$ suffices for (3).

In the next theorem, by virtue of Theorem C, we shall estimate the bounds of the operator convexity for convex functions.

Theorem 3.2. Let $A_j$ be positive operators on a Hilbert space $H$ satisfying $\text{Sp}(A_j) \subseteq [m, M]$ for some scalars $0 < m < M$ and $\omega_j \in \mathbb{R}_+$ such that $\sum_{j=1}^k \omega_j = 1$ ($j = 1, 2, \ldots, k$). If $f \in C([m, M])$ is a positive convex function, then

$$\frac{1}{\lambda(m, M, f)} f \left( \sum_{j=1}^k \omega_j A_j \right) \leq \sum_{j=1}^k \omega_j f(A_j) \leq \lambda(m, M, f) f \left( \sum_{j=1}^k \omega_j A_j \right)$$

(4)

holds for $\lambda(m, M, f) = \max_{m \leq t \leq M} \{(f(m) + \mu(t_m))/f(t)\}$, where $\mu = \frac{f(M) - f(m)}{M - m}$.

Proof. For each $\omega_j \in \mathbb{R}_+$ and unit vector $x \in H$ we put $x_j = \sqrt{\omega_j}x$ in Theorem C. Then we have

$$\sum_{j=1}^k \omega_j f(A_j)x_j \leq \lambda(m, M, f) f \left( \sum_{j=1}^k \omega_j A_jx_j \right).$$
Hence
\[
\left( \sum_{j=1}^{k} \omega_j f(A_j)x, x \right) \leq \lambda(m, M, f) \left( \sum_{j=1}^{k} \omega_j (A_j x, x) \right)
\]
and the last inequality holds by the convexity of \( f \). Therefore we have
\[
\sum_{j=1}^{k} \omega_j f(A_j) \leq \lambda(m, M, f) \left( \sum_{j=1}^{k} \omega_j A_j \right).
\]

Next, since \( f \) is convex, it follows from Jensen’s inequality that
\[
\left( \sum_{j=1}^{k} \omega_j f(A_j)x, x \right) = \sum_{j=1}^{k} \omega_j (f(A_j)x, x) \geq f \left( \sum_{j=1}^{k} \omega_j (A_j x, x) \right).
\]

Since \( 0 \leq m I \leq \sum_{j=1}^{k} \omega_j A_j \leq M I \), it follows from (3) for \( k = 1 \) that
\[
f \left( \sum_{j=1}^{k} \omega_j (A_j x, x) \right) = f \left( \left( \sum_{j=1}^{k} \omega_j A_j \right) x, x \right) \geq \frac{1}{\lambda(m, M, f)} f \left( \sum_{j=1}^{k} \omega_j A_j \right) x, x \right).
\]

Therefore we have
\[
\sum_{j=1}^{k} \omega_j f(A_j) \geq \frac{1}{\lambda(m, M, f)} f \left( \sum_{j=1}^{k} \omega_j A_j \right). \quad \square
\]

We have the following complementary result of Theorem 3.2 for concave functions.

**Theorem 3.3.** Let \( A_j \) be positive operators on a Hilbert space \( H \) satisfying \( \text{Sp}(A_j) \subseteq [m, M] \) for some scalars \( 0 < m < M \) and \( \omega_j \in \mathbb{R}_+ \) such that \( \sum_{j=1}^{k} \omega_j = 1(j = 1, 2, \ldots, k) \). If \( f \in \mathcal{C}(\{m, M\}) \) is a positive concave function, then
\[
\frac{1}{v(m, M, f)} f \left( \sum_{j=1}^{k} \omega_j A_j \right) \geq \sum_{j=1}^{k} \omega_j f(A_j) \geq v(m, M, f) f \left( \sum_{j=1}^{k} \omega_j A_j \right)
\]
(5)
holds for
\[ \nu(m, M, f) = \min_{m \leq t \leq M} \left\{ \frac{(f(m) + \mu(t - m))/f(t)}{M - m} \right\}, \]
where \( \mu = \frac{f(M) - f(m)}{M - m} \).

Proof of Theorem 3.1. This theorem follows from Theorems 3.2 and 3.3 for \( f(t) = t^p \). As a matter of fact, since \( f(t) = t^p \) is an operator concave function if \( 0 \leq p \leq 1 \), then we have the right hand inequality in (I) and by Theorem 3.3 we have the left hand inequality with \( \nu(m, M, f) = K(m, M, p) \). Since \( f(t) = t^p \) is a operator convex function if \( -1 \leq p < 0 \) or \( 1 \leq p \leq 2 \), then we have the left hand inequality (II) and by Theorem 3.2 we have the right hand inequality with \( \lambda(m, M, f) = K(m, M, p) \). Since \( f(t) = t^p \) is not operator convex though \( f \) is a convex function if \( p < -1 \) or \( p > 2 \) we obtain inequality (III) by Theorem 3.2. \( \square \)

For the sake of convenience we denote intervals from (i) to (iv) as in Table 1 (see Fig. 1).

<table>
<thead>
<tr>
<th>Interval</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>( r \leq s, s \notin (-1, 1), r \notin (-1, 1) ) or ( 1/2 \leq r \leq 1 \leq s ) or (-1 \leq s \leq -1/2 )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( s \geq 1, -1 &lt; r &lt; 1/2, r \neq 0 ) or (-1 &lt; r \leq 1, -1/2 &lt; s \leq 1, s \neq 0 )</td>
</tr>
<tr>
<td>(iii)</td>
<td>(-1 \leq -s \leq r \leq 1, r \neq 0 ) or (-1 \leq r \leq s \leq r/2 &lt; 0 )</td>
</tr>
<tr>
<td>(iv)</td>
<td>(-1/2 \leq r/2 &lt; s \leq -r \leq 1, s \neq 0 )</td>
</tr>
</tbody>
</table>

Fig. 1.
Our main result is given in the next theorem.

**Theorem 3.4.** Let $A_j$ be positive operators on a Hilbert space $H$ satisfying $\text{Sp}(A_j) \subseteq [m, M]$ for some scalars $0 < m < M$ and $\omega_j \in \mathbb{R}_+$ such that $\sum_{j=1}^k \omega_j = 1 (j = 1, 2, \ldots, k)$.

(i) If $r \leq s$, $s \notin (-1, 1)$, $r \notin (-1, 1)$ or $1/2 \leq r \leq 1$ or $r \leq -1$ or $s \leq -1/2$, then

$$A(\kappa, r, s)^{-1}M^{[s]}_k(A; \omega) \leq M^{[r]}_k(A; \omega) \leq M^{[s]}_k(A; \omega).$$

(ii) If $s \geq 1$, $-1 < r < 1/2$, $r \neq 0$ or $r \leq -1$, $-1/2 < s < 1$, $s \neq 0$, then

$$A(\kappa, r, s)^{-1}M^{[s]}_k(A; \omega) \leq M^{[r]}_k(A; \omega) \leq A(\kappa, r, s)M^{[s]}_k(A; \omega).$$

(iii) If $-1 \leq -s \leq r \leq s \leq 1$, $r \neq 0$ or $-1 \leq r \leq s \leq r/2 < 0$, then

$$A(\kappa, r, 1)^{-1}A(\kappa, r, s)^{-1}M^{[s]}_k(A; \omega) \leq M^{[r]}_k(A; \omega) \leq A(\kappa, r, 1)M^{[s]}_k(A; \omega).$$

(iv) If $-1/2 \leq r < 2 < s < -r \leq 1$, $s \neq 0$, then

$$A(\kappa, s, 1)^{-1}A(\kappa, r, s)^{-1}M^{[s]}_k(A; \omega) \leq M^{[r]}_k(A; \omega) \leq A(\kappa, s, 1)M^{[s]}_k(A; \omega),$$

where

$$A(\kappa, r, s) = \left\{ \frac{r(k^s - \kappa^r)}{(s - r)(\kappa^r - 1)} \right\}^{1/r} \left\{ \frac{s(k^{s} - r^s)}{(r - s)(r^s - 1)} \right\}^{-1}, \quad \kappa = \frac{M}{m}.$$ 

**Proof.** Mond and Pečarić proved in [11] that (i) holds. We proved in [12] that (i) and (ii) hold. Next we shall prove (iii) and (iv).

(iii) If $0 < r \leq s \leq 1$ then $0 < \frac{r}{r/s} \leq 1$. If we put $p = \frac{r}{r/s}$ in Theorem 3.1(I) and replace $A_j$ by $A_j^s$ ($j = 1, \ldots, k$) we obtain

$$K \left( m^s, M^s, \frac{r}{s} \right) \left( \sum_{j=1}^k \omega_j A_j^s \right)^{r/s} \leq \sum_{j=1}^k \omega_j A_j^s \leq \left( \sum_{j=1}^k \omega_j A_j^s \right)^{r/s}. $$

By raising above inequality to the power $1/r (\geq 1)$ it follows from Theorem A that

$$K \left( m^r, M^r, \frac{1}{r} \right)^{-1} K \left( m^s, M^s, \frac{r}{s} \right)^{1/r} M^{[s]}_k(A; \omega) \leq M^{[r]}_k(A; \omega).$$

Then, we have

$$A(\kappa, r, 1)^{-1}A(\kappa, r, s)^{-1}M^{[s]}_k(A; \omega) \leq M^{[r]}_k(A; \omega) \leq A(\kappa, r, 1)M^{[s]}_k(A; \omega).$$
because

\[ K \left( m', M', \frac{s}{r} \right) = \frac{m' (M')^\frac{r}{s} - M' (m')^\frac{r}{s}}{(\frac{r}{s} - 1) (M' - m')} \left( \frac{\frac{r}{s} - 1}{\frac{r}{s}} \right) \left( \frac{M' - m'}{m' M' - M' m'} \right) \]

\[ = \frac{m' M^s - M' m^s}{(\frac{r}{s} - 1) (M' - m')} \left( \frac{\frac{r}{s} - 1}{\frac{r}{s}} \right) \left( \frac{M' - m'}{m' M' - M' m'} \right) \]

\[ = \frac{r (\kappa^s - \kappa^r)}{(s - r)(\kappa^r - 1)} \left( \frac{s(\kappa^r - \kappa^s)}{(r - s)(\kappa^s - 1)} \right)^\frac{r}{s} \]

and

\[ K \left( m^s, M^s, \frac{r}{s} \right) = K \left( m', M', \frac{s}{r} \right)^{-\frac{1}{r}} \]

\[ = \left\{ \frac{r (\kappa^s - \kappa^r)}{(s - r)(\kappa^r - 1)} \right\}^{-\frac{1}{r}} \left\{ \frac{s(\kappa^r - \kappa^s)}{(r - s)(\kappa^s - 1)} \right\}^{\frac{1}{r}} \]

\[ = A(\kappa, r, s)^{-1}. \]

If \(-1 \leq -s \leq r < 0\) then \(-1 \leq \xi < 0\), but if \(-1 \leq r \leq s \leq r/2 < 0\) then \(1 \leq \xi \leq 2\). If we put \(p = \xi/2\) in Theorem 3.1(II) and replace \(A_j\) by \(A_j^s\) \((j = 1, \ldots, k)\) we obtain

\[ \left( \sum_{j=1}^{k} \omega_j A_j^s \right)^{\frac{r}{s}} \leq \sum_{j=1}^{k} \omega_j A_j^s \leq K \left( m^s, M^s, \frac{r}{s} \right) \left( \sum_{j=1}^{k} \omega_j A_j^s \right)^{\frac{r}{s}} \]

if \(-1 \leq -s \leq r < 0\) and

\[ \left( \sum_{j=1}^{k} \omega_j A_j^s \right)^{\frac{r}{s}} \leq \sum_{j=1}^{k} \omega_j A_j^s \leq K \left( m^s, M^s, \frac{r}{s} \right) \left( \sum_{j=1}^{k} \omega_j A_j^s \right)^{\frac{r}{s}} \]

if \(-1 \leq r \leq s \leq r/2 < 0\). Using that \(K(M^*, m^s, \frac{s}{r}) = K(m^s, M^s, \frac{s}{r})\) and by raising above inequalities to the power \(1/r\) \((\leq -1)\), we obtain from Corollary 2.6

\[ K \left( m', M', \frac{1}{r} \right) M_k^{\frac{r}{s}}(A; \omega) \]

\[ \geq M_k^{\frac{r}{s}}(A; \omega) \]

\[ \geq K \left( m', M', \frac{1}{r} \right)^{-1} K \left( m^s, M^s, \frac{r}{s} \right)^{1/r} M_k^{\frac{r}{s}}(A; \omega) \]
if \(-1 \leq s \leq r < 0\) or \(-1 \leq r \leq s \leq r/2 < 0\). Since \(\Delta(\kappa, r, s) K(m^s, M^s, \frac{r}{s})^\frac{1}{r} = 1\), we have
\[
\Delta(\kappa, r, 1)^{-1} A(\kappa, r, s)^{-1} M_k^{[r]}(A; \omega) \leq M_k^{[r]}(A; \omega) \leq \Delta(\kappa, r, 1) M_k^{[r]}(A; \omega)
\]
if \(-1 \leq s \leq r \leq s \leq r/2 < 0\).

(iv) Next, let \(-1 \leq r < -s < 0\) or \(-1/2 \leq r/2 < s < 0\). Then \(-1 < \frac{s}{r} < 0\) or \(0 < \frac{s}{r} < \frac{1}{2}\). If we put \(p = \frac{s}{r}\) in Theorem 3.1(II) and (I) and replace \(A_j\) by \(A'_j\) \((j = 1, \ldots, k)\) we obtain
\[
\left( \sum_{j=1}^k \omega_j A'_j \right)^{s/r} \leq \left( \sum_{j=1}^k \omega_j A'_j \right)^{s/r} \leq \left( \sum_{j=1}^k \omega_j A'_j \right)^{s/r}
\]
if \(-1 \leq r < -s < 0\) and
\[
K \left( M', m'^r, \frac{s}{r} \right) \left( \sum_{j=1}^k \omega_j A'_j \right)^{s/r} \leq \left( \sum_{j=1}^k \omega_j A'_j \right)^{s/r}
\]
if \(-1/2 \leq r/2 < s < 0\). By raising above inequalities to the power \(1/s\) we obtain from Theorem A and Corollary 2.6 that
\[
K \left( m^s, M^s, \frac{1}{s} \right)^{-1} M_k^{[r]}(A; \omega)
\]
\[
\leq M_k^{[r]}(A; \omega)
\]
\[
\leq K \left( m^s, M^s, \frac{1}{s} \right) K \left( M', m'^r, \frac{s}{r} \right)^{1/s} M_k^{[r]}(A; \omega)
\]
if \(-1 \leq r < -s < 0\) and
\[
K \left( M', m'^r, \frac{s}{r} \right) K \left( M', m'^r, \frac{s}{r} \right)^{1/s} M_k^{[r]}(A; \omega)
\]
\[
\geq M_k^{[r]}(A; \omega)
\]
\[
\geq K \left( M', m'^r, \frac{1}{s} \right)^{-1} M_k^{[r]}(A; \omega)
\]
if \(-1/2 \leq r/2 < s < 0\). Since \(K(M', m'^r, \frac{1}{s}) = K(m^s, M^s, \frac{1}{s}) = \Delta(\kappa, 1, s)^{-1} = \Delta(\kappa, s, 1)\) we have
\[
\Delta(\kappa, s, 1)^{-1} M_k^{[r]}(A; \omega) \leq M_k^{[r]}(A; \omega) \leq \Delta(\kappa, s, 1) M_k^{[r]}(A; \omega)
\]
if \(-1/2 \leq r/2 < s < -r \leq 1, s \neq 0\). Then we have
\[
\Delta(\kappa, s, 1)^{-1} \Delta(\kappa, r, s)^{-1} M_k^{[r]}(A; \omega) \leq M_k^{[r]}(A; \omega) \leq \Delta(\kappa, s, 1) M_k^{[r]}(A; \omega)
\]
if \(-1/2 \leq r/2 < s < -r \leq 1, s \neq 0\). □
References

[6] T. Furuta, Basic property of generalized Kantorovich constant $K(h, p) = \frac{h^p - h}{p - 1} \left( \frac{p - 1}{h^p - 1} \right)^p$ and its applications, Acta (Szeged) Math. 70 (2004) 319–337.