Dominating cycles in regular 3-connected graphs

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Abstract

Let G be a 3-connected, k-regular graph on at most 4k vertices. We show that, for k > 63, every longest cycle of G is a dominating cycle. We conjecture that G is in fact hamiltonian.

1. Introduction

Various authors have investigated the existence of Hamilton cycles in 2-connected, k-regular graphs, see [6, 2, 8, 10, 4]. The strongest result given by Zhu, Liu and Yu in [10] is that all 2-connected, k-regular graphs, k ≥ 6, on at most 3k + 3 vertices are hamiltonian.

Haggkvist [8] conjectured that the upper bound on the number of vertices could be increased to (m + 1)k under the stronger hypothesis that the graph is m-connected, m ≥ 4. The following example, constructed independently by the present authors and also H.A. Jung, shows that Haggkvist’s conjecture is false.

For k = 4t, construct G from two disjoint copies of K_{k-1}, H_1 and H_2, and one copy of K_{k,k-1} by adding a set of k independent edges from H_1 ∪ H_2 to the k-set of the K_{k,k-1}, such that H_1 and H_2 are both incident with 2t-edges and then

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deleting two independent sets of $2t$-edges from $H_1$ and $H_2$ respectively in order to obtain a $k$-regular graph. Then $G$ has $4k + 1$ vertices, has connectivity equal to $k/2$, and is not hamiltonian since it is not 1-tough (deleting the $k$-set of the $K_{k,k-1}$ leaves $(k + 1)$-components).

Although the above example shows that Haggkvist's conjecture is wide of the mark for $m \geq 4$, we feel that it is close to the truth for $m = 3$.

**Conjecture 1.1.** For $k \geq 4$, every 3-connected, $k$-regular graph on at most $4k$ vertices is hamiltonian.

The purpose of this paper is to take a step along the way to proving Conjecture 1.1 by showing the following theorem.

**Theorem 1.2.** Let $G$ be a 3-connected, $k$-regular graph on at most $4k$ vertices. Then for $k \geq 63$, every longest cycle in $G$ is a dominating cycle.

Our result is closely related to the work of H.A. Jung. In [9] he has shown that if $G$ is a 3-connected graph of minimum degree at least $k$, on at most $4k - 6$ vertices, and $C$ is a longest cycle of $G$, then every component of $G - C$ has at most two vertices. It is also related to the following result of Fan.

**Theorem 1.3.** Let $G$ be a 3-connected, $k$-regular graph and $C$ be a longest cycle in $G$. Then $|V(C)| \geq \min\{|V(G)|, 3k\}$.

We shall use the terminology of [5]. In addition, if $G$ is a graph we shall use $|G|$ to denote $|V(G)|$ and $e(G)$ to denote $|E(G)|$. For $A, B \subseteq V(G)$, we shall use $e(A)$ to denote the number of edges of $G$ joining vertices of $A$, $E(A, B)$ and $e(A, B)$ to denote the set and the number of edges joining vertices of $A$ to vertices of $B$, respectively, where edges joining vertices of $A \cap B$ are counted twice. Thus

$$e(A, B) = e(A - B, B - A) + 2e(A \cap B) + e(A - B, B \cap A) + e(B - A, B \cap A).$$

For a cycle $C = c_1c_2c_3 \cdots c_mc_1$, we shall read subscripts modulo $m$ and denote the set $\{c_i, c_{i+1}, \ldots, c_j\}$ by $C[c_i, c_j]$. For $A \subset V(C)$, let $A^+ = \{c_{i+1}; c_i \in A\}$, $A^+ = \{c_{i+1}; c_i \in A\}$, $A^- = \{c_{i-1}; c_i \in A\}$ and $A^- = \{c_{i-1}; c_i \in A\}$.

We shall use the following results due to Bondy and Chvatal and Jung.

**Theorem 1.4 ([3]).** Let $G$ be a graph on $n$ vertices and $u, v, w, x \in V(G)$ such that $uv \notin E(G)$.

1. If $d(u) + d(v) \geq n$ then $G$ is hamiltonian if and only if $G + uv$ is hamiltonian.
2. If $d(u) + d(v) \geq n + 1$ then $G$ has a Hamilton $wx$-path if and only if $G + uv$ has a Hamilton $wx$-path.
Theorem 1.5 ([9]). Let $C$ be a longest cycle in an $m$-connected ($m \geq 2$) graph $G$ and $H$ be a Hamilton-connected component of $G - C$. Then there exists a vertex $v$ in $H$ such that

$$|C| \geq s(d_G(v) - s + 2) + (m - s)(|H| - s + 1)$$

for all $0 \leq s \leq |H| + 1$.

2. Preliminary lemmas

Lemma 2.1. Let $G$ be a connected graph such that for every longest path $P$ in $G$, the sum of the degrees of the end-vertices of $P$ is at least $|P| + 1$. Then $G$ is Hamilton-connected.

Proof. We first show that $G$ is hamiltonian. Choose a longest path $P$ in $G$ and let the ends of $P$ be $u$ and $v$. Let $G_1$ be the $|P|$-closure of $G[P]$. Since $u$ and $v$ have degree sum at least $|P| + 1$ in $G[P]$, $uv \in E(G_1)$, and thus $G_1$ is hamiltonian. By Theorem 1.4, $G[P]$ is hamiltonian. Since $G$ is connected and $P$ is a longest path in $G$, we deduce that $P$ is a Hamilton path in $G$ and $G$ is hamiltonian.

Now let $H$ be the $(n + 1)$-closure of $G$ and choose a Hamilton cycle $C = v_1v_2\cdots v_nv_1$ of $G$. Without loss of generality we may assume that $v_1$ is a vertex of minimum degree in $H$. Since $v_1v_2\cdots v_n$ is a Hamilton path in $G$, it follows from the condition of our lemma that

$$d_H(v_1) + d_H(v_n) > d_G(v_1) + d_G(v_n) > n + 1$$

Since $d_H(v_n) \geq d_H(v_1)$, we have $d_H(v_n) \geq (n + 1)/2$. Moreover for each edge $v_nv_i \in E(H)$, $2 \leq i \leq n - 1$, $H$ contains the Hamilton $v_1v_{i+1}$-path: $v_1v_2\cdots v_nv_{n-1}\cdots v_{i+1}$. Thus by Theorem 1.4, $G$ contains a Hamilton $v_1v_{i+1}$-path and $d_G(v_1) + d_G(v_{i+1}) \geq n + 1$. Hence $v_1v_{i+1} \in E(H)$ and thus $d_H(v_1) \geq d_H(v_n) \geq (n + 1)/2$. Since $v_1$ has minimum degree in $H$, we deduce that $H$ has minimum degree at least $(n + 1)/2$. Thus $H$ is complete and hence is Hamilton connected. By Theorem 1.4, $G$ is Hamilton connected. \qed

Definition 2.2. In the following four lemmas, let $C = c_1c_2\cdots c_mc_1$ be a longest cycle in a graph $G$ and $H$ be a component of $G - C$. Let $W \subseteq V(H)$ be such that any two vertices of $W$ are joined by a path in $H$ of length at least $l - 1$. Let $X \subseteq V(C)$ be such that each vertex of $X$ is joined to at least two vertices of $W$. Let $Q$ be the set of ordered pairs $(c_i, c_j)$ such that $c_i$ and $c_j$ are joined to two distinct vertices of $W$ and $C[c_{i+1}, c_{j-1}] \cap N(W) = \emptyset$.

Putting $x = |X|$, $X = \{u_1, u_2, \ldots, u_x\}$ (the subscripts of $u_i$ will be reduced modulo $x$ if needed throughout) and $q = |Q|$, we have $q \geq x$ for $x \geq 2$. Also if $(c_i, c_j) \in Q$ then $|C[c_{i+1}, c_{j-1}]| \geq l$ by the maximality of $C$. 


Lemma 2.3. If \( q \geq 2 \) then \( n \geq |H| + 2 |N_c(W)| + q(l - 1) \).

**Proof.** The result follows since

\[
V(H) \cup N_c(W) \cup (N_c(W))^+ \cup \left( \bigcup_{(c_i, c_j) \in \mathcal{Q}} C[c_{i+2}, c_{j-1}] \right) \subseteq V(G)
\]

and the above sets are disjoint pairwise by the maximality of \( C \). \( \square \)

Lemma 2.4. Suppose that \( c_i, c_j \) are disjoint elements of \( X \) and that \( c_{i+1} \) is joined to \( c_i \in C[c_{i+1}, c_{j-1}] \) by a path which is internally disjoint from \( C \cup H \). Then

(a) \( N(c_{i+1}) \cap C[c_{i+1}, c_{j-1}] = N(c_{i+1}) \cap C[c_{i+1}, c_{j-1}] = \emptyset \);

(b) if \( c_i \in (N(c_{i+1}) \cup N(c_{j-1})) \cap C[c_{i+1}, c_{j-1}] \) then

\[
N(c_{j-1}) \cap (C[c_i, c_{i-1}] \cup C[c_{i+1}, c_{i+i}]) \cap C[c_{i+1}, c_{i-1}] = \emptyset.
\]

**Proof.** If (a) or (b) do not hold, we easily construct a longer cycle than \( C \) and obtain a contradiction. \( \square \)

Lemma 2.5. Suppose \( c_i, c_j, c_g, c_h \in X \) such that \( c_j \in C[c_{i+1}, c_k] \) and \( c_h \in C[c_g+1, c_i] \). Let \( S = C[c_{g+1}, c_{h-1}] \). Then

(a) \( e(\{c_{i+1}, c_{j-1}\}, S) \leq |S| - 1 \);

(b) if \( c_{i+1} \) and \( c_{j-1} \) are joined by a path which is internally disjoint from \( C \cup H \), then \( e(\{c_{i+1}, c_{j-1}\}, S) \leq |S| - 1 \);

(c) if \( l = 2 \) and \( j = i + 3 \), then \( e(\{c_{i+1}, c_{j-1}\}, S) \leq (2(|S| - 2))/3 \).

**Proof.** Let \( A = N(c_{i+1}) \cap S \), \( B = N(c_{j-1}) \cap S \). Since \( C \) is a longest cycle, we deduce that \( |S| \geq l \). Hence, without loss of generality, we may assume that \( A \neq \emptyset \) and \( A \cap C[c_{g+1}, c_{g+i}] = \emptyset \). Let \( w = \min\{s \geq g + 1 : c_s \in A\} \). By Lemma 2.4(a) and (b),

\[
B \subseteq S - (A^{-} \cup C[c_{w-1}, c_{w-l}] \cup \{c_{h-1}\})
\]

Since \( A^{-} \cap C[c_{w-1}, c_{w-l}] = \{c_{w-1}\} \) and \( c_{h-1} \notin A^{-} \cup C[c_{w-1}, c_{w-l}] \), we have \( |B| \leq |S| - (|A| + l) \) and \( e(\{c_{i+1}, c_{j-1}\}, S) = |A| + |B| \leq |S| - l \). So (a) is proved. The proof of (b) is similar to the above.

Let us consider the case of (c). By the maximality of \( C \), \( A \), \( A^+ \) and \( B^- \) are pairwise disjoint. Therefore:

1. if \( A \subseteq B \), then \( c_{h-1} \notin A \cup B^{-} \cup A^+ \);
2. if \( A \subseteq B \neq \emptyset \), then choosing \( c_s \in A \cup B \) with \( s \) minimal, it follows that \( c_{s-1} \notin S - (A \cup B^{-} \cup A^+) \);
3. if \( B \subseteq A \), then \( c_{g+1} \notin A \cup B^{-} \cup A^+ \);
4. if \( B \subseteq B \neq \emptyset \), then choosing \( c_t \in B - A \), with \( t \) maximal, it follows that \( c_t \notin S - (A \cup B^{-} \cup A^+) \).
Since in any case at least two of $c_{i+1}$, $c_i$, $c_{i-1}$, $c_i$ are not in $A \cup B^- \cup A^+$, thus we deduce that

$$|A \cup B^- \cup A^+| = 2|A| + |B| \leq |S| - 2.$$ 

Similarly, $2|B| + |A| \leq |S| - 2$. Hence

$$3|B| + 3|A| \leq 2(|S| - 2),$$

and

$$e(\{c_{i+1}, c_{i-1}\}, S) = |A| + |B| \leq \frac{2(|S| - 2)}{3}. \quad \Box$$

**Lemma 2.6.** If $G$ is $k$-regular on $n \leq 4k$ vertices and $x \geq 2$, then

$$4k \geq x[(l+1)x - 3k + 3] + l + e(X, H).$$

**Proof.** Let $S_i = C[u_i^+, u_{i+1}^-], 1 \leq i \leq x$, be the segments on $C$ which are the sets of vertices of $C$ between the vertices of $X$. Let $S_i \cap X^+ = \{p_i\}$, $S_i \cap X^- = \{q_i\}$ and $s_i = |S_i|$. Using Lemma 2.5,

$$e(\{p_i, q_i\}, C - X) \leq \bigcup_{j \neq i} (s_j - l) + 2(s_i - 1) \leq m - x - l(x - 1) + s_i - 2,$$

and since $C$ is a longest cycle, $e(\{p_i, q_i\}, H) = 0$. Let $R = G - (C \cup H)$ and $|R| = r$. If no vertex of $R$ is adjacent to both $p_i$ and $q_i$, then

$$e(\{p_i, q_i\}, R) \leq r$$

and

$$e(\{p_i, q_i\}, G - X) \leq m + r - x - l(x - 1) + s_i - 2$$

$$\leq n - l - x - l(x - 1) + s_i - 2 = n - (l + 1)x + s_i - 2.$$

Thus

$$e(\{p_i, q_i\}, X) \geq 2k - n + (l + 1)x - s_i + 2 \geq (l + 1)x - 2k + 2 - s_i. \quad (1')$$

On the other hand, if some vertex of $R$ is adjacent to both $p_i$ and $q_i$, then by Lemma 2.5,

$$e(\{q_i^-, q_i\}, C - X) \leq \bigcup_{j \neq i} (s_j - l) + 2(s_i - 1).$$

Since $C$ is a longest cycle, and since $q_i^+$ is joined to two vertices of $W$, we have $e(\{q_i^-, q_i\}, H) = 0$ and $e(\{q_i^-, q_i\}, R) \leq r$. Using the argument given above, we deduce that

$$e(\{q_i^-, q_i\}, X) \geq (l + 1)x - 2k + 2 - s_i. \quad (2')$$

Using (1') or (2') and summing from $i = 1$ to $x$ gives

$$e(C - X, X) \geq x[(l + 1)x - 2k + 2] - (n - l - x)$$

$$\geq x[(l + 1)x - 2k + 3] - 4k + l.$$
Since $G$ is $k$-regular,
\[
x_k \geq e(X, G - X) \geq e(X, C - X) + e(X, H) \\
\geq x[(l + 1)x - 2k + 3] - 4k + l + e(X, H).
\]
and
\[
4k \geq x[(l + 1)x - 3k + 3] + l + e(X, H),
\]
as required. 

The following lemma extends a result obtained by Ash [1].

**Lemma 2.7.** Let $C = c_1c_2 \cdots c_mc_1$ be a longest cycle in a graph $G$ and $ab$ be an edge of $G - C$. Let $W_a \subseteq N_C(a)$, $W_b \subseteq N_C(b)$, $w_a = |W_a|$ and $w_b = |W_b|$. If $w_a \geq w_b$, then

(a) $e(W_a^+, W_b^-) \leq w_a + 2w_b - 2 \leq w_b + 2w_a - 2$, and

\[
\frac{3(w_a + w_b)}{2} - 2
\]
moreover if $v \in W_a^{+2} - W_b^-$, then

(b) $e(W_a^+ \cup \{v\}, W_b^-) \leq w_a + 1 + 2w_b - 2 \leq w_b + 1 + 2w_a - 2$. and

\[
\frac{3(w_a + w_b + 1)}{2} - 2.
\]

**Proof.** Since $C$ is a longest cycle of $G$, $W_a^+ \cap W_b^- = \emptyset$. Let $H$ be the bipartite subgraph of $G$ with $V(H) = W_a^+ \cup W_b^-$ and $E(H) = E(W_a^+, W_b^-)$. Using the maximality of $C$, we deduce that there does not exist edges $c_ic_{i+1}, c_ic_{i-1} \in E(H)$ where $c_i \in C[c_{i+1}, c_{i-1}]$ and $c_i \in C[c_{i+1}, c_{i-1}]$. It follows that $H$ may be drawn in the plane as an outerplanar bipartite graph. Using Euler’s formula and the facts that the outer face has at least $2w_a$ edges and all other faces at least four edges, we deduce that $e(H) \leq w_a + 2w_b - 2$. Statement (b) is proved similarly. 

3. Proof of Theorem 1.2

By contradiction. Suppose that $C$ is a longest cycle of $G$ such that $G - C$ contains a component $H$ with

$|H| = h$

$= \max\{|H'|: H' \text{ is a component of } G - C' \text{ for some longest cycle } C' \text{ of } G\} \geq 2.$

Let $R = G - (C \cup H)$ and $|R| = r$. By Theorem 1.3, $|C| \geq 3k$. Thus $h \leq k$ and $d_C(v) \geq 1$ for each $v \in V(H)$. Consider the following three cases.

**Case 3.1:** $H$ is not Hamilton connected.
By Lemma 2.1, we may choose a longest path $P$ in $H$, joining two vertices $a,b \in V(H)$ satisfying $d_H(a) + d_H(b) \leq |P|$. Using the notation of Definition 2.2, let $|P| = l$, $W = \{a, b\}$ and $X = N_C(a) \cap N_C(b)$.

Suppose $N_C(a) \neq N_C(b)$, and without loss of generality assume that $d_C(a) \leq d_C(b)$. Since $d_C(a), d_C(b) \geq 1$, we have $q \geq 2$. Moreover, it can be seen that $q \geq x + 1 + \sigma$, where $\sigma = 0$ if $N_C(a) \subset N_C(b)$ and otherwise $\sigma = 1$. By Lemma 2.3,

$$n \geq h + 2 |N_C(W)| + (l - 1)(x + 1 + \sigma)$$
$$\geq l + 2 |N_C(a) \cup N_C(b)| + 2 |N_C(a) \cap N_C(b)| + (l - 3)x + (l - 1)(1 + \sigma)$$
$$\geq l + 2(d_C(a) + d_C(b)) + (l - 3)x + (l - 1)(1 + \sigma)$$
$$\geq 2l + 2(2k - l) + (l - 3)x + \sigma(l - 1) - 1 \geq 4k - 1 + (l - 3)x + \sigma(l - 1).$$

Since $H$ is not Hamilton connected, we have $l \geq 3$. Since $n \leq 4k$, we must have $\sigma = 0$ and thus $N_C(a) \subset N_C(b)$. Furthermore, $(l - 3)x \leq 1$. If $l = 3$, then it follows that $d_H(a) - d_H(b) = 1$ and since $G$ is $k$-regular $d_C(a) = d_C(b)$. This contradicts the fact that $N_C(a)$ is a proper subset of $N_C(b)$ and hence $l \geq 4$. Since $(l - 3)x \leq 1$, we have $l = 4$ and $x = 1$. On the other hand, since $N_C(a) \subset N_C(b)$, we know $x = d_C(a)$. From $d_H(a) \leq l - d_H(b) \leq 3$, we obtain

$$k - d_C(a) = d_C(a) + d_H(a) \leq 4,$$

contrary to the hypothesis about $k$ and hence $N_C(a) = N_C(b)$. Since $d_H(a) + d_H(b) \leq l$, we have $d_C(a) + d_C(b) \geq 2k - l$ and so

$$x = d_C(a) = d_C(b) \geq k - \frac{l}{2} \geq \frac{k}{2} \geq 2.$$

Using Lemma 2.6,

$$4k \geq (k - \frac{l}{2})[(l + 1)(k - \frac{l}{2}) - 3k + 5] + l = f(l).$$

Since $3 \leq l \leq k$, we have

$$(l + 1)(k - \frac{l}{2}) \geq 4\left(k - \frac{3}{2}\right) = 4k - 6$$

and

$$k - \frac{l}{2} \geq \frac{k}{2}.$$

Therefore

$$f(l) \geq \frac{k(k - 1)}{2} + 3 > 4k$$

for $k \geq 9$ since $f(l)$ is a concave function. This contradicts (1) and completes the discussion of Case 3.1.

**Case 3.2:** $H$ is Hamilton connected and $h \geq 3$.

**Subcase 3.2.1:** $k - 2 \leq h \leq k$. By Lemma 2.3, we have

$$4k \geq n - h + 2 |N_C(H)| + q(h - 1).$$

(2)

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Since $G$ is 3-connected, $q \geq 3$. Since $|N_C(H)| \geq q$, $h \geq k - 2$ and $k \geq 6$, we deduce that $q = 3$ and $h \leq k - 1$. Using $H$ and the facts that $q = 3$ and $d_C(v) \geq k - h + 1$ for all $v \in V(H)$, we deduce that $|N_C(H)| = 3$. Let $N_C(H) = \{x_1, x_2, x_3\} \subseteq X$ and let $S_1, S_2, S_3$ be the segments of $C$ between the vertices of $X$. Put $s_i = |S_i|$ and without loss of generality assume that $s_i \leq s_j$ for $1 \leq i < j \leq 3$.

Suppose $h = k - 1$. Then $s_1 = s_2 = k - 1$ and by the maximality of $C$, $e(S_i, S_j) = 0$ for $1 \leq i < j \leq 3$. Since $S_1$ and $S_2$ can play the role of $H$ and $e(H, X) \geq 2(k - 1)$, we deduce that $e(S_1, X) \geq 2(k - 1)$ and $e(S_2, X) \geq 2(k - 1)$. This contradicts the fact that $e(X, H \cup S_1 \cup S_2) \leq 3k$.

Thus $h = k - 2$ and $e(H, X) = 3(k - 2)$. Since each vertex of $X$ is incident with two edges of $C$, it follows that $E(X) \subseteq (E(H, X) \cup E(C))$. Let $X^+ \cap S_i = \{p_i\}$ and $X^- \cap S_i = \{q_i\}$ for some $1 \leq i \leq 3$. By Lemma 2.5.(a), $e(\{p_i\}, S_j) = s_j - (k - 2)$ for $j \neq i$. Since $e(\{p_i\}, H) = 0$ and $e(\{p_i\}, X) = 1$, it follows that

$$e(\{p_i\}, G - S_j) \leq 4k - s_i - 3(k - 2) - 2 = k - s_i + 4.$$ 

Thus $d_s(p_i) \geq s_i - 4$ and by a similar argument $d_s(q_i) \geq s_i - 4$. We next show that $(N_s(p_i))^{-1} \cap N_s(q_i) \neq \emptyset$. If this were not the case, then since $|N_s(p_i))^{-1}| = d_s(p_i) - 1$ and $|N_s(q_i)) = d_s(q_i)$, we would have $s_i \geq d_s(p_i) + d_s(q_i) - 1 \geq 2s_i - 8$, and hence $s_i \leq 8$. This contradicts the fact that $s_i \geq h = k - 2 > 8$.

Choosing a vertex $t_i \in (N_s(p_i))^{-1} \cap (N_s(q_i))^{-1}$, we may use Lemma 2.4.(a) to deduce that $e(\{t_i\}, S_j) = 0$ for $j \neq i$. Since $e(\{t_i\}, H) = 0 = e(\{t_i\}, X)$, we have $d(t_i) = s_i - 1 + r$. Since $s_1 + s_2 + s_3 = n - (k - 2) - 3 - r \leq 3k - 1 - r$, it follows that $s_i \leq k - 1$ and since $|C| \geq 3k$, we have $r \leq 2$. Thus $s_1 = k - 1$, $r = 2$, $s_2 = s_3 = k - 1$ and $t_i$ is adjacent to both vertices $u_1, u_2$ of $G - (C \cup H)$. We may now use the fact that $t_1$ and $t_2$ are both adjacent to $u_1$ to construct a longer cycle than $C$. This completes the discussion of Subcase 3.2.1.

**Subcase 3.2.2:** $4 \leq h \leq k - 3$. By Theorem 1.5, taking $s = h + 1$, we have

$$4k \geq n \geq h + (h + 1)[k - (h + 1) + 2] = g(h).$$

Since $g(h)$ is concave and $g(4) = g(k - 3) = 5k - 11 > 4k$ for $k \geq 12$, we obtain an immediate contradiction.

**Subcase 3.2.3:** $h = 3$. Let $V(H) = \{u, v, w\}$. Using the terminology of Definition 2.2, we have $W = V(H)$, $l = 3$ and $q \geq |N_C(u) \cap N_C(v)|$. Also $q \geq 2$. So by Lemma 2.3,

$$4k \geq |H| + 2|N_C(H)| + 2q
\geq 3 + 2|N_C(u) \cup N_C(v)| + 2|N_C(u) \cap N_C(v)| + 2|N_C(w) - (N_C(u) \cup N_C(v))|
\geq 3 + 2(d_C(u) + d_C(v)) + 2|N_C(w) - (N_C(u) \cup N_C(v))|
\geq 4k - 1 + 2|N_C(w) - (N_C(u) \cup N_C(v))|.$$ 

Hence $N_C(w) \subseteq N_C(u) \cup N_C(v)$ and similarly $N_C(u) \subseteq N_C(v) \cup N_C(w)$ and $N_C(v) \subseteq N_C(u) \cup N_C(w)$. Using Lemma 2.6 with $X = N_C(H)$ and $x \geq k - 2$ gives

$$4k \geq x[(l + 1)x - 3k + 3] + l + e(H, X)
\geq (k - 2)(k - 5) + 3(k - 2) - k^2 - 4k + 7.$$
This contradicts the fact \( k \geq 63 \) and completes the proof of Case 3.2.

**Case 3.3:** \( h = 2 \).

Let \( V(H) = \{a, b\} \). For \( f \in \{a, b\} \), put

\[
X_f = N_C(f), \quad Y_f = X_f^+ \cap X_f^-, \quad X_{ab} = X_a \cap X_b,
\]

\[
Z_f = X_f^- \quad (X_f^+ \cup X_{ab}), \quad M_f = X_f^- \quad (X_f^+ \cup X_{ab}),
\]

\[
|X_f| = x_f, \quad |Y_f| = y_f, \quad |X_{ab}| = x_{ab}, \quad |Z_f| = z_f.
\]

Then \( |M_f| = x_f - y_f - x_{ab} = z_f \). Since \( C \) is a longest cycle it follows that

\[
4k - 2 \geq |C| \geq 2y_a + 2y_b + 3z_a + 3z_b + 3x_{ab}.
\]

Since

\[
y_a + z_a + x_{ab} = d_C(a) = k - 1 = d_C(b) = y_b + z_b + x_{ab},
\]

we have \( 4k - 2 \geq 3(k - 1) + y_b + z_a + 2z_b \). Hence

\[
z_b \leq \frac{k + 1}{2},
\]

and similarly

\[
z_a \leq \frac{k + 1}{2}.
\]

Let \( S \) be the set of segments of \( C \) between the vertices of \( X_a \cup X_b \) and let \( S_{ab} \) be the set of all \( S_i \in S \) such that \( S_i \) is a segment between two vertices of \( X_{ab} \). Let

\[
S = \{S_i \in S_{ab}: |S_i| = 2\}, \quad S^2 = \{S_i \in S_{ab}: |S_i| = 3\}, \quad S^4 = \{S_i \in S_{ab}: |S_i| \geq 4\},
\]

\[
l_2 = |S^2|, \quad l_3 = |S^3|, \quad l_4 = |S^4|
\]

\[
Z_{ab} = X_{ab}^+ - X_{ab}^-, \quad M_{ab} = X_{ab}^- - X_{ab}^+, \quad P = X_{ab}^+ \cap X_{ab}^-.
\]

Then \( |Z_{ab}| = |M_{ab}| = x_{ab} - l_2 \) and \( |P| = l_3 \). Put

\[
T = X_a^+ \cup X_a^- \cup X_b^+ \cup X_b^- \cup P \cup \{a, b\}.
\]

Using (3) it follows that

\[
|T| = 2z_a + 2x_{ab} + y_a + 2z_b + y_b + l_3 + 2 = 2k + z_a + z_b + l_3.
\]

Since \( e(T, G - T) = k |T| - 2e(T) \) and \( e(G - T, T) = k(n - |T|) - 2e(G - T) \), we may use (6) to deduce that

\[
n = 2 |T| - \frac{2(e(T) - e(G - T))}{k} = 4k + 2z_a + 2z_b + 2l_3 - \frac{2(e(T) - e(G - T))}{k}.
\]

We shall use the following results to obtain upper bound on \( e(T) \).

\[
e(\{a\}, \{b\}) = 1; \quad (8)
\]

\[
e(\{a, b\}, T - \{a, b\}) = 0; \quad (9)
\]

\[
e(X_a^+ \cup X_a^-) = 0; \quad (10)
\]

\[
e(X_b^+ \cup X_b^-) = 0; \quad (11)
\]

\[
\text{if } c_i, c_j \in E(X_a^+, X_b^+) \cup E(X_a^-, X_b^-), \text{ then } j = i + 1 \text{ and } c_i \in X_a^+; \quad (12)
\]

\[
\text{if } c_i, c_j \in E(X_a^-, X_b^+) \cup E(X_a^+, X_b^-), \text{ then } j = i + 1 \text{ and } c_j \in X_a^-; \quad (13)
\]
The statements (9)–(13) follows from the maximality of $C$.

Using (10)–(13), it follows that

$$e(\bar{X}^+_a \cup \bar{X}^+_b, \bar{X}^-_a \cup \bar{X}^-_b) = e(\bar{Z}^+_a \cup \bar{Z}^+_b, \bar{Z}^-_a \cup \bar{Z}^-_b, M_a \cup M_b \cup M_{ab}) + l_2.$$  

By Lemma 2.7,

$$e(\bar{Z}^+_a \cup \bar{Z}^+_b, M_b \cup M_{ab}) \leq \frac{3(|Z_b| + |Z_{ab}| + |M_b| + |M_{ab}|)}{2} - 2$$

$$= \frac{3(z_a + z_b + 2x_{ab} - 2l_2)}{2} - 2,$$  

and

$$e(\bar{Z}^+_a \cup \bar{Z}^+_b, M_a \cup M_{ab}) \leq \frac{3(z_a + z_b + 2x_{ab} - 2l_2)}{2} - 2.$$  

Furthermore,

$$e(\bar{Z}^+_a, M_a) \leq z_a^2$$  

and

$$e(\bar{Z}^+_b, M_b) \leq z_b^2.$$  

Using (14)–(18), we obtain

$$e(\bar{X}^+_a \cup \bar{X}^+_b, \bar{X}^-_a \cup \bar{X}^-_b) \leq z_a^2 + z_b^2 + 3(z_a + z_b + 2x_{ab} - 2l_2) - 4 + l_2.$$  

By (12) and (13),

$$e(P, \bar{X}^+_a \cup \bar{X}^-_a \cup \bar{X}^+_b \cup \bar{X}^-_b) = 2l_3$$  

and

$$e(P) \leq \frac{l_3^2}{2}.$$  

Using (8)–(11) and (19)–(21), we obtain that

$$e(I) \leq z_a^2 + z_b^2 + 3(z_a + z_b + 2x_{ab} - 2l_2) + l_2 + \frac{l_3^2}{2} + 2l_3 - 3$$

$$\leq z_a^2 + z_b^2 + 3z_a + 3z_b + 5(x_{ab} - l_2) + \frac{l_3^2}{2} + 2l_3 + k - 4,$$

since $x_{ab} < k$. Substituting into (7) gives

$$n \geq 4k - 2 + 2z_a \left(1 - \frac{z_a + 3}{k}\right) + 2z_b \left(1 - \frac{z_b + 3}{k}\right) + 2l_3 \left(1 - \frac{l_3 + 4}{2k}\right) - \frac{10(x_{ab} - l_2)}{k}.$$  

Subcase 3.3.1: $z_a \geq 1$ and $z_b \geq 1$. Using (4) and (5), and the facts that $l_2 \leq k - 3$, $x_{ab} - l_2 < k$, $n \leq 4k$, $z_a \geq 1$, $z_b \geq 1$ and $k \geq 63$, it follows from (22) that

$$n \geq 4k - 10.$$  

(23)
If $x_{ab} - l_2 \leq z_a + z_b + l_3$, then substituting into (22), we get
\[ n \geq 4k - 2 + (z_a + z_b + l_3) \left( \frac{k - 17}{k} \right) \]
and hence $z_a = z_b = 1$ and $l_3 = 0$, and also from (22) we have
\[ n \geq 4k - 2 + 4 \left(1 - \frac{4}{k}\right) - \frac{20}{k} > 4k, \]
a contradiction. So we assume that $x_{ab} > z_a + z_b + l_2 + l_3$. It follows that some vertex in $X_{ab}^+$ belongs to a segment $S' \in S_{ab}$ of at least four vertices (i.e. length at least three).

Let $S' \cap X_{ab}^+ = \{u\}$, $S' \cap X_{ab}^- = \{v\}$, and put $T_1 = T \cup \{u, v\}$. Applying a similar argument to the one used to obtain (7), we have
\[ n = 4k + 4 + 2z_a + 2z_b + 2l_3 - 2(G) - 4G - T_1 = \frac{2(e(T_1) - e(G - T_1))}{k}. \]
Also
\[ e(\{a, b\}, \{u, v\}) = 0, \quad (25) \]
\[ e(u, X_a^+ \cup X_b^-) = 1, \quad \text{by (12),} \quad (26) \]
\[ e(v, X_a^- \cup X_b^+) = 1, \quad \text{by (13),} \quad (27) \]
\[ e(X_a^+ \cup X_b^- \cup \{u\}, X_a^- \cup X_b^+ \cup \{v\}) \leq e(Z_a \cup Z_b \cup Z_{ab} \cup \{u\}, M_a \cup M_b \cup M_{ab} \cup \{v\}) + 3l_2, \quad (28) \]
by (10)–(13) and using the facts that $e(u, X_{ab}^- - M_{ab}) \leq |X_{ab}^+ - Z_{ab}| = l_2,$ and $e(v, X_{ab}^+ - Z_{ab}) \leq l_2$.

Using Lemma 2.7(a) and (b), we have
\[ e(Z_a \cup Z_{ab} \cup \{u\}, M_b \cup M_{ab}) \leq \frac{3(z_a + z_b + 2x_{ab} - 2l_2 + 1)}{2} - 2, \quad (29) \]
\[ e(Z_b, \{v\} \cup M_{ab}) \leq (x_{ab} - l_2) + 1 + 2z_b - 2, \quad (30) \]
\[ e(Z_{ab}, M_a \cup \{v\}) \leq (x_{ab} - l_2) + 2z_a + 1 - 2. \quad (31) \]
Furthermore,
\[ e(Z_a, M_a \cup \{v\}) \leq z_a^2 + z_a, \quad (32) \]
\[ e(Z_b \cup \{u\}, M_b) \leq z_b^2 + z_b, \quad (33) \]
\[ e(Z_b \cup \{u\}, M_a) \leq \frac{3(z_b + z_a + 1)}{2} - 2, \quad (34) \]
\[ e(\{u\}, \{v\}) \leq 1. \quad (35) \]
Combining (28)–(35) gives
\[ e(X_a^+ \cup X_b^- \cup \{u\}, \{v\} \cup X_a^- \cup X_b^+) \leq z_a^2 + z_b^2 + 6z_a + 6z_b + 5(x_{ab} - l_2) - 2 + 3l_2. \]
Using (8)–(11), (20), (21), (24)–(27) and the fact that \( e(\{u, v\}, P) \leq 2l_3 \), we have

\[
e(T_1) \leq z_a^2 + z_b^2 + 6z_a + 6z_b + 5(x_{ab} - l_2) - 2 + 3l_2 + \frac{l_3^2}{2} + 4l_3.
\]  

(36)

We next consider \( e(G - T) \). Clearly, any vertex of \( T_1 \) is incident to at most two edges on \( C \), and any segment of \( S^3 \) has two consecutive vertices in \( T_1 \) and any one in \( S^3 \) has three consecutive ones. So

\[
e(C - T_1) \geq |C| - 2 |T_1| + l_2 + 2l_3.
\]

Let \( R = G - (C \cup H) \) and choose \( w \in V(R) \). By the maximality of \( C \), \( e(w, X_a^e \cup X_b^e \cup \{u\}) \leq 1 \), \( e(w, X_a^e \cup X_b^e \cup \{v\}) \leq 1 \) and \( e(w, P) \leq 1 \). Since \( e(w, H) = 0 \), we easily deduce that \( e(R, C - T_1) \geq |R| \). Thus

\[
e(G - T_1) \geq |C| - 2 |T_1| + l_2 + 2l_3 + |R|
\]

\[
\geq n - 2 - (4k + 4 + 2z_a + 2z_b + 2l_3) + l_2 + 2l_3
\]

\[
\geq l_2 - 2z_a - 2z_b - 16. \quad \text{by (23).} \quad (37)
\]

Now (24), (36) and (37) imply that

\[
n \geq 4k + 4 + 2z_a \left(1 - \frac{z_a + 8}{k}\right) + 2z_b \left(1 - \frac{z_b + 8}{k}\right) + 2l_3 \left(1 - \frac{l_3 + 8}{2k}\right) - \frac{2(5x_{ab} - 3l_2 + 14)}{k}.
\]  

(38)

To complete the discussion of Case 3.3.1., we show that

\[
x_{ab} \leq 2l_2 + l_3 + z_a + z_b + 2. \quad (39)
\]

To accomplish this, we define \( t_2, t_3, t_4 \) to be the number of segments of \( S \) which follow a vertex of \( X_{ab} \) and have two, three, and at least four vertices, respectively. Then \( t_2 + t_3 + t_4 = x_{ab} \) and

\[
2y_a + 2y_b + 3z_a + 3z_b + 3t_2 + 4t_3 + 5t_4 \leq |C| \leq 4k - 2.
\]

Using (3), we deduce that \( z_a + z_b - t_2 + t_4 \leq 2 \). Thus

\[
x_{ab} = t_2 + t_3 + t_4 \leq 2l_2 + t_3 - z_a - z_b + 2.
\]

Since \( t_2 + t_3 \leq l_2 + l_3 + z_a + z_b \) and \( t_2 \leq l_2 + z_a + z_b \), (39) follows.

Using (39) and (3), we obtain

\[
10x_{ab} - 6l_2 + 28 \leq x_{ab} + 6k + 3l_3 + 28
\]

\[
\leq 7k + 3l_3 + 26 \quad (z_a \geq 1 \text{ implies } x_{ab} \leq k - 2).
\]

Substituting into (38) gives

\[
n \geq 4k - 3 + 2z_a \left(1 - \frac{z_a + 8}{k}\right) + 2z_b \left(1 - \frac{z_b + 8}{k}\right) + 2l_3 \left(1 - \frac{l_3 + 11}{2k}\right) - \frac{26}{k}.
\]
Since  
\[ z_a \left( 1 - \frac{z_a + 8}{k} \right) \geq 1 - \frac{9}{k} \text{ for all } 1 \leq z_a \leq \frac{k+1}{2} \]
and  
\[ z_b \left( 1 - \frac{z_b + 8}{k} \right) \geq 1 - \frac{9}{k} \text{ for all } 1 \leq z_b \leq \frac{k+1}{2} , \]
the above inequality implies  
\[ n \geq 4k - 3 + 4 \left( 1 - \frac{9}{k} \right) - \frac{26}{k} = 4k + 1 - \frac{62}{k} > 4k , \]
a contradiction and this completes the discussion of Subcase 3.3.1.

**Subcase 3.3.2:** \( \min \{ z_a, z_b \} = 0 \). Assuming \( z_a = 0 \), it follows that \( y_a = 0 \). Hence \( X_a = X_{ab} = X_b = X \) and \( z_b = y_b = 0 \).

For any \( S_i \in S^2 \), by the hypothesis of the maximality of \( |H| \), we have \( e(S_i, R) = 0 \). Using Lemma 2.5.(c),
\[ e(S_i, S_j) \leq \frac{2(|S_i| - 2)}{3} \]
for any \( S_j \in S_{ab} \setminus \{ S_i \} \). Therefore, considering \( e(S_i) = 1 \), we deduce that
\[ e(S_i, X) \geq 2(k - 1) - \frac{2(4k - 2 - r - (k - 1) - 2(k - 1))}{3} \geq \frac{4(k - 2)}{3} . \] (40)

Using (40), and (1'), (2') of the proof of Lemma 2.6, we have  
\[ (k - 1)k \geq e(G - X, X) \geq e(H, X) + e(S^2, X) + e(S^3 \cup S^4, X) \]
\[ \geq 2(k - 1) + l_2 \left( \frac{4k - 8}{3} \right) + (l_3 + l_4)(k - 1) - (n - 2 - (k - 1) - 2l_2) . \]

From \( l_2 + l_3 + l_4 = x_{ab} = k - 1 \), it follows that  
\[ k^2 + 1 \geq l_2 \left( \frac{4k - 2}{3} \right) + (k \quad 1 \quad l_2)(k \quad 1) \]
and \( 6k \geq l_2(k + 1) \). So we obtain that \( l_2 \leq 5 \).

Since \( 2l_2 + 3l_3 + 4l_4 \leq |C - X| \leq 4k - 2 - (k - 1) \), we have \( k - 3 \leq l_3 + 2l_2 \).
From (22), we may easily deduce that
\[ l_3 \left( 1 - \frac{l_3 + 4}{2k} \right) \leq 6 \]
and so \( l_3 \leq 12 \). Therefore we have \( k \leq 3 + 12 + 10 = 25 \), contrary to the hypothesis of \( k \geq 63 \).

We finish the discussion of Subcase 3.3.2 and therefore complete the proof of Theorem 1.2. \( \square \)
References