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Monotone Methods for a Discrete Boundary Problem

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Abstract—This paper is motivated by recent interests in space discrete Nagumo equations and is concerned with the existence of solutions of a nonlinear discrete boundary value problem. Monotone methods are used to derive the existence theorems. These methods, as is well known, provide constructive schemes for calculating the solutions.

Keywords—Nonlinear difference equations, Boundary value problems, Upper and lower solutions, Minimal and maximal solutions.

1. INTRODUCTION

Interests in stability and spatial chaos in discrete or semidiscrete dynamical systems can be found in several recent studies [1–4]. For instance, the semidiscrete Nagumo equation

$$u_n' = d\left(u_{n-1} - 2u_n + u_{n+1}\right) + f\left(u_n\right),$$

or the fully discrete Nagumo equation

$$u_i(j+1) - u_i(j) = \kappa\left(u_{i-1}(j) - 2u_i(j) + u_i(j-1)\right) + \alpha f\left(u_i(j)\right),$$

have been studied in [1–3], respectively. These studies show that discrete analogs of partial differential equations yield interesting dynamical systems in their own right. In this paper, we are concerned with the existence of steady-state or time independent solutions of dynamical systems, such as the ones indicated above. More specifically, denoting $v_{k+1} - v_k$ by Δv_k and $v_{k+1} - 2v_k + v_{k-1}$ by $\Delta^2 v_{k-1}$, respectively, we will seek solutions of a class of discrete boundary value problem of the form

$$\Delta^2 v_{k-1} + f(k, v_k) = 0, \qquad k = 1, 2, \dots, n, \tag{1.1}$$

where $f(k,v)$ is a real function defined for $k = 1, \dots, n$ and $v \in R$. In general, such solutions are subject to boundary conditions. In these paper, we will consider a standard type of boundary conditions of the form

$$v_0 = 0 = v_{n+1}, \tag{1.2}$$

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Discrete boundary value problems such as (1.1),(1.2) have already been studied by many authors (see, for example, [5–8]), since these problems are also natural consequences of discretization techniques of differential boundary problems. Besides, they arise in the study of solid state physics, chemical reactions, population dynamics, etc.

As is well known, there are several techniques which are often employed in boundary value problems. These include or involve the method of *a priori* estimates, contraction mappings theorems, the Brouwer fixed point theorem, the method of perturbation, etc. Here, we will employ the method of upper and lower solutions (see, e.g., [9–11]). Such methods, as is well known, will provide constructive schemes for calculating the desired solutions.

2. UPPER AND LOWER SOLUTIONS

In the sequel, a sequence $u = \{u_a, u_{a+1}, \dots, u_b\}$ is said to be less than or equal to another sequence $w = \{w_a, w_{a+1}, \dots, w_b\}$ (denoted by $u \le w$) if each of the components of u is less than or equal to the corresponding ones of w. A solution of the boundary value problem (1.1),(1.2) is a real sequence of the form $\{u_0, u_1, \dots, u_{n+1}\}$ such that (1.1),(1.2) is satisfied. A real sequence $w = \{w_0, w_1, \dots, w_{n+1}\}$ is called an upper solution of (1.1),(1.2) if

$$\Delta^2 w_{k-1} + f(k, w_k) \le 0, \qquad k = 1, \dots, n, \tag{2.1}$$

and

$$w_0 \ge 0, \qquad w_{n+1} \ge 0. \tag{2.2}$$

Similarly, a real sequence $u = \{u_0, u_1, \dots, u_{n+1}\}$ is called a lower solution of (1.1),(1.2) if

$$\Delta^2 u_{k-1} + f(k, u_k) \ge 0, \qquad k = 1, 2, \dots, n, \tag{2.3}$$

and

$$u_0 \le 0, \qquad u_{n+1} \le 0. \tag{2.4}$$

There is a maximum principle as follows.

LEMMA 2.1. *Let $\{v_0, v_1, \dots, v_{n+1}\}$ be a real sequence which satisfies the recurrence relation*

$$\Delta^2 v_{k-1} + F(k, v_k) \ge 0, \qquad k = 1, 2, \dots, n, \tag{2.5}$$

where for each $k \in \{1, \dots, n\}$, $F(k, v) \le 0$, whenever $v \ge 0$. If

$$M = \max\{v_0, \dots, v_{n+1}\} \ge 0,$$

then $v_k < M$ for $1 \le k \le n$, unless $v_k = M$ for $0 \le k \le n+1$.

PROOF. Suppose to the contrary that $v_j = M$ for some $j \in \{1, \dots, n\}$. Then $\Delta^2 v_{j-1} = v_{j-1} - 2v_j + v_{j+1} \ge 0$, which implies $v_{j-1} = v_j = v_{j+1} = M$. The proof can now be completed by showing inductively that $v_{j-1} = v_{j-2} = \dots = v_0$ and $v_{j+1} = v_{j+2} = \dots = v_{n+1}$. ∎

As an application, if $v = \{v_0, \dots, v_{n+1}\}$ is a lower solution of (1.1),(1.2), where $f(k, v) \le 0$ for $1 \le k \le n$ and $v \ge 0$, then $v_k \le 0$ for $0 \le k \le n+1$. Indeed, if $\max\{v_0, \dots, v_{n+1}\} > 0$, then it is either v_0 or v_{n+1}, which is impossible.

As another application, we will show that a lower solution is less than or equal to any upper solution when $f(k, v)$ is nonincreasing in v.

LEMMA 2.2. *Assume that $f(k, v)$ is nonincreasing in v for $1 \le k \le n$. Then, for any lower solution $u = \{u_0, \dots, u_{n+1}\}$ and upper solution $w = \{w_0, \dots, w_{n+1}\}$ of (1.1),(1.2), we have $u_k \le w_k$ for $0 \le k \le n+1$.*

PROOF. We obtain from (2.1) and (2.3), that

$$\Delta^2 (u_{k-1} - w_{k-1}) + f(k, u_k) - f(k, w_k) \ge 0, \qquad 1 \le k \le n, \tag{2.6}$$

and

$$u_0 - w_0 \le 0, \qquad u_{n+1} - w_{n+1} \le 0.$$

Thus, the sequence $\{v_k\}$ defined by $v_k = u_k - w_k$ for $0 \le k \le n+1$, satisfies the recurrence relation (2.6) which is of the form (2.5). By Lemma 2.1, we see that $v_k = u_k - w_k \le 0$ for $0 \le k \le n+1$. ∎

We remark that, in general, a lower solution may not be less than or equal to an upper solution. As we will see in Section 3, it is useful to know that a lower solution is less than or equal to an upper solution. Thus, an additional result is derived in addition to Lemma 2.2.

LEMMA 2.3. *Let $w = \{w_0, \dots, w_{n+1}\}$ be an upper solution of (1.1),(1.2) such that there is a positive sequence $z = \{z_0, \dots, z_{n+1}\}$ satisfying*

$$\Delta^2(\lambda z_{k-1}) < f(k, w_k) - f(k, w_k + \lambda z_k), \qquad 1 \le k \le n,$$

for any $\lambda > 0$. Then $v \le w$, for any lower solution $v = \{v_0, \dots, v_{n+1}\}$ of (1.1),(1.2).

PROOF. Suppose to the contrary that v is a lower solution of (1.1),(1.2) such that

$$v_j - w_j = \max_{1 \le i \le n} \{v_i - w_i\} > 0,$$

then $v_j = w_j + \lambda^* z_j$, for some $\lambda^* > 0$. Furthermore,

$$0 \ge \Delta^2(v_{j-1} - w_{j-1} - \lambda^* z_{j-1}) > -f(j, v_j) + f(j, w_j) + f(j, w_j + \lambda^* z_j) - f(j, w_j) = 0,$$

which is a contradiction. ∎

Another comparison theorem for lower and upper solutions is as follows, the proof of which is elementary.

LEMMA 2.4. *Assume that $f_1(k,v) \le f(k,v) \le f_2(k,v)$ for $1 \le k \le n$. Then, an upper solution of*

$$\Delta^2 w_{k-1} + f_2(k, w_k) = 0, \qquad 1 \le k \le n, \quad w_0 = 0 = w_{n+1},$$

is also an upper solution of (1.1),(1.2), a lower solution of

$$\Delta^2 u_{k-1} + f_1(k, u_k) = 0, \qquad 1 \le k \le n, \quad u_0 = 0 = u_{n+1},$$

is also a lower solution of (1.1),(1.2).

Next, we derive an existence theorem for upper and lower solutions. Before doing so, let us recall [5] that the boundary problem (1.1),(1.2) is equivalent to the matrix problem

$$Ax = F, \tag{2.7}$$

where $A = (a_{ij})$, is defined by

$$a_{ij} = \begin{cases} 2, & i = j, \\ -1, & |i-j| = 1, \\ 0, & \text{otherwise}, \end{cases}$$

$x = \mathrm{col}(x_1, x_2, \dots, x_n)$ and $F = \mathrm{col}(f(1, x_1), \dots, f(n, x_n))$. The boundary problem (1.1),(1.2) is equivalent to (2.7), in the sense that a vector $x = \mathrm{col}(x_1, \dots, x_n)$ is a solution of (2.7) if, and only if, the sequence $\{0, x_1, \dots, x_n, 0\}$ is a solution of (1.1),(1.2).

THEOREM 2.1. *If $f(k,v) \geq L$ (or $f(k,v) \leq L$) for $1 \leq k \leq n$, then (1.1),(1.2) has a lower solution (respectively, an upper solution).*

PROOF. Consider the system of linear equations

$$\Delta^2 v_{k-1} + L = 0, \qquad 1 \leq k \leq n, \tag{2.8}$$

$$v_0 = 0 = v_{n+1}. \tag{2.9}$$

As we have just mentioned, this system is equivalent to a matrix problem of the form (2.7), where F is now $\operatorname{col}(L, \dots, L)$. Note that the matrix is nonsingular thus, this system will have a unique solution, which in turn gives rise to a solution v of the linear system. In view of Lemma 2.3, v is also a lower solution of (1.1),(1.2). ∎

Next, suppose there is a nonnegative constant c such that $f(k,c) \leq 0$ for $1 \leq k \leq n$. Then the sequence $v = \{c, c, \dots, c\}$ satisfies

$$\Delta^2 v_{k-1} + f(k, v_k) = f(k,c) \leq 0,$$

for $1 \leq k \leq n$. That is, v is an upper solution of (1.1),(1.2). The following is now clear.

THEOREM 2.2. *Suppose there is a nonnegative (or nonpositive) constant c such that $f(k,c) \leq 0$ (respectively, $f(k,c) \geq 0$) for $1 \leq k \leq n$, then $v = \{c, \dots, c\}$ is an upper solution (respectively, a lower solution) of (1.1),(1.2).*

Other existence theorems for lower and upper solutions of (1.1),(1.2), can be obtained by means of the comparison Lemma 2.3. For instance, we may look for eigensolutions of the linear eigenvalue problem

$$\Delta^2 x_{k-1} + \lambda q_k x_k = 0, \qquad 1 \leq k \leq n, \tag{2.10}$$

$$x_0 = 0 = x_{n+1}, \tag{2.11}$$

which is equivalent to the matrix eigenvalue problem

$$Ax = \lambda \operatorname{diag}(q_1, \dots, q_n) x.$$

Since A is a symmetric matrix, this matrix eigenvalue problem has real eigenvalues and their corresponding eigenvectors give rise to lower and an upper solutions of (1.1),(1.2).

THEOREM 2.3. *Let λ be an eigenvalue of the eigenvalue problem (2.10),(2.11) and let $u = \{u_0, \dots, u_{n+1}\}$ be its corresponding eigenvector. Suppose further that $f(k,x) \leq \lambda q_k x$ ($\lambda q_k x \leq f(k,x)$) for $1 \leq k \leq n$. Then u is an upper solution (respectively, a lower solution) of (1.1),(1.2).*

3. EXISTENCE OF SOLUTIONS

In this section, we will derive several existence theorems for the solutions of the boundary problem (1.1),(1.2).

LEMMA 3.1. *Suppose $|f(k,v)| \leq M$ for $1 \leq k \leq n$ and $v \in R$. Suppose further that $f(k,\cdot)$ is continuous. Then the boundary problem (1.1),(1.2) has a solution.*

PROOF. Let $G = (g_{ij})$ be the inverse of the matrix A in (2.7). Then, we may rewrite (2.7) as a fixed point problem

$$x = Tx, \tag{3.1}$$

where $T : R^n \to R^n$ is defined by

$$(Tx)_i = \sum_{j=1}^{n} g_{ij} f(j, x_j), \qquad 1 \leq i \leq n.$$

Let

$$K = \max_{1\le i\le n} \sum_{j=1}^{n} |g_{ij}|,$$

and let

$$\Omega = \{x = \mathrm{col}\,(x_1, \ldots, x_n) \mid \|x\| \le KM\},$$

where $\|x\| = \max_{1\le i\le n} |x_i|$. It is easy to see that Ω is a bounded, convex and closed subset of R^n. Furthermore, T transforms Ω into Ω, in a continuous manner in view of the assumptions imposed on f. By the Brouwer fixed point theorem, there exists a vector $x^* = \mathrm{col}\,(x_1^*, \ldots, x_n^*) \in \Omega$ such that $x^* = Tx^*$. But then $\mathrm{col}\,(0, x_1^*, \ldots, x_n^*, 0)$ is a solution of (1.1),(1.2) as required. ∎

THEOREM 3.1. *Suppose that $f(k, \cdot)$ is continuous. Then for any lower solution $u = \{u_0, \ldots, u_{n+1}\}$ and upper solution $w = \{w_0, \ldots, w_{n+1}\}$ of (1.1),(1.2) satisfying $u \le w$, there is a solution $v = \{v_0, \ldots, v_{n+1}\}$ of (1.1),(1.2) which satisfies $u \le v \le w$.*

PROOF. Consider the boundary value problem

$$\Delta^2 x_{k-1} + \Phi(k, x_k) = 0, \qquad k = 1, 2, \ldots, n, \tag{3.2}$$

$$x_0 = 0 = x_{n+1}, \tag{3.3}$$

where

$$\Phi(k, x) = \begin{cases} \dfrac{f(k, w_k) + (w_k - x)}{(1 + x^2)}, & x > w_k, \\ f(k, x), & u_k \le x \le w_k, \\ \dfrac{f(k, u_k) + (u_k - x)}{(1 + x^2)}, & x < u_k, \end{cases} \tag{3.4}$$

for $1 \le k \le n$. Clearly, the function Φ is bounded for $1 \le k \le n$ and $x \in R$, and is continuous in x. Thus, by Lemma 3.1, there exists a solution $v = \{v_0, \ldots, v_{n+1}\}$ of the boundary problem (3.2) and (3.3).

We assert that the solution v satisfies $u \le v \le w$ so that it is also a solution of (1.1),(1.2) in view of the definition of Φ. Indeed, suppose to the contrary that $v_j - w_j = \max_{1\le i\le n}\{v_i - w_i\} > 0$. Then

$$0 \ge \Delta^2(v_j - w_j) \ge f(j, w_j) - \Phi(j, v_j) = \frac{v_j - w_j}{1 + v_j^2} > 0,$$

which is a contradiction. Similarly, we may show that $u \le v$. ∎

As an example, suppose

$$0 \le f(k, v) \le 4 \sin^2\left(\frac{\pi}{2(n+1)}\right) v, \qquad 1 \le k \le n.$$

Then the zero sequence is a lower solution of (1.1),(1.2) by Theorem 2.2, and the sequence $w = \{w_0, \ldots, w_{n+1}\}$ defined by

$$w_k = \sin\frac{k\pi}{n+1}, \qquad 0 \le k \le n+1,$$

is an upper solution of (1.1),(1.2) since it satisfies

$$\Delta^2 w_{k-1} + 4\sin^2\left(\frac{\pi}{2(n+1)}\right) w_k = 0, \qquad 1 \le k \le n,$$

and $w_0 = 0 = w_{n+1}$. In view of Theorem 3.2, (1.1),(1.2) has a solution $v = \{v_0, \ldots, v_{n+1}\}$ which satisfies

$$0 \le v_k \le \sin\frac{k\pi}{n+1}, \qquad 0 \le k \le n+1.$$

Next, we derive an existence theorem when a Lipschitz condition is satisfied.

THEOREM 3.2. *Suppose $f(k,\cdot)$ is continuous. Suppose further that there exist a lower solution $u = \{u_0, \dots, u_{n+1}\}$, an upper solution $w = \{w_0, \dots\ w_{n+1}\}$, and a positive sequence $\{p_k\}_{k=1}^n$, such that $u \le w$ and for each $k \in \{1, \dots, n\}$, the following one-sided Lipschitz condition*

$$f(k,x) - f(k,y) \ge -p_k(x-y), \tag{3.5}$$

holds whenever $u_k \le y \le x \le w_k$. Then the boundary problem (1.1),(1.2) has a solution u^* and a solution w^* such that $u^* \le w^*$.

PROOF. For any sequence $\eta = \{\eta_0, \dots, \eta_{n+1}\}$ which satisfies $u \le \eta \le w$, consider the following boundary problem

$$\Delta^2 v_{k-1} + f(k,\eta_k) - p_k(v_k - \eta_k) = 0, \qquad 1 \le k \le n, \tag{3.6}$$

$$v_0 = 0 = v_{n+1}. \tag{3.7}$$

This problem is equivalent to the matrix problem

$$Bx = H, \tag{3.8}$$

where the matrix $B = (b_{ij})$ is defined by

$$b_{ij} = \begin{cases} 2 + p_i, & i = j, \\ -1, & |i-j| = 1, \\ 0, & \text{otherwise}, \end{cases}$$

$x = \operatorname{col}(x_1, \dots, x_n)$, $H = \operatorname{col}(f(1,\eta_1) + p_1\eta_1, \dots, f(n,\eta_n) + p_n\eta_n)$. Since $p_k > 0$ for $1 \le k \le n$, B is a strictly diagonally dominant matrix, and hence, is nonsingular. Thus (3.8), and also (3.6),(3.7), have unique solutions.

Let us define the sector

$$\Omega = \{\eta = \{\eta_0, \dots, \eta_{n+1}\} \mid u \le \eta \le w\},$$

and let $\Gamma : \Omega \to R^{n+2}$ be defined by

$$\Gamma\eta = \mu,$$

where $\mu = \{\mu_0, \dots, \mu_{n+1}\}$ is the unique solution of the boundary problem (3.6),(3.7). We assert that $u \le \Gamma u$ and $\Gamma w \le w$. Indeed, let $\Gamma u = \xi = \{\xi_0, \dots, \xi_{n+1}\}$, and suppose to the contrary that

$$v_j - \xi_j = \max_{1 \le i \le n} \{v_i - \xi_i\} > 0,$$

then

$$0 \ge \Delta^2(v_j - \xi_j) \ge -f(j,v_j) + (f(j,v_j) - p_j(\xi_j - v_j)) = p_j(v_j - \xi_j) > 0,$$

which is a contradiction. Similarly, we may show that $\Gamma w \le w$.

Next, we assert that for any ξ, $\psi \in \Omega$, and $\xi \le \psi$, we have $\Gamma\xi \le \Gamma\psi$. Indeed, let $\Gamma\xi = \tau = \{\tau_0, \dots, \tau_{n+1}\}$ and $\Gamma\psi = \rho = \{\rho_0, \dots, \rho_{n+1}\}$, and suppose to the contrary that

$$\tau_j - \rho_j = \max_{1 \le i \le n} \{\tau_i - \rho_i\} > 0,$$

then we have

$$\begin{aligned} 0 \ge \Delta^2(\tau_{j-1} - \rho_{j-1}) &= \{-f(j,\xi_j) + p_j(\tau_j - \xi_j)\} + \{f(j,\psi_j) - p_j(\rho_j - \psi_j)\} \\ &= \{f(j,\psi_j) - f(j,\xi_j)\} + p_j\{\tau_j - \xi_j - \rho_j + \psi_j\} \\ &\ge -p_j(\psi_j - \xi_j) + p_j(\tau_j - \rho_j) + p_j(\psi_j - \xi_j) \\ &= p_j(\tau_j - \rho_j) > 0, \end{aligned}$$

which is a contradiction.

Therefore, if we define two sequences as follows:

$$u^{(0)} = u, \quad u^{(j+1)} = \Gamma u^{(j)}, \qquad \text{for } j \geq 0,$$

and

$$w^{(0)} = w, \quad w^{(j+1)} = \Gamma w^{(j)}, \qquad \text{for } j \geq 0,$$

then we have

$$u = u^{(0)} \leq u^{(1)} \leq \cdots \leq w^{(1)} \leq w^{(0)} = w.$$

It follows that the limits $\lim_{j\to\infty} u^{(j)} = u^*$ and $\lim_{j\to\infty} w^{(j)} = w^*$ exist. Furthermore, by means of the continuity of the function $f(k,\cdot)$, we easily see that they are solutions of (1.1),(1.2). ∎

We remark that the solutions u^* and w^* of (1.1),(1.2) found in the above theorem are minimal and maximal in the sense that if v is any solution of (1.1),(1.2) which satisfies $u \leq v \leq w$, then $u^* \leq v \leq w^*$. Indeed, note that $\Gamma v = v$. Thus $u^{(1)} = \Gamma u^{(0)} \leq \Gamma v = v$, and by induction, $u^{(j)} \leq v$ for $j \geq 1$. This shows that $u^* \leq v$. Similarly, $v \leq w^*$.

4. BOUNDARY PROBLEMS OF TWO INDEPENDENT VARIABLES

Results analogous to those stated in the previous sections can be derived for boundary value problems involving nonlinear partial difference equations. To save space, the terminologies in [5] will be employed. Let S be a net in the lattice plane

$$\{z = (i,j) \mid i,j \text{ are integers}\},$$

and let ∂S be its exterior boundary. Consider the partial difference boundary value problem

$$Dv(z) + f(z, v(z)) = 0, \qquad z \in S, \tag{4.1}$$

$$v(z) = 0, \qquad z \in \partial S, \tag{4.2}$$

where $f(z,v)$ is a real function defined for $z \in S$ and $v \in R$, and D is the discrete Laplacian defined by

$$\begin{aligned} Dv(i,j) &= \Delta_1^2 v(i-1,j) + \Delta_2^2 v(i,j-1) \\ &= v(i+1,j) + v(i-1,j) + v(i,j+1) + v(i,j-1) - 4v(i,j). \end{aligned}$$

As in Section 2, a real double sequence $u = \{u(z)\}_{z\in S}$ is said to be less than or equal to another double sequence $w = \{w(z)\}_{z\in S}$, if $u(z) \leq w(z)$ for $z \in S$. A solution of (4.1),(4.2) is a real double sequence $v = \{v(z)\}_{z\in S\cup\partial S}$ such that (4.1),(4.2) is satisfied. A real double sequence $w = \{w(z)\}_{z\in S\cup\partial S}$ is called an upper solution of (4.1),(4.2) if

$$Dw(z) + f(z, w(z)) \leq 0, \qquad z \in S,$$

and

$$w(z) \geq 0, \qquad z \in \partial S.$$

Similarly, a real double sequence $u = \{u(z)\}_{z\in S\cup\partial S}$ is called a lower solution of (4.1),(4.2) if

$$Du(z) + f(z, u(z)) \geq 0, \qquad z \in S,$$

and

$$u(z) \leq 0, \qquad z \in \partial S.$$

LEMMA 4.1. *Let $v = \{v(z)\}_{z \in S \cup \partial S}$ be a real double sequence which satisfies the recurrence relation*

$$Dv(z) + F(z, v(z)) \geq 0, \qquad z \in S,$$

where for each $z \in S$, $F(z, v) \leq 0$, whenever $v \geq 0$. If

$$M = \max\{v(z) \mid z \in S \cup \partial S\} \geq 0,$$

then $v(z) < M$ for $z \in S$, unless $v(z) = M$ for $z \in S \cup \partial S$.

PROOF. Suppose to the contrary that $v(z^*) = M$ for some $z^* \in S$. We assert that for any other point z in $S \cup \partial S$, $v(z^*) = v(z)$. Indeed, let $z^{(1)} = z^*$, $z^{(2)}, \dots, z^{(n)} = z$ be a path of points contained in $S \cup \partial S$. Since $v(z^*) = M$, the values of the neighboring points of v are not greater than M. Consequently, $Dv(z^{(1)}) \leq 0$, which together with $Dv(z^{(1)}) \geq -F(z^{(1)}, v(z^{(1)})) \geq 0$, imply $Dv(z^{(1)}) = 0$. But then the values of v at the four neighbors of $z^{(1)}$, in particular, $z^{(2)}$, are equal to M. If $z^{(2)} \neq z$, we may repeat the above argument repeatedly to conclude our proof. ∎

As an immediate application, let us consider the following linear system

$$\begin{aligned} Dv(z) + p(z)v(z) &= 0, \qquad z \in S, \\ v(z) &= 0, \qquad z \in \partial S, \end{aligned}$$

where $p(z) \leq 0$ for $z \in S$. We assert that this system can have the trivial solution only. Indeed, if $v = \{v(z)\}_{z \in S \cup \partial S}$ is a nontrivial solution, we may assume that it's maximum

$$v(z^*) = \max_{z \in S \cup \partial S} v(z) > 0.$$

But in view of Lemma 4.1, $z^* \in \partial S$ so that $v(z^*) = 0$. This contradiction completes the proof of our assertion.

The following is now clear.

LEMMA 4.2. *Let S be a net, and $p(z) \leq 0$ for $z \in S$. Then the linear system*

$$\begin{aligned} Dv(z) + p(z)v(z) &= q(z), \qquad z \in S, \\ v(z) &= h(z), \qquad z \in \partial S, \end{aligned}$$

has a unique solution.

By means of these two lemmas, results similar to Lemma 2.2, Theorem 2.1, and Theorem 3.1 can easily be formulated and proved. Furthermore, results similar to Lemma 2.4, Theorem 2.2, Theorem 2.3, Lemma 3.1, and Theorem 3.2, can also be formulated easily and proved. In particular, we have the following existence criteria.

THEOREM 4.1. *Suppose $f(z, \cdot)$ is continuous. Then for any lower solution $u = \{u(z)\}$ and upper solution $w = \{w(z)\}$ of (4.1),(4.2) satisfying $u \leq w$, there exists a solution $v = \{v(z)\}$ of (4.1),(4.2) which satisfies $u \leq v \leq w$.*

THEOREM 4.2. *Suppose $f(z, \cdot)$ is continuous. Suppose further that there exist a lower solution $u = \{u(z)\}$, an upper solution $w = \{w(z)\}$, and a positive function $p = \{p(z)\}_{z \in S}$ such that $u \leq w$, and for each $z \in S$, $f(z, \rho) - f(z, \tau) \geq -p(z)(\rho - \tau)$ holds whenever $u(z) \leq \tau \leq \rho \leq w(z)$. Then the boundary problem (4.1),(4.2) has a solution u^* and a solution w^*, such that $u^* \leq v \leq w^*$ for any solution v of (4.1),(4.2) satisfying $u \leq v \leq w$.*

REFERENCES

1. S.N. Chow and W. Shen, Dynamics in a discrete Nagumo equation-spatial chaos, (preprint).
2. B. Zinner, Existence of travelling wavefront solutions for the discrete Nagumo equation, *J. Diff. Eq.* **96**, 1–27, (1992).
3. B. Zinner, Stability of travelling wavefronts for the discrete Nagumo equation, *SIAM J. Math. Anal.* **22**, 1016–1020, (1991).
4. J.W. Cahn, S.N. Chow and E.S. Van Vleck, Spatially discrete nonlinear diffusion equations, *Rocky Mountain J. Math.* **25**, 87–118, (1995).
5. S.S. Cheng and S.S. Lin, Existence and uniqueness theorems for nonlinear difference boundary value problems, *Utilitas Math.* **39**, 167–186, (1991).
6. R.P. Agarwal, On boundary value problems for second order discrete systems, *Applicable Analysis* **20**, 1–17, (1985).
7. A. Lasota, A discrete boundary value problem, *Annales Polonici Mathecatici* **20**, 183–190, (1968).
8. R.A. Usmani and R.P. Agarwal, On the numerical solution of two point boundary value problems, *Appl. Math. Comp.* **25**, 247–264, (1988).
9. G.S. Ladde, V. Lakshmikantham and A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman Advanced Publishing Program, (1985).
10. Y.B. Chen and W. Zhuang, On monotone iterative method for periodic boundary problems of nonlinear integro-differential equations, *Nonlinear Analysis* **22**, 295–303, (1994).
11. Q. Sheng and R.P. Agarwal, Monotone methods for higher order partial difference equations, *Computers Math. Applic.* **28** (1-3), 291–307, (1994).