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ScienceDirect
Journal of Computational and Applied Mathematics 215 (2008) 220–229
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

# Analytical solution of the linear fractional differential equation by Adomian decomposition method☆

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Received 13 March 2006; received in revised form 8 April 2007

**Abstract**

In this paper, we consider the $n$-term linear fractional-order differential equation with constant coefficients and obtain the solution of this kind of fractional differential equations by Adomian decomposition method. With the equivalent transmutation, we show that the solution by Adomian decomposition method is the same as the solution by the Green's function. Finally, we illustrate our result with some examples.


*MSC:* 26A33; 65L05

*Keywords:* Fractional derivative; Fractional integral; Adomian decomposition method; Green's function

## 1. Introduction

Nowadays there is increasing attention paid to fractional differential equations and their applications in different research areas. It is well known that these equations are concluded from many physical and chemical problems [15] such as the motion of a large thin plate in a Newtonian fluid, the process of cooling a semi-infinite body by radiation, the $PI^{\lambda}D^{\mu}$ controllers for the control of dynamical systems, the phenomena in electromagnetic acoustic viscoelasticity, electrochemistry and material science and so on. And these equations are more adequate for modeling physical and chemical process than integer-order differential equations.

So far there have been several fundamental works on the fractional derivative and fractional differential equations, written by Oldham and Spanier [14], Miller and Ross [12], Poldubny [15] and others. These works are an introduction to the theory of the fractional derivative and fractional differential equations and provide a systematic understanding of the fractional calculus such as the existence and the uniqueness, some analytical methods for solving fractional differential equations, namely the Green's function method, the Mellin transform method, the power series and Yu. I. Babenko's symbolic method. In particular, restrictions on the initial conditions, the hypotheses on the behavior of the right-hand side function $f$ and the existence of the Green's function are treated in [15, Chapters III and V]. Besides

☆ Project Supported by a National Key Basic Research Project of China (Grant no. 2004CB318000) and the Natural Science Foundation of China (Grant no. 10371090) and the Natural Science Foundation of Zhejiang Province (Grant no. M103043).
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doi:10.1016/j.cam.2007.04.005

there are many extensive research papers on fractional differential equations including the development of an effective method for solving fractional differential equations. It should be mentioned that the work of Diethelm et al. is devoted to establishing numerical solutions of several classes of linear and nonlinear fractional differential equations [5–10]. Trinks and Ruge have proposed a numerical scheme for the solution of Bagley–Torvik equation [18]. And at present an attempt has been made to obtain the solutions of fractional differential equations by the decomposition method, as the decomposition method provides an effective procedure for the analytical solution of large classes of linear and nonlinear differential equations [1–4,13,16,17]. However, in spite of extensive studies for fractional differential equations, the literature is rather sparse for generalized-order fractional differential equations.

In this paper, we consider a class of linear fractional differential equations of arbitrary order with constant coefficients and make use of Adomian decomposition method to obtain the solutions of these fractional differential equations. We verify that the solutions obtained by Adomian decomposition method are the same as the solutions expressed by the Green's function, which are convergent as the solutions expressed by the Green's function are convergent expansions in power series [15]. And we show the convergence of the solution by Adomian decomposition method.

The paper is organized as follows. In Section 2, we give definitions of the fractional derivative and fractional integral with some basic properties. In Section 3, the analytical solution of the linear fractional differential equation of arbitrary order is given by Adomian decomposition method and verified to the same as the one by the Green's function method. In Section 4, the convergence of the solution is considered. And we show some concrete examples in Section 5.

## 2. Preliminaries and notations

In this section we give the definition of the Riemann–Liouville fractional derivative and fractional integral with some basic properties. At the same time we review the application of Adomian decomposition method to differential equations and the properties of multiple infinite sums.

**Definition 1** (*Podlubny [15]*). The Riemann–Liouville fractional derivative of order $p$ with respect to the variable $t$ and with the starting point at $t = a$ is

$$ {}_aD_t^p f(t) = \begin{cases} \dfrac{1}{\Gamma(-p+m+1)} \dfrac{d^{m+1}}{dt^{m+1}} \int_a^t (t-\tau)^{m-p} f(\tau)\,\mathrm{d}\tau, & 0 \leqslant m \leqslant p < m+1, \\ \dfrac{d^m f(t)}{dt^m}, & p = m+1 \in N. \end{cases} $$

**Definition 2** (*Podlubny [15]*). The Riemann–Liouville fractional integral of order $p$ is

$$ {}_aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau)\,\mathrm{d}\tau, \quad p > 0. $$

For convenience, we denote ${}_aD_t^p$ by $D^p$, i.e. $D^p \equiv {}_aD_t^p$, for $p$ is real.

**Lemma 3** (*Podlubny [15]*). *If $f(t)$ is continuous, then the following relationship holds*:

$$ {}_aD_t^p({}_aD_t^{-q} f(t)) = {}_aD_t^{p-q} f(t). $$

**Lemma 4** (*Podlubny [15]*). *If $f(t)$ is continuous, then the following relationship holds*:

$$ {}_aD_t^{-p}({}_aD_t^{-q} f(t)) = {}_aD_t^{-p-q} f(t) = {}_aD_t^{-q}({}_aD_t^{-p} f(t)). $$

*Adomian decomposition method*: [2–4] Let us consider the differential equation in the form

$$ Lu + Ru + Nu = g, \tag{1} $$

where $L$ is an invertible linear operator, $R$ is the remainder of the linear operator and $N$ is a nonlinear operator. Applying the inverse operator $L^{-1}$ to both sides of Eq. (1), we obtain

$$u = \Phi + L^{-1}g - L^{-1}Ru - L^{-1}Nu, \tag{2}$$

where $\Phi$ arises from the given initial condition.

The Adomian decomposition method assumes the solution $u$ described by the series

$$u = \sum_{n=0}^{\infty} u_n. \tag{3}$$

By substituting the above equation in Eq. (2), we obtain the recursive relationship:

$$u_0 = \Phi + L^{-1}g,$$
$$u_n = -L^{-1}Ru_{n-1} - L^{-1}Nu_{n-1}, \quad n \geqslant 1. \tag{4}$$

Hence the general solution $u$ is obtained since each term $u_i$ is calculated.

The following properties of multiple infinite sums are useful to get the main result in Section 3.

**Proposition 5.**

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1,k_2,\ldots,k_{n-1},k_n} = \sum_{m=0}^{\infty} \sum_{\substack{k_1,\ldots,k_{n-1},k_n \geqslant 0 \\ k_1+\cdots+k_{n-1}+k_n=m}} a_{k_1,k_2,\ldots,k_{n-1},k_n}.$$

**Proof.** It is well known that an infinite double sums can be written in terms of a single infinite series:

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1,k_2} = \sum_{m=0}^{\infty} \sum_{\substack{k_1,k_2 \geqslant 0 \\ k_1+k_2=m}} a_{k_1,k_2}.$$

So the identity is true when $n = 2$. Now, let us assume that

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_{n-1}=0}^{\infty} a_{k_1,k_2,\ldots,k_{n-1}} = \sum_{s=0}^{\infty} \sum_{\substack{k_1,\ldots,k_{n-1} \geqslant 0 \\ k_1+\cdots+k_{n-1}=s}} a_{k_1,k_2,\ldots,k_{n-1}}.$$

Then

$$\begin{aligned}
\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1,k_2,\ldots,k_{n-1},k_n} &= \sum_{k_n=0}^{\infty} \sum_{k_1=0}^{\infty} \cdots \sum_{k_{n-1}=0}^{\infty} a_{k_1,k_2,\ldots,k_{n-1},k_n} \\
&= \sum_{k_n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{\substack{k_1,\ldots,k_{n-1} \geqslant 0 \\ k_1+\cdots+k_{n-1}=s}} a_{k_1,k_2,\ldots,k_{n-1},k_n} \\
&= \sum_{m=0}^{\infty} \sum_{\substack{k_n+s=m \\ k_n \geqslant 0, s \geqslant 0}} \sum_{\substack{k_1,\ldots,k_{n-1} \geqslant 0 \\ k_1+\cdots+k_{n-1}=s}} a_{k_1,k_2,\ldots,k_{n-1},k_n} \\
&= \sum_{m=0}^{\infty} \sum_{\substack{k_1,\ldots,k_{n-1},k_n \geqslant 0 \\ k_1+\cdots+k_{n-1}+k_n=m}} a_{k_1,k_2,\ldots,k_{n-1},k_n}. \qquad \square
\end{aligned}$$

Thus, it is easy to see

**Proposition 6.**

$$\sum_{m=0}^{\infty}\sum_{\substack{k_1,\ldots,k_{n-1},k_n\geqslant 0\\ k_1+\cdots+k_{n-1}+k_n=m}} a_{k_1,k_2,\ldots,k_{n-1},k_n}=\sum_{s=0}^{\infty}\sum_{\substack{k_1,\ldots,k_{n-1}\geqslant 0\\ k_1+\cdots+k_{n-1}=s}}\sum_{k_n=0}^{\infty} a_{k_1,k_2,\ldots,k_{n-1},k_n}.$$

## 3. The analytical solution of the linear constant coefficients fractional differential equation

In this section, we make use of Adomian decomposition method to present the analytical solution of the linear fractional-order differential equation with constant coefficients. After some transmutations and calculations, we show that the solution by Adomian decomposition method is the same as the solution by the Green's function.

Consider the following $n$-term fractional-order differential equation with constant coefficients

$$a_n D^{\beta_n} y(t)+a_{n-1}D^{\beta_{n-1}}y(t)+\cdots+a_1 D^{\beta_1}y(t)+a_0 D^{\beta_0}y(t)=f(t), \tag{5}$$

$$y^{(i)}(0)=0 \quad \text{for } i=0,1,\ldots,n, \tag{6}$$

where $D^p \equiv {}_0D_t^p$, $n+1>\beta_n \geqslant n>\beta_{n-1}>\cdots>\beta_1>\beta_0$ and $a_i\,(i=0,1,\ldots,n)$ is a real constant.

By applying the inverse operator $D^{-\beta_n}$ to Eq. (5), based on Lemma 3 and the initial condition (6), we have

$$y(t)+\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n}y(t)+\cdots+\frac{a_0}{a_n}D^{\beta_0-\beta_n}y(t)=\frac{1}{a_n}D^{-\beta_n}f(t), \tag{7}$$

Using the Adomian decomposition method and Lemma 4, we obtain the recursive relationship:

$$\begin{aligned}
y_0(t)&=\frac{1}{a_n}D^{-\beta_n}f(t),\\
y_1(t)&=-\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n}+\cdots+\frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)y_0(t),\\
y_2(t)&=(-1)^2\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n}+\cdots+\frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)^2 y_0(t),\\
&\cdots\quad\cdots\\
y_s(t)&=(-1)^s\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n}+\cdots+\frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)^s y_0(t),\\
&\cdots\quad\cdots
\end{aligned} \tag{8}$$

Adding all terms of the recursion, we obtain the solution by Adomian decomposition method

$$\begin{aligned}
y(t)&=\sum_{s=0}^{\infty}y_s(t)\\
&=\sum_{s=0}^{\infty}(-1)^s\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n}+\cdots+\frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)^s y_0(t)\\
&=\frac{1}{a_n}\sum_{s=0}^{\infty}(-1)^s\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n}+\cdots+\frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)^s D^{-\beta_n}f(t)
\end{aligned}$$

$$= \frac{1}{a_n} \sum_{s=0}^{\infty} (-1)^s \sum_{\substack{k_0, k_1, \ldots, k_{n-1} \geqslant 0 \\ k_0 + k_1 + \cdots + k_{n-1} = s}} \frac{s!}{k_0! k_1! \cdots k_{n-1}!} \left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}} \left(\frac{a_{n-2}}{a_n}\right)^{k_{n-2}} \cdots \left(\frac{a_0}{a_n}\right)^{k_0}$$

$$\times D^{k_{n-1}(\beta_{n-1}-\beta_n)+k_{n-2}(\beta_{n-2}-\beta_n)+\cdots+k_0(\beta_0-\beta_n)-\beta_n} f(t)$$

$$= \frac{1}{a_n} \sum_{s=0}^{\infty} (-1)^s \sum_{\substack{k_0, k_1, \ldots, k_{n-1} \geqslant 0 \\ k_0 + k_1 + \cdots + k_{n-1} = s}} \frac{s!}{k_0! k_1! \cdots k_{n-1}!} \left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}} \left(\frac{a_{n-2}}{a_n}\right)^{k_{n-2}} \cdots \left(\frac{a_0}{a_n}\right)^{k_0}$$

$$\times \frac{\int_0^t (t-\tau)^{\beta_n + k_{n-1}(\beta_n - \beta_{n-1}) + k_{n-2}(\beta_n - \beta_{n-2}) + \cdots + k_0(\beta_n - \beta_0) - 1} f(\tau)\, \mathrm{d}\tau}{\Gamma(\beta_n + k_{n-1}(\beta_n - \beta_{n-1}) + k_{n-2}(\beta_n - \beta_{n-2}) + \cdots + k_0(\beta_n - \beta_0))}. \tag{9}$$

Utilizing the properties of the infinite sums (Proposition 5 and 6), the above solution can be rewritten in an equivalent form:

$$y(t) = \frac{1}{a_n} \sum_{k_{n-1}=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\substack{k_0, k_1, \ldots, k_{n-2} \geqslant 0 \\ k_0 + k_1 + \cdots + k_{n-2} = m}} (-1)^{k_{n-1}+m} \frac{(k_{n-1}+m)!}{k_0! k_1!, \ldots, k_{n-1}!}$$

$$\times \left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}} \left(\frac{a_{n-2}}{a_n}\right)^{k_{n-2}} \cdots \left(\frac{a_0}{a_n}\right)^{k_0}$$

$$\times \frac{\int_0^t (t-\tau)^{\beta_n + k_{n-1}(\beta_n - \beta_{n-1}) + k_{n-2}(\beta_n - \beta_{n-2}) + \cdots + k_0(\beta_n - \beta_0) - 1} f(\tau)\, \mathrm{d}\tau}{\Gamma(\beta_n + k_{n-1}(\beta_n - \beta_{n-1}) + k_{n-2}(\beta_n - \beta_{n-2}) + \cdots + k_0(\beta_n - \beta_0))}$$

$$= \int_0^t \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0, k_1, \ldots, k_{n-2} \geqslant 0 \\ k_0 + k_1 + \cdots + k_{n-2} = m}} \frac{m!}{k_0! k_1!, \ldots, k_{n-2}!}$$

$$\times \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n}\right)^{k_i} (t-\tau)^{(\beta_n - \beta_{n-1})m + \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j - 1}$$

$$\times \sum_{k_{n-1}=0}^{\infty} (-1)^{k_{n-1}} \left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}} \frac{(k_{n-1}+m)!}{(k_{n-1})!}$$

$$\times \frac{(t-\tau)^{k_{n-1}(\beta_n - \beta_{n-1})}}{\Gamma(k_{n-1}(\beta_n - \beta_{n-1}) + m(\beta_n - \beta_{n-1}) + \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j)} f(\tau)\, \mathrm{d}\tau$$

$$= \int_0^t \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0, k_1, \ldots, k_{n-2} \geqslant 0 \\ k_0 + k_1 + \cdots + k_{n-2} = m}} (m; k_0, k_1, \ldots, k_{n-2})$$

$$\times \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n}\right)^{k_i} (t-\tau)^{(\beta_n - \beta_{n-1})m + \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j - 1}$$

$$\times E^{(m)}_{\beta_n - \beta_{n-1}, \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j} \left(-\frac{a_{n-1}}{a_n} (t-\tau)^{\beta_n - \beta_{n-1}}\right) f(\tau)\, \mathrm{d}\tau, \tag{10}$$

where $(m; k_0, k_1, \ldots, k_{n-2})$ are the multinomial coefficients and the function

$$G_n(t) = \frac{1}{a_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{\substack{k_0,k_1,\ldots,k_{n-2}\geqslant 0\\ k_0+k_1+\cdots+k_{n-2}=m}}(m; k_0, k_1, \ldots, k_{n-2})$$
$$\times \prod_{i=0}^{n-2}\left(\frac{a_i}{a_n}\right)^{k_i} t^{(\beta_n-\beta_{n-1})m+\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j-1}$$
$$\times E^{(m)}_{\beta_n-\beta_{n-1},\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j}\left(-\frac{a_{n-1}}{a_n}t^{\beta_n-\beta_{n-1}}\right) \tag{11}$$

is the fractional Green's function for Eq. (5) and $E^{(i)}_{\lambda,\mu}(y)$ is Mittag-Leffler function:

$$E^{(i)}_{\lambda,\mu}(y) = \frac{d^i}{dy^i}E_{\lambda,\mu}(y) = \sum_{j=0}^{\infty}\frac{(i+j)!y^j}{j!\Gamma(\lambda j+\lambda i+\mu)}.$$

So it is readily seen that the solution by Adomian decomposition method is actually the solution by the Green's function [15], i.e.

$$y(t) = \int_0^t G_n(t-\tau)f(\tau)\,\mathrm{d}\tau.$$

## 4. Analysis of the convergence

In this section, we show the convergence of the solution by Adomian decomposition method.
Based on Eq. (8), we have

$$|y_s(t)| = |(-1)^s\left(\frac{a_{n-1}}{a_n}D^{\beta_{n-1}-\beta_n}+\cdots+\frac{a_0}{a_n}D^{\beta_0-\beta_n}\right)^s y_0(t)|$$
$$= |(-1)^s\sum_{\substack{k_0,k_1,\ldots,k_{n-1}\geqslant 0\\ k_0+k_1+\cdots+k_{n-1}=s}}\frac{s!}{k_0!k_1!\cdots k_{n-1}!}\left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}}\left(\frac{a_{n-2}}{a_n}\right)^{k_{n-2}}\cdots\left(\frac{a_0}{a_n}\right)^{k_0}$$
$$\times D^{k_{n-1}(\beta_{n-1}-\beta_n)+k_{n-2}(\beta_{n-2}-\beta_n)+\cdots+k_0(\beta_0-\beta_n)}y_0(t)|$$
$$= \left|\sum_{\substack{k_0,k_1,\ldots,k_{n-1}\geqslant 0\\ k_0+k_1+\cdots+k_{n-1}=s}}\frac{s!}{k_0!k_1!\cdots k_{n-1}!}\left(\frac{a_{n-1}}{a_n}\right)^{k_{n-1}}\left(\frac{a_{n-2}}{a_n}\right)^{k_{n-2}}\cdots\left(\frac{a_0}{a_n}\right)^{k_0}\right.$$
$$\left.\times\frac{\int_0^t(t-\tau)^{k_{n-1}(\beta_n-\beta_{n-1})+k_{n-2}(\beta_n-\beta_{n-2})+\cdots+k_0(\beta_n-\beta_0)-1}y_0(\tau)\,\mathrm{d}\tau}{\Gamma(k_{n-1}(\beta_n-\beta_{n-1})+k_{n-2}(\beta_n-\beta_{n-2})+\cdots+k_0(\beta_n-\beta_0))}\right|$$

$$\geqslant L \cdot \sum_{\substack{k_0,k_1,\ldots,k_{n-1}\geqslant 0\\ k_0+k_1+\cdots+k_{n-1}=s}} \frac{s!}{k_0!k_1!\cdots k_{n-1}!} \cdot \frac{(M|t|^{(\beta_n-\beta_{n-1})})^{k_{n-1}}\ldots(M|t|^{(\beta_n-\beta_0)})^{k_0}}{\Gamma(k_{n-1}(\beta_n-\beta_{n-1})+\cdots+k_0(\beta_n-\beta_0)+1)}$$

$$= L \cdot \sum_{\substack{k_0,k_1,\ldots,k_{n-1}\geqslant 0\\ k_0+k_1+\cdots+k_{n-1}=s}} \frac{(s;k_0,k_1,\ldots,k_{n-1})\prod_{i=0}^{n-1} z_i^{k_i}}{\Gamma(1+\sum_{i=0}^{n-1}k_i\gamma_i)}, \tag{12}$$

where $M=\max\{|(a_{n-1})/a_n|,\ |(a_{n-2})/a_n|,\ldots,|a_0/a_n|\}$, $L=\max_{t\in[0,T]}|y_0(t)|$, $\gamma_i=\beta_n-\beta_i (i=0,\ldots,n-1)$, $z_i=M|t|^{\gamma_i} (i=0,\ldots,n-1)$,$(s;k_0,k_1,\ldots,k_{n-1})$ are multinomial coefficients.

Hence, $\sum_{s=0} y_s(t) \leqslant L\cdot E_{(\gamma_0,\ldots,\gamma_{n-1}),\beta}(z_0,\ldots,z_{n-1})$, where $\beta=1$ and $E_{(\gamma_0,\ldots,\gamma_{n-1}),\beta}(z_0,\ldots,z_{n-1})$ is the multivariate Mittag–Leffler function

$$E_{(\gamma_0,\ldots,\gamma_{n-1}),\beta}(z_0,\ldots,z_{n-1}) = \sum_{s=0}^{\infty} \sum_{\substack{k_0,k_1,\ldots,k_{n-1}\geqslant 0\\ k_0+k_1+\cdots+k_{n-1}=s}} \frac{(s;k_0,k_1,\ldots,k_{n-1})\prod_{i=0}^{n-1} z_i^{k_i}}{\Gamma(\beta+\sum_{i=0}^{n-1}k_i\gamma_i)}$$

which is convergent [11]. So we obtain the convergence of the solution by Adomian decomposition method.

## 5. Illustration

In this section, we give some examples to demonstrate the conclusion that the solution by Adomian decomposition method is actually the solution by the Green's function.

**Example 1.** Relaxation–oscillation equation [15]

Let us consider an initial value problem for the relaxation–oscillation equation

$$\begin{aligned} &{}_0D_t^{\alpha} y(t) + Ay(t) = f(t) \quad (t>0),\\ &y^{(k)}(0)=0 \quad (k=0,1,\ldots,n-1), \end{aligned} \tag{13}$$

where $n-1 \leqslant \alpha < n$.

Following the above procedure of solving the $n$-term linear fractional-order differential equation and using Adomian decomposition method, we have

$$\begin{aligned} &y_0 = D^{-\alpha} f(t),\\ &y_1 = -AD^{-\alpha} y_0 = -AD^{-2\alpha} f(t),\\ &y_2 = -AD^{-\alpha} y_1 = (-A)^2 D^{-3\alpha} f(t),\\ &\ldots \quad \ldots\\ &y_n = -AD^{-\alpha} y_{n-1} = (-A)^n D^{-(n+1)\alpha} f(t),\\ &\ldots \quad \ldots \end{aligned} \tag{14}$$

Thus, we obtain the solution $y(t)$ by Adomian decomposition method which is the same as the solution by the Green's function:

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n \\ &= \sum_{n=0}^{\infty} (-A)^n D^{-(n+1)\alpha} f(t) \\ &= \sum_{n=0}^{\infty} (-A)^n \frac{1}{\Gamma((n+1)\alpha)} \int_0^t (t-\tau)^{(n+1)\alpha-1} f(\tau)\,\mathrm{d}\tau \\ &= \int_0^t \sum_{n=0}^{\infty} (t-\tau)^{(\alpha-1)} \frac{(-A)^n (t-\tau)^{n\alpha}}{\Gamma(n\alpha+\alpha)} f(\tau)\,\mathrm{d}\tau \\ &= \int_0^t G_2(t-\tau) f(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{15}$$

where $G_2(t) = t^{\alpha-1} E_{\alpha,\alpha}(-At^{\alpha})$.

**Example 2.** Bagley–Torvik equation [15,16]

Let us consider the following initial value equation problem for the inhomogeneous Bagley–Torvik equation

$$\begin{aligned} &Ay''(t) + BD^{3/2} y(t) + Cy(t) = f(t) \quad (t > 0), \\ &y(0) = 0, \quad y'(0) = 0. \end{aligned} \tag{16}$$

Following the above procedure of solving the $n$-term linear fractional-order differential equation and using Adomian decomposition method, we have

$$\begin{aligned} y_0 &= \frac{1}{A} D^{-2} f(t), \\ y_1 &= -\frac{C}{A}\left(\frac{B}{C} I + D^{-3/2}\right) D^{-1/2} y_0, \\ y_2 &= -\frac{C}{A}\left(\frac{B}{C} I + D^{-3/2}\right) D^{-1/2} y_1 = \left(-\frac{C}{A}\right)^2 \left(\frac{B}{C} I + D^{-3/2}\right)^2 D^{-2/2} y_0, \\ y_3 &= -\frac{C}{A}\left(\frac{B}{C} I + D^{-3/2}\right) D^{-1/2} y_2 = \left(-\frac{C}{A}\right)^3 \left(\frac{B}{C} I + D^{-3/2}\right)^3 D^{-3/2} y_0, \\ &\cdots \quad \cdots \\ y_n &= -\frac{C}{A}\left(\frac{B}{C} I + D^{-3/2}\right) D^{-1/2} y_{n-1} = \left(-\frac{C}{A}\right)^n \left(\frac{B}{C} I + D^{-3/2}\right)^n D^{-n/2} y_0, \\ &\cdots \quad \cdots \end{aligned} \tag{17}$$

Thus, we obtain the solution $y(t)$ by Adomian decomposition method which is the same as the solution by the Green's function:

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} y_n \\
&= \sum_{n=0}^{\infty}\left(-\frac{C}{A}\right)^n\left(\frac{B}{C}I + D^{-32}\right)^n D^{-n/2}y_0 \\
&= \sum_{n=0}^{\infty}\left(-\frac{C}{A}\right)^n \sum_{i=0}^{n} C_n^{n-i}\left(\frac{B}{C}I\right)^{n-i} D^{-3/2i}D^{-n/2}\frac{1}{A}D^{-2}f(t) \\
&= \frac{1}{A}\sum_{n=0}^{\infty}\left(-\frac{C}{A}\right)^n \sum_{i=0}^{n}\frac{n!}{(n-i)!i!}\left(\frac{B}{C}\right)^{n-i} D^{-(3i/2+n/2+2)}f(t) \\
&= \frac{1}{A}\int_0^t \sum_{n=0}^{\infty}\left(-\frac{C}{A}\right)^n \sum_{i=0}^{n}\frac{n!}{(n-i)!i!}\left(\frac{B}{C}\right)^{n-i}\frac{1}{\Gamma(3i/2+n/2+2)}(t-\tau)^{3i/2+n/2+2-1}f(\tau)\,\mathrm{d}\tau \\
&= \frac{1}{A}\int_0^t \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(-\frac{C}{A}\right)^{i+j}\frac{(i+j)!}{j!i!}\left(\frac{B}{C}\right)^{j}\frac{1}{\Gamma(3i/2+(i+j)/2+2)}(t-\tau)^{3i/2+(i+j)/2+2-1}f(\tau)\,\mathrm{d}\tau \\
&= \frac{1}{A}\int_0^t \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{C}{A}\right)^i t^{2i+1}\sum_{j=0}^{\infty}\frac{(i+j)!(-\frac{B}{A}\sqrt{t-\tau})^j}{j!\Gamma(j/2+i/2+3i/2+2)}f(\tau)\,\mathrm{d}\tau \\
&= \int_0^t G_3(t-\tau)f(\tau)\,\mathrm{d}\tau, \qquad (18)
\end{aligned}
$$

where $G_3(t) = (1/A)\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}(\frac{C}{A})^i t^{2i+1}E^{(i)}_{1/2,2+3i/2}(-\frac{B}{A}\sqrt{t})$.

## 6. Conclusion

In this paper, we analyze the solution of the $n$-term linear fractional-order differential equation with constant coefficients by Adomian decomposition method. And we verify that the solution by Adomian decomposition method is actually the same as the solution by the Green's function.

## Acknowledgements

The authors would like to express their thanks to the referee and Professor B. Sommeijer for their valuable suggestions and comments.

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