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Advances in Mathematics 230 (2012) 372–389

ADVANCES IN Mathematics

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Convex normality of rational polytopes with long edges ☆

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Received 26 December 2009; accepted 12 December 2011

Available online 16 December 2011

Communicated by Ezra Miller

Abstract

We introduce the property of *convex normality* of rational polytopes and give a dimensionally uniform lower bound for the edge lattice lengths, guaranteeing the property. As an application, we show that if every edge of a lattice d-polytope P has lattice length $\geqslant 4d(d+1)$ then P is normal. This answers in the positive a question raised in 2007. If P is a lattice simplex whose edges have lattice lengths $\geqslant d(d+1)$ then P is even covered by lattice parallelepipeds. For the approach developed here, it is necessary to involve rational polytopes even for the application to lattice polytopes.
Published by Elsevier Inc.

MSC: primary 52B20; secondary 05E40, 11H06, 13F99, 14M25

Keywords: Convex polytope; Lattice polytope; Rational polytope; Normal polytope; Integrally closed polytope

1. Integrally closed polytopes

All our polytopes are assumed to be convex. For a polytope P the set of its vertices will be denoted by $\mathrm{vert}(P)$.

A polytope $P \subset \mathbb{R}^d$ is *lattice* if $\mathrm{vert}(P) \subset \mathbb{Z}^d$, and P is *rational* if $\mathrm{vert}(P) \subset \mathbb{Q}^d$.

Let $P \subset \mathbb{R}^d$ be a lattice polytope and denote by L the subgroup of $\mathbb{Z}^d$, affinely generated by the lattice points in P; i.e.,

☆ Partially supported by NSF grant DMS-1000641.
E-mail address: soso@sfsu.edu.

0001-8708/$ – see front matter Published by Elsevier Inc.
doi:10.1016/j.aim.2011.12.003

$$L = \sum_{x,y \in P \cap \mathbb{Z}^d} \mathbb{Z}(x-y) \subset \mathbb{Z}^d.$$

Definition 1.1. (See [9, Def. 2.59].) Let $P \subset \mathbb{R}^d$ be a lattice polytope.

(a) P is *integrally closed* if the following condition is satisfied:

$$c \in \mathbb{N},\ z \in cP \cap \mathbb{Z}^d \quad \Longrightarrow \quad \exists x_1, \dots, x_c \in P \cap \mathbb{Z}^d, \quad x_1 + \dots + x_c = z.$$

(b) P is *normal* if for some (equivalently, every) point $t \in P \cap \mathbb{Z}^d$ the following condition is satisfied:

$$c \in \mathbb{N},\ z \in cP \cap (ct + L) \quad \Longrightarrow \quad \exists x_1, \dots, x_c \in P \cap \mathbb{Z}^d, \quad x_1 + \dots + x_c = z.$$

(Observe, $P \cap (t + L) = P \cap \mathbb{Z}^d$.)

The normality property is invariant under *affine isomorphisms of lattice polytopes*, and the property of being integrally closed is invariant under affine changes of coordinates, leaving the lattice structure $\mathbb{Z}^d \subset \mathbb{R}^d$ invariant.

A lattice polytope $P \subset \mathbb{R}^d$ is integrally closed if and only if it is normal and L is a direct summand of $\mathbb{Z}^d$. Obvious examples of normal but not integrally closed polytopes are the s.c. *empty* lattice simplices of large volume. No classification of such simplices is known in dimensions $\geqslant 4$, the main difficulty being the lack of satisfactory a characterization of their lattice widths; see [15,23]. For recent advances in the field see [3,4].

A normal polytope $P \subset \mathbb{R}^d$ can be made into a full-dimensional integrally closed polytope by changing the lattice of reference $\mathbb{Z}^d$ to L, the ambient Euclidean space $\mathbb{R}^d$ to the subspace $\mathbb{R}L$, and shifting P so that $0 \in P$. In particular, normal and integrally closed polytopes refer to the same isomorphism classes of lattice polytopes. In the literature, however, the difference between 'normal' and 'integrally closed' is sometimes blurred.

Normal/integrally closed polytopes enjoy popularity in algebraic combinatorics and they have been showcased on recent workshops [1,2]. These polytopes represent the homogeneous case of the Hilbert bases of finite positive rational cones and the connection to algebraic geometry is that they define projectively normal embeddings of toric varieties. There are many challenges of number theoretic, ring theoretic, homological, and K-theoretic nature, concerning the associated objects: Ehrhart series', rational cones, toric rings, and toric varieties; see [9].

If a lattice polytope is covered by (in particular, subdivided into) integrally closed polytopes, then it is integrally closed as well. The simplest integrally closed polytopes one can think of are *unimodular simplices*, i.e., the lattice simplices $\Delta = \operatorname{conv}(x_1, \dots, x_k) \subset \mathbb{R}^d$, $\dim \Delta = k - 1$, with $x_1 - x_j, \dots, x_{j-1} - x_j, x_{j+1} - x_j, \dots, x_k - x_j$ a part of a basis of $\mathbb{Z}^d$ for some (equivalently, every) j.

Unimodular simplices are the smallest 'atoms' in the world of normal polytopes. But not all 3-dimensional integrally closed polytopes are triangulated into unimodular simplices [16]. (The first such example in dimension 4 was given in [11, Prop. 1.2.4].) Moreover, not all 5-dimensional integrally closed polytopes are covered by unimodular simplices [7] – contrary to what had been conjectured before [22]. Further 'negative' results, such as [6] and [10], the latter disproving an additive version of the unimodular cover property that was conjectured in [12], contributed to the current thinking in the area that there is no succinct geometric characterization of the normality property. One could even conjecture that in higher dimensions the situation gets as bad as it can; see the discussion at the end of [2, p. 2313].

'Positive' results in the field mostly concern special classes of lattice polytopes that are normal, or have unimodular triangulations or unimodular covers. Knudsen–Mumford's classical theorem ([9, Sect. 3B], [17, Chap. III]) says that every lattice polytope P has a multiple cP for some $c \in \mathbb{N}$ that is triangulated into unimodular simplices. Whether the factor c can be chosen uniformly w.r.t. dimension seems to be a very hard problem. More recently, it was shown in [8] that there exists a dimensionally uniform exponential lower bound for unimodularly covered dilated polytopes. By improving one crucial step in [8], von Thaden was able to cut down the bound to a degree 6 polynomial function in the dimension [9, Sect. 3C], [24].

For polytopes, arising in a different context and admitting unimodular triangulations as certificate of normality, see [5,18,19,21]; for other techniques for establishing normality, along with its higher homological analogues, see [20].

The results above on dilated polytopes yield no new examples of normal polytopes, though. In fact, an easy argument ensures that for any lattice d-polytope P all multiples cP, $c \geqslant d-1$, are integrally closed [11, Prop. 1.3.3], [14]. However, that argument does not allow a modification that would apply to lattice polytopes with long edges of independent lengths.

The following conjecture was proposed in [2, p. 2310]:

Conjecture. *Simple lattice polytopes with 'long' edges are normal, where 'long' means some invariant, uniform in the dimension.*

More precisely, let P be a simple lattice polytope. Let k be the maximum over the heights of Hilbert basis elements of tangent cones to vertices of P. Then, if any edge of P has length $\geqslant k$, the polytope P should be normal.

Here: (i) the length is measured in the lattice sense, (ii) 'tangent cones' is the same as corner cones, and (iii) the heights of Hilbert basis elements of corner cones are normalized w.r.t. the extremal generators of the cones (leading, in particular, to non-integral rational heights).

The second part of the conjecture is a far reaching extension of the following well-known problem, a.k.a. *Oda's question*, that has attracted much interest recently: are all *smooth polytopes* normal? A lattice polytope $P \subset \mathbb{R}^d$ is called smooth if the primitive (i.e., with coprime components) edge vectors at every vertex of P define a part of a basis of $\mathbb{Z}^d$. Smooth polytopes correspond to the projective embeddings of smooth projective toric varieties and they are simple polytopes with $k = 1$. Oda's question still remains wide open. The fact that so far no smooth polytope just without a unimodular triangulation has been found illustrates how limited our understanding in the area is. The second part of the conjecture yields also a dimensionally uniform bound, mentioned in the first part. In fact, it is known that, for every $d \geqslant 2$, the normalized heights of Hilbert basis elements of a simplicial rational d-cone are at most $d-1$; see, for instance, [9, Prop. 2.43(d)].

Another motivation for the conjecture above is the following question in toric geometry, discussed in [2, p. 2310]: are all line bundles over a projective toric variety, deep enough inside the *nef cone*, projectively normal? If so, how deep is 'deep enough'? See [13, Chap. 6] for generalities on the nef cones of toric varieties. It is not difficult to show that if an ample line bundle $\mathcal{L}$ over a projective toric variety X is on lattice depth l inside the nef cone $\mathrm{Nef}(X)$ w.r.t. every facet of the cone, then the edges of the lattice polytope P of $\mathcal{L}$ are all of lattice lengths $\geqslant l$.

In this paper we introduce the notion of *k-convex-normality*, $k \in \mathbb{Q}_{\geqslant 2}$, which is a 'convex-rational' version of Definition 1.1. Next is the main result of the paper:

Theorem 1.2. *Let P be a rational (not necessarily simple) polytope of dimension d whose every edge has lattice length $\geqslant d(d+1)k$. Then P is k-convex-normal.*

Although k-convex-normality concerns the dilated polytopes cP with $c \in [2,k]_{\mathbb{Q}}$, when applied to lattice polytopes this is enough to cover the factors $c \in \mathbb{N}$ in Definition 1.1, even with $k = 4$. As an application to lattice polytopes, we prove the first part of the conjecture above in the following strong form:

Theorem 1.3. *Let P be a (not necessarily simple) lattice polytope of dimension d.*

(a) *If every edge of P has lattice length $\geqslant 4d(d+1)$ then P is integrally closed.*
(b) *If P is a simplex and every edge of P has lattice length $\geqslant d(d+1)$ then P is covered by lattice parallelepipeds. In particular, P is integrally closed.*

In particular, if a line bundle $\mathcal{L}$ over a projective toric variety is on lattice depth $\geqslant 4d(d+1)$ w.r.t. every facet of the nef cone, then $\mathcal{L}$ is projectively normal.

For the reader's convenience we now give a brief outline of the proof of Theorem 1.2. Let P be a rational polytope with long edges. Assuming Theorem 1.2 is true in dimension $d-1$, we first show that the neighborhood of a certain width of the boundary surface of any multiple cP with $c \in [2,k]_{\mathbb{Q}}$ behaves as if P were convex-normal. Then it is shown that the complement of this neighborhood is covered by certain parallel translates of lattice parallelepipeds inside cP. This does not require the inductive assumption and is achieved by propagating 'corner parallelepipedal covers' deep inside cP. Actually, the situation is more subtle, the reason being that the width of the mentioned boundary of cP depends on P and does not grow along with c. As a result, one needs that the inductively covered boundary neighborhood and the region, covered by the parallelepipeds, overlap in certain nontrivial way.

1.1. Notation and terminology

The affine and convex hulls of a subset $X \subset \mathbb{R}^d$ will be denoted, respectively, by $\operatorname{aff}(X)$ and $\operatorname{conv}(X)$.

The relative interior $\operatorname{int}(P)$ of a polytope $P \subset \mathbb{R}^d$ is by definition the absolute interior of P in $\operatorname{aff}(P)$.

Let $H \subset \mathbb{R}^d$ be an affine hyperplane. The one of the two half-spaces, bounded by H and clear from the context, will be denoted by H^+.

For a polytope $P \subset \mathbb{R}^d$ the set of its facets will be denoted by $\mathbb{F}(P)$. (Recall, $\operatorname{vert}(P)$ is the set of vertices of P.) If $\dim P = d$ and $F \in \mathbb{F}(P)$, the half-space H_F^+ and hyperplane H_F are defined from the unique irredundant representation of the form ([9, Thm. 1.6], [25, Thm. 2.15(7)])

$$P = \bigcap_{\mathbb{F}(P)} H_F^+, \quad H_F = \operatorname{aff}(F).$$

A polytope is *simple* if its edge directions at every vertex are linearly independent.

A *parallelepiped* is by definition the Minkowski sum of segments of linearly independent directions.

Cones $C \subset \mathbb{R}^d$ are always assumed to be finite and positive, i.e., they are intersections of finitely many homogeneous half-spaces and contain no nontrivial subspaces. A cone is *simplicial* if its edge directions are linearly independent.

Let $C \subset \mathbb{R}^d$ be a rational cone, i.e., C is the intersection of half-spaces with rational boundary hyperplanes. Then the primitive lattice points on the one-dimensional faces (*rays*) of C are called the *extremal generators* of C.

A *d-polytope* or *d-cone* is the same as a d-dimensional polytope or, respectively, d-dimensional cone.

$\mathbb{R}_+$, $\mathbb{Q}_+$, and $\mathbb{Z}_+$ refer to the corresponding sets of nonnegative numbers.

For an interval $I \subset \mathbb{R}$ and number $\lambda \in \mathbb{R}$ we let

$$I_{\mathbb{Q}} = I \cap \mathbb{Q}, \qquad I_{\mathbb{N}} = I \cap \mathbb{N}, \qquad \mathbb{Q}_{\geqslant\lambda} = [\lambda, \infty)_{\mathbb{Q}}, \qquad \mathbb{Q}_{>\lambda} = (\lambda, \infty)_{\mathbb{Q}},$$
$$\mathbb{N}_{\geqslant\lambda} = [\lambda, \infty)_{\mathbb{N}}, \qquad \mathbb{N}_{>\lambda} = (\lambda, \infty)_{\mathbb{N}}.$$

For a subset $X \subset \mathbb{R}^d$ we put $\mathbb{R}_+X = \{\lambda x \mid \lambda \in \mathbb{R}_+,\ x \in X\}$.

The *lattice length* of a rational segment $[x, y] \subset \mathbb{R}^d$, $x, y \in \mathbb{Q}^d$, is the ratio of its Euclidean length and that of the primitive integer vector in the direction of $y - x$.

For a rational polytope P, by $\mathsf{E}(P)$ we denote the minimum of the lattice lengths of the edges of P.

The Euclidean distance between a point $x \in \mathbb{R}^d$ and an affine hyperplane $H \subset \mathbb{R}^d$ is denoted by $\|x, H\|$.

2. Convex normality

For a polytope $P \subset \mathbb{R}^d$ and a rational number $c \geqslant 1$ denote

$$\mathbb{U}_{\mathrm{vert}}(P, c) = \bigcup_{\substack{v \in \mathrm{vert}(P) \\ x \in (c-1)P \cap ((c-1)v + \mathbb{Z}^d)}} x + P.$$

Obviously, $\mathbb{U}_{\mathrm{vert}}(P, c) \subset cP$.

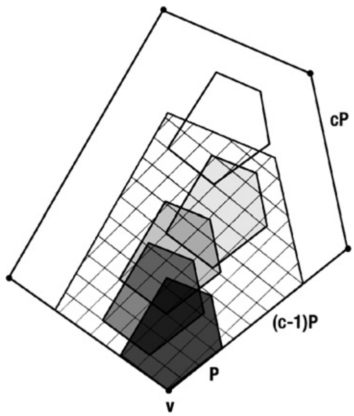
Crucial in our approach to the normality property is the following notion that mixes just the optimal amounts of discreteness and continuity:

Definition 2.1. Assume $d \in \mathbb{N}$, $k \in \mathbb{Q}_{\geqslant 2}$, and P is a rational d-polytope. P is said to be *k-convex-normal* if the following equality is satisfied for all $c \in [2, k]_{\mathbb{Q}}$:

$$\mathbb{U}_{\mathrm{vert}}(P, c) = cP. \qquad (\mathrm{CN}(d, k))$$

Here is a convenient equivalent reformulation. For $c \in \mathbb{Q}_{\geqslant 2}$ and $v \in \mathrm{vert}(P)$ denote by $Q(v)$ the parallel translate of $(c-1)P$ that moves $(c-1)v$ to cv. Put

$$R(v,c)=\bigcup_{x\in Q(v)\cap(cv+\mathbb{Z}^d)}(x-v+P):$$



Then P is convex-normal iff for all $c\in[2,k]_{\mathbb{Q}}$ we have $cP=\bigcup_{\mathrm{vert}(P)}R(v,c)$.

Informally, convex normality is a measure of density of the point configuration $P\cap\mathbb{Z}^d$ w.r.t. P. For instance, the unimodular simplices of dimension $\geqslant 2$ are *not* convex-normal, but their high multiples are convex-normal. More importantly for our goals, all lattice parallelepipeds are convex-normal; see Lemma 2.2(a) below.

It is easily observed that a unimodular integral change of coordinates respects the property $\mathrm{CN}(d,k)$, and the same is true for rational parallel translations. Also, one can show (although we do not need it) that $cP=\mathbb{U}_{\mathrm{vert}}(P,c)$ for any rational d-polytope P and any real number $c\in[1,\frac{d+1}{d}]$.

Lemma 2.2. (a) *Let $\square$ be a rational parallelepiped. If $\mathsf{E}(\square)\geqslant 1$ then $c\square=\mathbb{U}_{\mathrm{vert}}(\square,c)$ for every $c\in\mathbb{Q}_{\geqslant 1}$. If $\mathsf{E}(\square)<1$ then $\mathbb{U}_{\mathrm{vert}}(\square,c)\neq c\square$ for all $c\in\mathbb{Q}_{>2}$, sufficiently close to* 2.

(b) *For every natural number d, any $(d-1)$-convex-normal lattice d-polytope is integrally closed.*

Proof. (a) Assume $\mathsf{E}(\square)\geqslant 1$. First consider the case $\dim\square=1$. We can assume $\square=[0,l]$ for some $l\in\mathbb{Q}_{\geqslant 1}$. If $c<2$ then

$$[0,cl]=[0,l]\cup\big[(c-1)l,cl\big]\subset\mathbb{U}_{\mathrm{vert}}\big([0,l],c\big).$$

If $c\geqslant 2$ then $[0,(c-1)l]\cup[l,cl]=[0,cl]$ and, simultaneously, the inequality $l\geqslant 1$ implies the mutually symmetric inclusions

$$\big[0,(c-1)l\big]\subset\bigcup_{x\in[0,(c-1)l]\cap\mathbb{Z}^d}x+[0,l],$$

$$[l,cl]\subset\bigcup_{x\in[0,(c-1)l]\cap((c-1)l+\mathbb{Z}^d)}x+[0,l].$$

Consider the case $\dim\square=d>1$. We can assume $\square\subset\mathbb{R}^d$. Without loss of generality we can further assume that $\square=\prod_{i=1}^d[0,l_i]$ for some $l_i\in\mathbb{Q}_{>1}$. In fact, one first applies a parallel translation that moves a vertex of $\square$ to 0, then applies an appropriate rational change of coordinates that transforms the primitive lattice edge vectors of $\square$, emerging from 0, into the standard basic vectors of $\mathbb{R}^d$, and, finally, changes the lattice of reference to the integer lattice w.r.t. to the new coordinates. The new lattice is a parallel translate of a subgroup of the old copy of $\mathbb{Z}^d$. In particular, $\mathbb{U}_{\mathrm{vert}}(\square, c)$, constructed w.r.t. the 'new $\mathbb{Z}^d$' is a subset of the one constructed w.r.t. to the 'old $\mathbb{Z}^d$'. Also, the condition $\mathsf{E}(\square) \geqslant 1$ remains valid w.r.t. to the new lattice of reference.

For $\delta \in \{0, 1\}^d$ denote by $v^\delta(\square)$ the vertex of $\square$ whose ith coordinate is 0 iff the ith component of δ is 0. Pick $z = (z_1, \dots, z_d) \in c\square$. By the one-dimensional case, for every component z_i we can fix $\delta_i \in \{0, 1\}$ so that

$$z_i \in \bigcup_{\xi \in [0,(c-1)l_i] \cap (\delta_i (c-1) l_i + \mathbb{Z}^d)} \xi + [0, l_i].$$

Then

$$z \in \bigcup_{x \in (c-1)\square \cap (v^\delta((c-1)\square) + \mathbb{Z}^d)} x + \square, \quad \delta = (\delta_1, \dots, \delta_d).$$

Now assume $\mathsf{E}(\square) < 1$. Without loss of generality we can assume $\dim \square = 1$ and, moreover, $\square = [0, l]$. Pick an arbitrary $\varepsilon \in \mathbb{Q}_{>0}$ with $\varepsilon < l^{-1} - 1$ and let $c = 2 + \varepsilon$. Then $[0, (c-1)l] \cap \mathbb{Z} = \{0\}$ and $[0, (c-1)l] \cap ((c-1)l + \mathbb{Z}) = \{(c-1)l\}$, and, consequently,

$$\frac{cl}{2} \in [0, cl] \setminus \mathbb{U}_{\mathrm{vert}}\big([0, l], c\big).$$

(b) Notice that lattice segments ($d = 1$) and lattice polygons ($d = 2$) are vacuously $(d-1)$-convex-normal. So the statement includes the known fact that all lattice segments and lattice polygons are integrally closed; see [9, Cor. 2.54].

Let P be a lattice d-polytope. Then

$$v + \mathbb{Z}^d = cv + \mathbb{Z}^d = \mathbb{Z}^d \quad \text{for all } v \in \mathrm{vert}(P) \text{ and } c \in \mathbb{N}. \tag{1}$$

Assume P is a lattice d-polytope, satisfying $\mathrm{CN}(d, d-1)$, and let $c \in [2, d-1]_{\mathbb{N}}$. Then, in view of (1), for every $z \in cP \cap \mathbb{Z}^d$ there exist $x \in (c-1)P \cap \mathbb{Z}^d$ and $x_c \in x + P$ with $z = x + x_c$. Then, necessarily, $x_c \in P \cap \mathbb{Z}^d$, and the descending induction from c to 1 implies $z = x_1 + \dots + x_c$ with $x_1, \dots, x_c \in P \cap \mathbb{Z}^d$.

Now assume $c \in \mathbb{N}_{\geqslant d}$ and $z \in cP \cap \mathbb{Z}^d$. Then, by [9, Thm. 2.52] (an essentially equivalent result, but stated for the *normalization of the polytopal monoid* of P instead of the integral closure in $\mathbb{Z}^d$, is [11, Cor. 1.3.4]), there exist a natural number $1 \leqslant c_0 \leqslant d-1$, a lattice point $x_0 \in c_0 P \cap \mathbb{Z}^d$, and a family of lattice points $x_i \in P \cap \mathbb{Z}^d$, $i = 1, \dots, c - c_0$, such that $z = x_0 + x_1 + \dots + x_{c-c_0}$. So the general case reduces to the case $c \leqslant d - 1$. $\square$

3. CN in dimension $d-1 \Rightarrow$ boundary CN in dimension d

For a polytope P and a vertex $v \in \mathrm{vert}(P)$ we let $\mathbb{F}(P)^v$ denote the facets F of P that are visible from v, i.e., $v \notin F$.

For two polytopes $Q, P \subset \mathbb{R}^d$ with $\dim Q = d-1$ and $\dim P = d$ the Euclidean width of P w.r.t. $\mathrm{aff}(Q)$ will be denoted by $\mathrm{width}_Q(P)$.

For a d-polytope $P \subset \mathbb{R}^d$, a facet $F \subset P$, and a real number $\varepsilon > 0$ we define the *ε-layer along F inside P* to be the polytope

$$F_P(\varepsilon) = \bigl\{x \in P\colon \ \|x, H_F\| \leqslant \varepsilon\bigr\}.$$

If $\varepsilon < \mathrm{width}_F(P)$ then $F_P(\varepsilon)$ has a facet, different from F and parallel to F. It will be denoted by $F_P(\varepsilon)^+$.

Definition 3.1. Assume $k \in \mathbb{Q}_{\geqslant 2}$ and $P \subset \mathbb{R}^d$ is a rational d-polytope. P is said to be *k-boundary-convex-normal* if the following condition is satisfied for every $c \in [2,k]_{\mathbb{Q}}$ and every $F \in \mathbb{F}(P)$:

$$\bigl((cF)_{cP}\bigr)(\varepsilon_F) \subset \mathbb{U}_{\mathrm{vert}}(P,c), \quad \varepsilon_F = \frac{\mathrm{width}_F(P)}{d+1}. \qquad (\mathrm{BCN}(d,k))$$

Lemma 3.2. *Let $d \in \mathbb{N}_{\geqslant 2}$, $k \in \mathbb{Q}_{\geqslant 2}$, and $\lambda \in \mathbb{Q}_{>0}$. Assume every rational $(d-1)$-polytope Q with $\mathsf{E}(Q) \geqslant \frac{d}{d+1}\lambda$ satisfies $\mathrm{CN}(d-1, k+\frac{k-1}{d})$. Let P be a rational d-polytope with $\mathsf{E}(P) \geqslant \lambda$, $w \in \mathrm{vert}(P)$, and $F \in \mathbb{F}(P)^w$. Then for the rational d-pyramid $\Delta = \mathrm{conv}(w,F)$ and every $c \in [2,k]_{\mathbb{Q}}$ we have*

$$\bigl((cF)_{c\Delta}\bigr)(\varepsilon) \subset \mathbb{U}_{\mathrm{vert}}(P,c), \quad \varepsilon = \frac{\|w, H_F\|}{d+1}.$$

Proof. We can assume $P \subset \mathbb{R}^d$. Denote:

$$\Pi = \bigcup_{\substack{v \in \mathrm{vert}(F) \\ x \in (c-1)F \cap ((c-1)v + \mathbb{Z}^d)}} x + F_\Delta(\varepsilon).$$

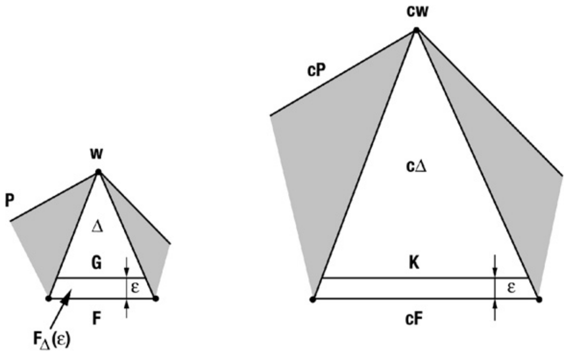
Since $\Pi \subset \mathbb{U}_{\mathrm{vert}}(P,c)$, it is enough to show

$$\bigl((cF)_{c\Delta}\bigr)(\varepsilon) \subset \Pi. \qquad (2)$$

Let $G = F_\Delta(\varepsilon)^+ \in \mathbb{F}(F_\Delta(\varepsilon))$. Then G is a homothetic image of F with factor $d/(d+1)$. In particular, G is a rational $(d-1)$-polytope whose every edge has lattice length $\geqslant \frac{d}{d+1}\lambda$. By the assumption, G satisfies $\mathrm{CN}(d-1, k+\frac{k-1}{d})$.

The rational polytope $K = ((cF)_{c\Delta})(\varepsilon)^+$ is a homothetic image of F with factor $\frac{cd+c-1}{d+1}$. So K is a homothetic image of G with factor

$$c_1 = \frac{cd+c-1}{d+1} \cdot \frac{d+1}{d} = c + \frac{c-1}{d} \in \left[2 + \frac{1}{d}, k + \frac{k-1}{d}\right]_{\mathbb{Q}}:$$



The polytope $(c-1)F$ is a rational homothetic image of G with factor $\frac{(d+1)(c-1)}{d}$. In particular, $(c_1-1)G=(c-1)F$ and, by the inductive assumption on rational $(d-1)$ polytopes with lattice edge lengths $\geqslant \frac{d}{d+1}\lambda$, we have

$$\bigcup_{\substack{v\in \operatorname{vert}(F)\\ x\in (c-1)F\cap((c-1)v+\mathbb{Z}^d)}} x+G=K,$$

or, equivalently, $K\subset \Pi$. To put in other words, the lid of the truncated pyramid $((cF)_{c\Delta})(\varepsilon)$ is covered by the relevant parallel translates of the lid of the smaller truncated pyramid $F_\Delta(\varepsilon)$.

Pick a point $z\in ((cF)_{c\Delta})(\varepsilon)$. The ray $cw+\mathbb{R}_+(z-cw)$ intersects K at some point z_K. Let $z_K\in x+G$ for some x as in the index set in the definition of Π. Then

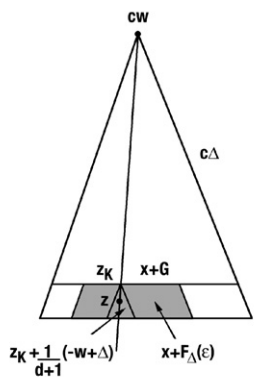
$$\big(z_K+\mathbb{R}_+(-w+\Delta)\big)\cap\big((cF)_{c\Delta}\big)(\varepsilon)=z_K+\frac{1}{d+1}(-w+\Delta)$$

and

$$F_\Delta(\varepsilon)=G+\frac{1}{d+1}(-w+\Delta).$$

Therefore,

$$z\in z_K+\frac{1}{d+1}(-w+\Delta)\subset x+G+\frac{1}{d+1}(-w+\Delta)=x+F_\Delta(\varepsilon)\subset \Pi:$$



□

Remark 3.3. (a) In the proof of Lemma 3.2 there are two places that make it necessary to involve rational polytopes in our induction on dimension: the polytope G, to which the assumption on $(d-1)$-polytopes is applied, is usually not lattice even if P is, and the number c_1 is usually not an integer.

(b) If one defined the convex normality by the 'dual' equalities:

$$cP = \bigcup_{\substack{v\in \operatorname{vert}(P)\\ x\in P\cap(v+\mathbb{Z}^d)}} x+(c-1)P \quad \text{for all } c\in[2,k]_{\mathbb{Q}},$$

then the lower bound for the analogue of c_1 in the proof of Lemma 3.2 would have been $2-\frac{1}{d+1}$, blocking the possibility for induction on d.

Lemma 3.4. *Let $d\in\mathbb{N}_{\geqslant 2}$, $k\in\mathbb{Q}_{\geqslant 2}$, and $\lambda\in\mathbb{Q}_{>0}$. If every rational $(d-1)$-polytope Q with $\mathsf{E}(Q)\geqslant\frac{d}{d+1}\lambda$ satisfies $\mathrm{CN}(d-1,k+\frac{k-1}{d})$ then every rational d-polytope P with $\mathsf{E}(P)\geqslant\lambda$ satisfies $\mathrm{BCN}(d,k)$.*

Proof. Let P be a rational d-polytope with edge lengths $\geqslant\lambda$, $F\in\mathbb{F}(P)$, and

$$\varepsilon_F=\frac{\operatorname{width}_F(P)}{d+1}.$$

Fix a vertex $w\in\operatorname{vert}(P)\setminus F$ with $\|w,H_F\|=\operatorname{width}_F(P)$. Such exists because $\operatorname{width}_F(P)=\max_{\operatorname{vert}(P)}(\|v,H_F\|)$.

For every facet $G\in\mathbb{F}(P)^w$ denote

$$\Delta(G)=\operatorname{conv}(w,G)\quad\text{and}\quad \varepsilon_{w,G}=\frac{\|w,H_G\|}{d+1}.$$

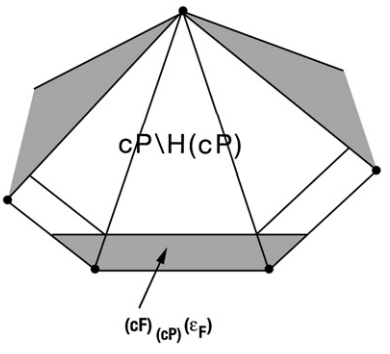
By Lemma 3.2, for every $c\in[2,k]_{\mathbb{Q}}$ we have the inclusion

$$\bigcup_{G\in\mathbb{F}(P)^w}(cG)_{c\Delta(G)}(\varepsilon_{w,G})\subset\mathbb{U}_{\operatorname{vert}}(P,c).$$

But for every $c\in[2,k]_{\mathbb{Q}}$ we also have

$$(cF)_{cP}(\varepsilon_{w,F})=(cF)_{cP}(\varepsilon_F)\subset cP\setminus\mathsf{H}(cP)=\bigcup_{G\in\mathbb{F}(P)^w}(cG)_{c\Delta(G)}(\varepsilon_{w,G}),$$

where $\mathsf{H}(cP)$ denotes the homothetic image of cP, centered at cw and with factor $\frac{cd+c-1}{cd+c}$. The inclusion in the middle, essentially, amounts to the convexity of cP:



□

4. Deep parallelepipedal covers from vertices

Fix a rational d-polytope $P \subset \mathbb{R}^d$, a rational number $l \geqslant 1$, and a vertex $v \in P$.

For a system of positive rational numbers $\bar{\varepsilon} = (\varepsilon_F)_{F\in\mathbb{F}(P)^v}$ we denote

$$P - \bar{\varepsilon} \cdot \mathbb{F}(P)^v = \overline{P \setminus \bigcup_{\mathbb{F}(P)^v} F_P(\varepsilon_F)},$$

the 'bar' on the right-hand side referring to the closure in the Euclidean topology.

Pick a simplicial d-cone of the form $C = \mathbb{R}_+(v_1 - v) + \cdots + \mathbb{R}_+(v_d - v) \subset \mathbb{R}^d$ with $v_1, \ldots, v_d \in \operatorname{vert}(P)$.

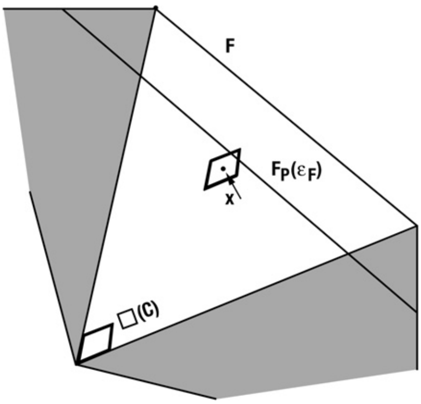
Let x_i be the primitive integer vector in the direction of $v_i - v$ and $\square(C) \subset C$ be the parallelepiped, spanned over 0 by the x_i, $i = 1, \ldots, d$. Denote by $P(\square(C))$ the union of the integral parallel translates of $\square(C)$ of type $v + \sum_{i=1}^d a_i x_i + \square(C)$, $a_1, \ldots, a_d \in \mathbb{Z}_+$, which fall inside P.

Lemma 4.1. *If* $\mathsf{E}(P) \geqslant ld(d+1)$ *and* $\varepsilon_F = \frac{\operatorname{width}_F(P)}{l(d+1)}$ *for every* $F \in \mathbb{F}(P)^v$ *then*

$$\bigl(P - \bar{\varepsilon} \cdot \mathbb{F}(P)^v\bigr) \cap C \subset P\bigl(\square(C)\bigr).$$

Proof. By shifting P by $-v$, we can assume $v = 0$.

Pick $x \in (P - \bar{\varepsilon} \cdot \mathbb{F}(P)^0) \cap C$. There exist $b_1, \ldots, b_d \in \mathbb{Z}_+$ with $x \in \sum_{i=1}^d b_i x_i + \square(C)$. We want to show $\sum_{i=1}^d b_i x_i + \square(C) \subset P$. By the choice of x, it is enough to show that $\operatorname{width}_F(\square(C)) \leqslant \varepsilon_F$ for every $F \in \mathbb{F}(P)^0$:



Consider the simplices $\Delta_1 = \operatorname{conv}(0, x_1, \dots, x_d)$ and $\Delta_2 = \operatorname{conv}(0, v_1, \dots, v_d)$. Then $\square(C) \subset d\Delta_1$ and $ld(d+1)\Delta_1 \subset \Delta_2$. Therefore, for every $F \in \mathbb{F}(P)^0$ we have

$$\operatorname{width}_F\bigl(\square(C)\bigr) \leqslant \operatorname{width}_F(d\Delta_1) \leqslant \frac{\operatorname{width}_F(\Delta_2)}{l(d+1)} \leqslant \frac{\operatorname{width}_F(P)}{l(d+1)} = \varepsilon_F. \qquad \square$$

Corollary 4.2. *In the situation of Lemma* 4.1*, the union of all lattice parallelepipeds inside P contains $P - \bar{\varepsilon} \cdot \mathbb{F}(P)^v$.*

Proof. This follows from Lemma 4.1 and the existence of a cover of the form $\mathbb{R}_+(P - v) = \bigcup_J C_j$, where the $C_j \subset \mathbb{R}^d$, $j \in J$, are *simplicial* d-cones, spanned by extremal generators of the cone $\mathbb{R}_+(P - v)$ – *the Carathéodory Theorem* for cones; see [9, Thm. 1.55]. One can even choose the cover to be a triangulation of $\mathbb{R}_+P$; see [9, Thm. 1.54], [25, Prop. 1.15(i)]. $\square$

5. Recursion rules for CN

Let $d \in \mathbb{N}$, $k \in \mathbb{Q}_{\geqslant 2}$, and P denote a general rational d-polytope. Define:

$$\mathfrak{cn}(d,k) = \inf\bigl(l \in \mathbb{Q} \bigm| \mathsf{E}(P) \geqslant l \;\Rightarrow\; P \text{ satisfies } \mathrm{CN}(d,k)\bigr),$$
$$\mathfrak{bcn}(d,k) = \inf\bigl(l \in \mathbb{Q} \bigm| \mathsf{E}(P) \geqslant l \;\Rightarrow\; P \text{ satisfies } \mathrm{BCN}(d,k)\bigr).$$

It is not a priori clear that these are finite numbers. What makes them finite and, in fact, the whole strategy work is the following recursion rules:

Lemma 5.1. *For $d \in \mathbb{N}_{\geqslant 2}$ and $k \in \mathbb{Q}_{\geqslant 2}$ we have*:

(a) $\mathfrak{cn}(1,k) \leqslant 1$.
(b) $\mathfrak{bcn}(d,k) \leqslant \frac{d+1}{d}\,\mathfrak{cn}(d-1, k + \frac{k-1}{d})$,
(c) $\mathfrak{cn}(d,k) \leqslant \max(kd(d+1), \mathfrak{bcn}(d,k))$.

Proof. (a) This follows from the first half of Lemma 2.2(a).

One can say more: $\mathfrak{cn}(1,2) = 0$ and, by the second half of Lemma 2.2(a), $\mathfrak{cn}(1,k) = 1$ for $k > 2$.

(b) This follows from Lemma 3.4.

(c) We will use the following Minkowski sum formula for two homothetic parallelepipeds $\square_1, \square_2 \subset \mathbb{R}^d$, with $\square_1$ at most as large as $\square_2$:

$$\square_1 + \square_2 = \bigcup_{v \in \operatorname{vert}(\square_1)} v + \square_2. \tag{3}$$

Let $P \subset \mathbb{R}^d$ be a rational d-polytope with $\mathsf{E}(P) > \max(kd(d+1), \mathfrak{bcn}(d,k))$. We want to show that P satisfies $\mathrm{CN}(d,k)$.

Pick $v \in \operatorname{vert}(P)$. Applying the parallel translation by $-v$, there is no loss of generality in assuming $v = 0$.

Fix a cover of the form $\mathbb{R}_+ P = \bigcup_J C_j$, where the C_j, $j \in J$, are simplicial d-cones, spanned by extremal rays of $\mathbb{R}_+ P$; see the proof of Corollary 4.2.

Assume $c \in [2,k]_{\mathbb{Q}}$. Because $c - 1 \geqslant 1$ we have $\mathsf{E}((c-1)P) \geqslant \mathsf{E}(P) > kd(d+1)$ and by (twofold application of) Lemma 4.1, for every $j \in J$ we have the inclusions:

$$\begin{gathered}\bigl(P - \bar{\varepsilon} \cdot \mathbb{F}(P)^0\bigr) \cap C_j \subset P\bigl(\square(C_j)\bigr), \\ \bigl((c-1)P - \bar{\varepsilon} \cdot \mathbb{F}\bigl((c-1)P\bigr)^0\bigr) \cap C_j \subset \bigl((c-1)P\bigr)\bigl(\square(C_j)\bigr),\end{gathered} \tag{4}$$

notation as in Lemma 4.1 with $\bar{\varepsilon} = (\varepsilon_F)_{\mathbb{F}(P)^0}$, $\varepsilon_F = \frac{\operatorname{width}_F(P)}{k(d+1)}$.

For $t \in \mathbb{Q}_{>0}$ denote $t\bar{\varepsilon} = (t\varepsilon_F)_{F \in \mathbb{F}(P)^0}$. Because $c - 1 \geqslant 1$, we have

$$(c-1)P - (c-1)\bar{\varepsilon} \cdot \mathbb{F}\bigl((c-1)P\bigr)^0 \subset (c-1)P - \bar{\varepsilon} \cdot \mathbb{F}\bigl((c-1)P\bigr)^0,$$

which, together with the second inclusion in (4), gives

$$\bigl((c-1)P - (c-1)\bar{\varepsilon} \cdot \mathbb{F}\bigl((c-1)P\bigr)^0\bigr) \cap C_j \subset \bigl((c-1)P\bigr)\bigl(\square(C_j)\bigr). \tag{5}$$

Pick $j \in J$. Denote by A, resp. by B, the set of parallelepipeds of type

$$\sum_{i=1}^d a_i x_{ji} + \square(C_j), \quad a_1, \dots, a_d \in \mathbb{Z}_+,\ x_{j1}, \dots, x_{jd} \text{ – the extremal generators of } C_j,$$

which fall inside $(c-1)P$, resp. inside P. Then we have

$$\begin{aligned}
&\bigl((c-1)P\bigr)\bigl(\square(C_j)\bigr) + P\bigl(\square(C_j)\bigr) \\
&\quad= \bigcup_{(\square_1,\square_2) \in A \times B} \square_1 + \square_2 = \bigcup_{\substack{\square_1 \in A \\ x \in \operatorname{vert}(\square_1)}} \bigcup_{\square_2 \in B} x + \square_2 \\
&\quad= \bigcup_{x \in ((c-1)P)(\square(C_j)) \cap \mathbb{Z}^d} \bigcup_{\square \in B} x + \square = \bigcup_{x \in ((c-1)P)(\square(C_j)) \cap \mathbb{Z}^d} x + P\bigl(\square(C_j)\bigr) \\
&\quad\subset \bigcup_{x \in ((c-1)P)(\square(C_j)) \cap \mathbb{Z}^d} x + (P \cap C_j) \subset \bigcup_{x \in ((c-1)P) \cap \mathbb{Z}^d} x + (P \cap C_j),
\end{aligned}$$

where the second and third equalities follow from (3). We record:

$$\big((c-1)P\big)\big(\square(C_j)\big)+P\big(\square(C_j)\big)\subset\bigcup_{x\in((c-1)P)\cap\mathbb{Z}^d}x+(P\cap C_j). \tag{6}$$

On the other hand, for every $j\in J$, the following equality holds true for reasons of homothety (w.r.t. to the origin):

$$\begin{aligned}&\big((c-1)P-(c-1)\bar{\varepsilon}\cdot\mathbb{F}\big((c-1)P\big)^0\big)\cap C_j+\big(P-\bar{\varepsilon}\cdot\mathbb{F}(P)^0\big)\cap C_j\\&\quad=\big(cP-c\bar{\varepsilon}\cdot\mathbb{F}(cP)^0\big)\cap C_j.\end{aligned} \tag{7}$$

Then, integrating over $j\in J$, the first inclusion in (4), (5), (6), and (7) imply

$$cP-c\bar{\varepsilon}\cdot\mathbb{F}(cP)^0\subset\bigcup_{x\in(c-1)P\cap\mathbb{Z}^d}x+P. \tag{8}$$

For every $F\in\mathbb{F}(P)^0$ we have $c\varepsilon_F\leqslant\frac{\mathrm{width}_F(P)}{d+1}$. Therefore,

$$cP-\bar{\sigma}\cdot\mathbb{F}(cP)^0\subset cP-c\bar{\varepsilon}\cdot\mathbb{F}(cP)^0, \tag{9}$$

where

$$\bar{\sigma}=(\sigma_F)_{F\in\mathbb{F}(P)^0},\quad \sigma_F=\frac{\mathrm{width}_F(P)}{d+1}.$$

Because $\mathsf{E}(P)>\mathfrak{bcn}(d,k)$, (8) and (9) together imply $\mathrm{CN}(d,k)$ for P. $\square$

Corollary 5.2. (a) *For all $d\in\mathbb{N}_{\geqslant 2}$ and $k\in\mathbb{Q}_{\geqslant 2}$ we have*

$$\mathfrak{cn}(d,k)\leqslant\max\bigg(d(d+1)k,\frac{d+1}{d}\,\mathfrak{cn}\bigg(d-1,k+\frac{k-1}{d}\bigg)\bigg).$$

(b) *For all $d\in\mathbb{N}$ and $k\in\mathbb{Q}_{\geqslant 2}$ we have $\mathfrak{cn}(d,k)<\infty$.*

The part (a) follows from Lemma 5.1(b), (c), and the part (b) follows from the part (a) and Lemma 5.1(a).

Remark 5.3. (a) In the proof above we used twice that $c-1\geqslant 1$. This explains why in Definition 1.1 we choose $k\geqslant 2$ and $c\in[2,k]_{\mathbb{Q}}$, and not $k\geqslant 1$ and $c\in[1,k]_{\mathbb{Q}}$.

(b) We have *not* shown that $\lim_{k\to\infty}\mathfrak{cn}(d,k)<\infty$.

6. Proof of the main result

6.1. Proof of Theorem 1.2

The limit case will be taken care of by

Lemma 6.1. *Let $d \in \mathbb{N}$ and $k \in \mathbb{Q}_{\geqslant 2}$. If P is a rational d-polytope with $\mathsf{E}(P) = \mathfrak{cn}(k, d)$ then P satisfies* $\mathrm{CN}(d, k)$.

Proof. We can assume $P \subset \mathbb{R}^d$. On the one hand, for all $c \in [2, k]_{\mathbb{Q}}$ and all sufficiently small $\varepsilon \in \mathbb{Q}_{>0}$, depending on k and P (but not on c), the following holds true for any vertex $v \in P$: the set

$$(c-1)(1+\varepsilon)P \cap \big((c-1)(1+\varepsilon)v + \mathbb{Z}^d\big) \subset \mathbb{R}^d$$

is the parallel translate by $\varepsilon(c-1)v$ of the set

$$(c-1)P \cap \big((c-1)v + \mathbb{Z}^d\big) \subset \mathbb{R}^d.$$

On the other hand, the polytope $(1+\varepsilon)P$ is a homothetic image of P, approximating P as $\varepsilon \to 0$. Consequently, since the unions of only finitely many polytopes are involved, for every number $c \in [2, k]_{\mathbb{Q}}$, the complement $cP \setminus \mathbb{U}_{\mathrm{vert}}(P, c)$ is a closed measurable set in $\mathbb{R}^d$ that can be approximated measure-wise with arbitrary precision by sets of the form $c(1+\varepsilon)P \setminus \mathbb{U}_{\mathrm{vert}}((1+\varepsilon)P, c)$, $\varepsilon \in \mathbb{Q}_{>0}$. But the latter are all empty sets. $\square$

Now we turn to Theorem 1.2 proper. By Corollary 5.2(b), the function $\mathfrak{cn}(d, k) : \mathbb{N} \times \mathbb{Q}_{\geqslant 2} \to \mathbb{R}_+$ is well defined. For any fixed $d \in \mathbb{N}$ the function $\mathfrak{cn}(d, k) : \mathbb{Q}_{\geqslant 2} \to \mathbb{R}_+$ is non-decreasing. So, by Corollary 5.2(a), for all $d \in \mathbb{N}_{\geqslant 2}$ and $k \in \mathbb{Q}_{\geqslant 2}$ we have the (simpler) inequalities:

$$\mathfrak{cn}(d, k) \leqslant \max\left(d(d+1)k, \frac{d+1}{d}\,\mathfrak{cn}\left(d-1, \frac{d+1}{d} \cdot k\right)\right).$$

By induction on i, based on iterative use of this inequality, we derive

$$\mathfrak{cn}(d, k) \leqslant \max_{i=1,\ldots,d-1}\left(\left\{\frac{d+1-j}{d+2-j} \cdot (d+1)^2 k\right\}_{j=1}^{i}, \frac{d+1}{d+1-i}\,\mathfrak{cn}\left(d-i, \frac{d+1}{d+1-i} \cdot k\right)\right).$$

Therefore,

$$\begin{aligned}\mathfrak{cn}(d, k) &\leqslant \max\left(\left\{\frac{d+1-j}{d+2-j} \cdot (d+1)^2 k\right\}_{j=1}^{d-1}, \frac{d+1}{2}\,\mathfrak{cn}\left(1, \frac{d+1}{2} \cdot k\right)\right) \\ &\leqslant \max\left(d(d+1)k, \frac{d+1}{2}\right) = d(d+1)k.\end{aligned}$$

This already proves the version of Theorem 1.2 with the strict inequality $\mathsf{E}(P) > d(d+1)k$, and the non-strict inequality is covered by Lemma 6.1. $\square$

6.2. Proof of Theorem 1.3(a)

All we need is

Lemma 6.2. *Every lattice d-polytope P with $\mathsf{E}(P) \geqslant \mathfrak{cn}(d,4)$ is integrally closed.*

Proof. Let $P \subset \mathbb{R}^d$ be as in the lemma. We show the equality in Definition 1.1(a) by induction on the factors $c \in \mathbb{N}$. Assume it has been shown for all factors $< c$.

For every $n \in \mathbb{N}$ denote

$$I_n = \left[2^n, 2^{n+1}\right]_{\mathbb{N}}, \qquad P_n = 2^{n-1}P, \qquad L_n = 2^{n-1}\mathbb{Z}^d \subset \mathbb{Z}^d.$$

Then P_n is a rational polytope with $\mathsf{E}_n(P) \geqslant \mathfrak{cn}(d,4)$, where the subindex in E_n indicates that the lattice lengths are measured w.r.t. L_n.

Let $c \in I_n$ for some $n \in \mathbb{N}$, and pick $z \in cP \cap \mathbb{Z}^d$. We have

$$cP = \begin{cases} c'P_n \text{ with } c' = c2^{-n+1} \in [2,4]_{\mathbb{Q}} & \text{if } n > 1, \\ c \in [2,4]_{\mathbb{Q}} & \text{if } n = 1. \end{cases}$$

If $n > 1$ then P_n satisfies $\mathrm{CN}(d,4)$ w.r.t. the lattice L_n; one invokes Lemma 6.1 in the limit case $\mathsf{E}(P) = \mathfrak{cn}(d,4)$. So $z = x + y$ for some $x \in (c'-1)P_n \cap ((c'-1)v + L_n)$, $v \in \operatorname{vert} P_n$, and $y \in P_n$. Then, necessarily, $y \in P_n \cap \mathbb{Z}^d$. In particular, $((c - 2^{n-1})P) \cap \mathbb{Z}^d + (2^{n-1}P) \cap \mathbb{Z}^d = (cP) \cap \mathbb{Z}^d$.

If $n = 1$ then we have $z \in ((c-1)P) \cap \mathbb{Z}^d + P$; again, Lemma 6.1 is invoked in the limit case $\mathsf{E}(P) = \mathfrak{cn}(d,4)$. Therefore, $((c-1)P) \cap \mathbb{Z}^d + P \cap \mathbb{Z}^d = (cP) \cap \mathbb{Z}^d$.

In both cases the induction assumption applies. $\square$

6.3. Proof of Theorem 1.3(b)

Lattice parallelepipeds are integrally closed – a consequence of Lemma 2.2(a). Therefore, we only need to show that a rational simplex P with $\mathsf{E}(P) \geqslant d(d+1)$ is covered by lattice parallelepipeds. In view of Corollary 4.2, it is enough to show that we have the cover

$$\bigcup_{v \in \operatorname{vert}(P)} P - \bar{\varepsilon}(v) \cdot \mathbb{F}(P)^v = P,$$

where $\bar{\varepsilon}(v) = (\varepsilon_F)_{F \in \mathbb{F}(P)^v}$ for every $v \in \operatorname{vert}(P)$ and $\bar{\varepsilon}_F = \frac{\operatorname{width}_F(P)}{d+1}$ for every $F \in \mathbb{F}(P)$. (Notation as in that corollary.)

Since P is a simplex, for every vertex $v \in \operatorname{vert}(P)$ the polytope $P - \bar{\varepsilon} \cdot \mathbb{F}(P)^v$ is the homothetic image of P with factor $\frac{d}{d+1}$ and centered at v. Therefore, the desired covering follows from the fact that at least one of the barycentric coordinates of each point $x \in P$ w.r.t. the vertices of P is $\geqslant \frac{1}{d+1}$. $\square$

Remark 6.3. (a) The equality $\bigcup_{v \in \operatorname{vert}(P)} P - \bar{\varepsilon}(v) \cdot \mathbb{F}(P)^v = P$ does not hold true for general polytopes, not even in dimension 2. This explains the need of $\mathrm{BCN}(d,k)$ in the proof of Theorem 1.2.

(b) We have the following minor improvement of Theorem 1.2 in dimensions $d = 3, 4$: every lattice d-polytope P with $\mathsf{E}(P) \geqslant d(d^2 - 1)$ is integrally closed. In fact, Theorem 1.2 and Lemma 2.2(b) imply the version of Theorem 1.3(a) with the inequality $\mathsf{E}(P) \geqslant d(d^2 - 1)$, which is a better estimate than $\mathsf{E}(P) \geqslant 4d(d + 1)$ for $d = 3, 4$.

Notice. The results in this paper extend to all polytopes whose edges are parallel to rational directions and all real factors $\geqslant 2$. For the approach developed above, the most general setting possible is when one fixes an arbitrary finitely generated additive subgroup $\Lambda \subset \mathbb{R}^d$ (no longer a discrete subset of $\mathbb{R}^d$ if rank $\Lambda > d$) and studies polytopes $P \subset \mathbb{R}^d$ whose edge directions are parallel to elements of Λ.

A notable exception from the arguments above that go through when rank $\Lambda > d$ is the proof of Lemma 6.1.

Acknowledgments

We thank (i) the referee, whose thorough study of the paper resulted in a number of substantial expositional improvements and inclusion of pictures, and (ii) Diane Fenster, who created the pictures used in this paper.

References

[1] Workshop: Combinatorial challenges in toric varieties, The workshop held April 27 to May 1, 2009, organized by Joseph Gubeladze, Christian Haase, and Diane Maclagan, American Institute of Mathematics, Palo Alto, http://www.aimath.org/pastworkshops/toricvarieties.html.
[2] Mini-workshop: Projective normality of smooth toric varieties, Oberwolfach Rep. 4 (3) (2007) 2283–2319, Abstracts from the mini-workshop held August 12–18, 2007, organized by Christian Haase, Takayuki Hibi and Diane Maclagan.
[3] Margherita Barile, Dominique Bernardi, Alexander Borisov, Jean-Michel Kantor, On empty lattice simplices in dimension 4, Proc. Amer. Math. Soc. 139 (2011) 4247–4253.
[4] Victor Batyrev, Johannes Hofscheier, A generalization of a theorem of G.K. White, preprint http://arxiv.org/abs/1004.3411.
[5] Matthias Beck, Serkan Hoşten, Cyclotomic polytopes and growth series of cyclotomic lattices, Math. Res. Lett. 13 (2006) 607–622.
[6] Winfried Bruns, On the integral Carathéodory property, Experiment. Math. 16 (2007) 359–365.
[7] Winfried Bruns, Joseph Gubeladze, Normality and covering properties of affine semigroups, J. Reine Angew. Math. 510 (1999) 161–178.
[8] Winfried Bruns, Joseph Gubeladze, Unimodular covers of multiples of polytopes, Doc. Math. 7 (2002) 463–480.
[9] Winfried Bruns, Joseph Gubeladze, Polytopes, Rings, and K-Theory, Springer Monogr. Math., Springer, Dordrecht, 2009.
[10] Winfried Bruns, Joseph Gubeladze, Martin Henk, Alexander Martin, Robert Weismantel, A counterexample to an integer analogue of Carathéodory's theorem, J. Reine Angew. Math. 510 (1999) 179–185.
[11] Winfried Bruns, Joseph Gubeladze, Ngô Viêt Trung, Normal polytopes, triangulations, and Koszul algebras, J. Reine Angew. Math. 485 (1997) 123–160.
[12] W. Cook, J. Fonlupt, A. Schrijver, An integer analogue of Carathéodory's theorem, J. Combin. Theory Ser. B 40 (1986) 63–70.
[13] David Cox, John Little, Hal Schenck, Toric Varieties, Grad. Stud. Math., American Mathematical Society, Providence, RI, 2011.
[14] Günter Ewald, Uwe Wessels, On the ampleness of invertible sheaves in complete projective toric varieties, Results Math. 19 (1991) 275–278.
[15] Christian Haase, Günter M. Ziegler, On the maximal width of empty lattice simplices, European J. Combin. 21 (2000) 111–119.
[16] J.-M. Kantor, K.S. Sarkaria, On primitive subdivisions of an elementary tetrahedron, Pacific J. Math. 211 (2003) 123–155.

[17] G. Kempf, Finn Faye Knudsen, D. Mumford, B. Saint-Donat, Toroidal Embeddings. I, Lecture Notes in Math., vol. 339, Springer-Verlag, Berlin, 1973.
[18] Tomonori Kitamura, Gröbner bases associated with positive roots and Catalan numbers, Osaka J. Math. 42 (2) (2005) 421–433.
[19] Hidefumi Ohsugi, Takayuki Hibi, Unimodular triangulations and coverings of configurations arising from root systems, J. Algebraic Combin. 14 (3) (2001) 199–219.
[20] Sam Payne, Lattice polytopes cut out by root systems and the Koszul property, Adv. Math. 220 (2009) 926–935.
[21] Victor Reiner, Volkmar Welker, On the Charney–Davis and Neggers–Stanley conjectures, J. Combin. Theory Ser. A 109 (2) (2005) 247–280.
[22] Andras Sebö, Hilbert bases, Carathéodory's theorem and combinatorial optimization, in: Proc. of the IPCO Conference, Waterloo, Canada, 1990, pp. 431–455.
[23] András Sebő, An introduction to empty lattice simplices, in: Integer Programming and Combinatorial Optimization, Graz, 1999, in: Lecture Notes in Comput. Sci., vol. 1610, Springer, Berlin, 1999, pp. 400–414.
[24] Michael von Thaden, Improved bounds for covers and triangulations of lattice polytopes, Dissertation, Osnabrück, 2007.
[25] Günter Ziegler, Lectures on Polytopes, revised edition, Grad. Texts in Math., vol. 152, Springer-Verlag, New York, 1998.