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On the existence of some ARCH(∞) processes

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Abstract

A new sufficient condition for the existence of a stationary causal solution of an ARCH(∞) equation is provided. This condition allows us to consider coefficients with power-law decay, so that it can be applied to the so-called FIGARCH processes, whose existence is thus proved.

MSC: 60G10; 62M10

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1. Introduction

It can arguably be said that autoregressive conditionally heteroskedastic (ARCH) and long memory processes are two success stories of the nineties, so that they were bound to meet. Their tentative offspring was the FIGARCH process, introduced by Baillie et al. [1] without proving its existence, which has remained controversial up to now. More precisely, the FIGARCH(p, d, q) process is the solution of the equations

$$X_n = \sigma_n z_n, \tag{1}$$

$$\sigma_n^2 = a_0 + \left\{ I - (I - L)^d \frac{\theta(L)}{\phi(L)} \right\} X_n^2, \tag{2}$$

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where $\{z_n\}$ is an i.i.d. sequence with zero mean and unit variance, $a_0 > 0$, $d \in (0, 1)$, L is the backshift operator and $(I - L)^d$ is the fractional differencing operator:

$$(I - L)^d = I + \sum_{j=1}^{\infty} \frac{(-d)(1-d)\cdots(j-1-d)}{j!} L^j,$$

and θ and ϕ are polynomials such that $\theta(0) = \phi(0) = 1$, $\phi(z) \neq 0$ for all complex number z in the closed unit disk and the coefficients of the series expansion of $1 - (1 - z)^d \theta(z)/\phi(z)$ are nonnegative. Then the coefficients $\{a_j\}_{j\geq 1}$ defined by $\sum_{j=1}^{\infty} a_j L^j = I - (I - L)^d \theta(L)/\phi(L)$ satisfy $a_j \sim c j^{-d-1}$ for some constant $c > 0$ and $\sum_{j=1}^{\infty} a_j = 1$.

These processes are subcases of what can be called IARCH(∞), defined as solutions of Eq. (1) and

$$\sigma_n^2 = a_0 + \sum_{j=1}^{\infty} a_j X_{n-j}^2, \tag{3}$$

for some sequence $\{a_j\}$ such that $a_0 > 0$ and $\sum_{j=1}^{\infty} a_j = 1$. The letter I stands for integrated, in analogy to ARIMA processes. An important property of such processes is that a stationary solution necessarily has infinite variance. Indeed, if $\sigma^2 = \mathbb{E}[\sigma_n^2] < \infty$, then $\mathbb{E}[X_n^2] = \sigma^2$ and (3) implies $\sigma^2 = a_0 + \sigma^2$, which is impossible. If the condition $\sum_{j=1}^{\infty} a_j = 1$ is not imposed, a solution to Eqs. (1) and (3) is simply called an ARCH(∞) process.

A solution of an ARCH(∞) equation is said to be causal with respect to the i.i.d. sequence $\{z_n\}$ if for all n, σ_n is $\mathcal{F}_{n-1}^z$ measurable, where $\mathcal{F}_n^z$ is the sigma-field generated by $\{z_n, z_{n-1}, \ldots\}$. Note that to avoid trivialities, here and in the following, σ_n is the positive square root of σ_n^2. There exists an important literature on ARCH(∞), IARCH(∞) and FIGARCH processes. For a recent review, see for instance [4]. The known conditions for the existence of stationary causal conditions for ARCH equations are always a compromise between conditions on the distribution of the innovation sequence $\{z_n\}$ and summability conditions on the coefficients $\{a_j, j \geq 1\}$. Giraitis and Surgailis [5] provide a necessary and sufficient condition for the solution to have finite fourth moment. The only rigorous result in the IARCH(∞) case was obtained by Kazakevičius and Leipus [6]. They prove the existence of a causal stationary solution under the condition that the coefficients a_j decay geometrically fast, which rules out FIGARCH processes, and a mild condition on the distribution of z_0.

The purpose of this paper is to provide a new sufficient condition for the existence of a stationary solution to an ARCH(∞) equation, which allows power-law decay of the coefficients a_j, even in the IARCH(∞) case. This condition is stated in Section 2. It is applied to the IARCH(∞) case in Section 3 and the existence of a stationary solution to the FIGARCH equation is proved. Further research directions are given in Section 4. In particular, the memory properties of FIGARCH processes are still to be investigated. This is an important issue, since the original motivation for these processes was the modelling of long memory in volatility.

2. A sufficient condition for the existence of ARCH(∞) processes

Theorem 1. *Let $\{a_j\}_{j\geq 0}$ be a sequence of nonnegative real numbers and $\{z_k\}_{k\in\mathbb{Z}}$ a sequence of i.i.d. random variables. For $p > 0$, define*

$$A_p = \sum_{j=1}^{\infty} a_j^p \quad \text{and} \quad \mu_p = \mathbb{E}[z_0^{2p}].$$

If there exists $p \in (0, 1]$ such that

$$A_p \mu_p < 1, \tag{4}$$

then there exists a strictly stationary solution of the ARCH(∞) equation:

$$X_n = \sigma_n z_n, \tag{5}$$

$$\sigma_n^2 = a_0 + \sum_{j=1}^{\infty} a_j X_{n-j}^2, \tag{6}$$

given by (5) and

$$\sigma_n^2 = a_0 + a_0 \sum_{k=1}^{\infty} \sum_{j_1,\dots,j_k \ge 1} a_{j_1} \cdots a_{j_k} z_{n-j_1}^2 \cdots z_{n-j_1-\dots-j_k}^2. \tag{7}$$

The process $\{X_n\}$ so defined is the unique causal stationary solution to Eqs. (5) and (6) such that $\mathbb{E}[|X_n|^{2p}] < \infty$.

Proof. Define $\xi_k = z_k^2$, so that $\mathbb{E}[\xi_k^p] = \mu_p$, and define the $[0, \infty]$-valued r.v.

$$S_0 = a_0 + a_0 \sum_{k=1}^{\infty} \sum_{j_1,\dots,j_k \ge 1} a_{j_1} \cdots a_{j_k} \xi_{-j_1} \cdots \xi_{-j_1-\dots-j_k}. \tag{8}$$

Since $p \in (0, 1]$, we apply the inequality $(a + b)^p \le a^p + b^p$ valid for all $a, b \ge 0$ to S_0^p:

$$S_0^p \le a_0^p + a_0^p \sum_{k=1}^{\infty} \sum_{j_1,\dots,j_k \ge 1} a_{j_1}^p \cdots a_{j_k}^p \xi_{-j_1}^p \cdots \xi_{-j_1-\dots-j_k}^p.$$

Then, by independence of the ξ_j's, we obtain

$$\begin{aligned} \mathbb{E}[S_0^p] &\le a_0^p + a_0^p \sum_{k=1}^{\infty} \sum_{j_1,\dots,j_k \ge 1} a_{j_1}^p \cdots a_{j_k}^p \mathbb{E}[\xi_{-j_1}^p \cdots \xi_{-j_1-\dots-j_k}^p] \\ &= a_0^p \left[1 + \sum_{k=1}^{\infty} (\mu_p A_p)^k \right] = \frac{a_0^p}{1 - A_p \mu_p}, \end{aligned} \tag{9}$$

where we used (4). This bound shows that $S_0 < \infty$ a.s. and the sequence

$$S_n = a_0 + a_0 \sum_{k=1}^{\infty} \sum_{j_1,\dots,j_k \ge 1} a_{j_1} \cdots a_{j_k} \xi_{n-j_1} \cdots \xi_{n-j_1-\dots-j_k}, \quad n \in \mathbb{Z},$$

is a sequence of a.s. finite r.v.'s. Since only nonnegative numbers are involved in the summation, we may write

$$\begin{aligned} \sum_{j=1}^{\infty} a_j S_{n-j} \xi_{n-j} &= a_0 \sum_{j_0=1}^{\infty} a_{j_0} \xi_{n-j_0} \\ &\quad + a_0 \sum_{j_0=1}^{\infty} a_{j_0} \xi_{n-j_0} \sum_{k=1}^{\infty} \sum_{j_1,\dots,j_k \ge 1} a_{j_1} \cdots a_{j_k} \xi_{n-j_0-j_1} \cdots \xi_{n-j_0-j_1-\dots-j_k} \\ &= a_0 \sum_{k=0}^{\infty} \sum_{j_0, j_1,\dots,j_k \ge 1} a_{j_0} \cdots a_{j_k} \xi_{n-j_0} \cdots \xi_{n-j_0-j_1-\dots-j_k}. \end{aligned}$$

Hence $\{S_n, n \in \mathbb{Z}\}$ satisfies the recurrence equation

$$S_n = a_0 + \sum_{j=1}^{\infty} a_j S_{n-j} \xi_{n-j}.$$

The technique of infinite chaotic expansions used here is standard; it was already used in the proof of [7, Theorem 2.1]. This proves the existence of a strictly stationary solution for (5) and (6) by setting $\sigma_n^2 = S_n$ and $X_n = \sigma_n z_n$. Using (9), we moreover have $\mathbb{E}[|X_n|^{2p}] \leq \mu_p a_0^p / (1 - A_p \mu_p)$.

Suppose now that $\{X_n\}$ is a strictly stationary causal solution of the ARCH(∞) equations (5) and (6). Then, for any $q \geq 1$, the following expansion holds:

$$\sigma_n^2 = a_0 + a_0 \sum_{k=0}^{q} \sum_{j_1,\ldots,j_k \geq 1} a_{j_1} \cdots a_{j_k} \xi_{n-j_1} \cdots \xi_{n-j_1-\cdots-j_k} \tag{10}$$

$$+ \sum_{j_1,\ldots,j_{q+1} \geq 1} a_{j_1} \cdots a_{j_{q+1}} \xi_{n-j_1} \cdots \xi_{n-j_1-\cdots-j_q} X^2_{n-j_1-\cdots-j_{q+1}}. \tag{11}$$

The last display implies that the series on the right-hand side of (10) converges to S_n as $q \to \infty$. Denote by $R_{n,q}$ the remainder term in (11). Since $\{X_n\}$ is a causal solution, $X_{n-j_1-\cdots-j_{q+1}}$ is independent of $\xi_{n-j_1} \cdots \xi_{n-j_1-\cdots-j_q}$ for all $j_1, \ldots, j_{q+1} \geq 1$. Hence, for any $p \leq 1$,

$$\mathbb{E}[R_{n,q}^p] \leq (A_p \mu_p)^q \mathbb{E}[X_0^{2p}].$$

If assumption (4) holds and $\mathbb{E}[X_0^{2p}] < \infty$, then $\mathbb{E}[\sum_{q \geq 1} R_{n,q}^p] < \infty$ so that, as $q \to \infty$, $R_{n,q} \to 0$ a.s., implying $\sigma_n^2 = S_n$ a.s. □

3. IARCH(∞) processes

IARCH (Integrated ARCH) processes are particular ARCH(∞) processes for which $A_1 \mu_1 = 1$, or, equivalently up to a scale factor,

$$A_1 = 1 \quad \text{and} \quad \mu_1 = 1. \tag{12}$$

To the best of our knowledge, the only rigorous general result on IARCH(∞) processes was obtained by Kazakevičius and Leipus [6]. See [4] for a recent review. In Theorem 2.1 of [6], it is proved that if

$$\mathbb{E}[|\log(z_0)|^2] < \infty, \tag{13}$$

$$\sum_i a_i q^i < \infty \quad \text{for some } q > 1, \tag{14}$$

hold, then there exists a unique stationary causal solution to the ARCH(∞) equations (5) and (6). Condition (13) on the distribution of z_0 is mild, but the condition (14) rules out power-law decay of the coefficients $\{a_j\}$.

Theorem 1 yields the following sufficient condition for the existence of a IARCH(∞) process.

Corollary 2. *If $A_1 = 1$ and $\mu_1 = 1$, (4) holds for some $p \in (0, 1]$ if and only if there exists $p^* < 1$ such that $A_{p_*} < \infty$ and*

$$\sum_{j=1}^{\infty} a_i \log(a_i) + \mathbb{E}[z_0^2 \log(z_0^2)] \in (0, \infty]. \tag{15}$$

Then, the process defined by (5) and (7) is a solution of the ARCH(∞) *equation and* $\mathbb{E}[|X_n|^q] < \infty$ *for all* $q \in [0, 2)$ *and* $\mathbb{E}[X_n^2] = \infty$.

Proof. Since $a_i \leq 1$ for all $i \geq 1$, it holds that $\sum_{j=1}^{\infty} a_i \log(a_i) \leq 0$ and the convexity of the function $x \mapsto x \log(x)$ implies $\mathbb{E}[z_0^2 \log(z_0^2)] \geq 0$.

First assume that there exists $p \in (0, 1]$ such that (4) holds. Since $A_1 = \mu_1 = 1$, then necessarily, $p < 1$ and for all $q \in [p, 1]$, $A_q < \infty$. Thus we can define the function $\phi : [p, 1] \to \mathbb{R}$ by

$$\phi(q) = \log(A_q \mu_q) = \log \sum_{j=1}^{\infty} a_j^q + \log \mathbb{E}[z_0^{2q}].$$

Hölder inequality implies that the functions $q \mapsto \log \sum_{j=1}^{\infty} a_j^q$ and $q \mapsto \log \mathbb{E}[z_0^{2q}]$ are both convex on $[p, 1]$. Thus ϕ is also convex on $[p, 1]$ and, since $\phi(p) < 0$ and $\phi(1) = 0$, the left derivative of ϕ at 1, which is given by the left-hand side of (15), is positive (possibly infinite).

Conversely suppose that there exists $p^* < 1$ such that $A_{p^*} < \infty$ and that (15) holds. Then ϕ is a convex function on $[p^*, 1]$ and (15) implies that $\phi(q) < 0$ for $q < 1$ sufficiently close to 1.

By convexity of ϕ and since $\phi(1) = 0$, we also get that $A_p \mu_p < 1$ implies $A_q \mu_q < 1$ for all $q \in [p, 1)$. Then, by Theorem 1, the process $\{X_n,\ n \in \mathbb{Z}\}$ defined by (7) and (6) is a solution to the ARCH(∞) equation and satisfies $\mathbb{E}[|X_0|^q] < \infty$ for all positive $q < 2$. □

Comments on Corollary 2. (i) Condition (15) is not easily comparable to conditions (13) and (14) of [6]. Condition (15) is not necessary for proving the existence of a causal stationary solution if the coefficients a_j decay geometrically fast (in particular if there are only finitely many nonvanishing coefficients), as a consequence of [6, Theorem 2.1]; however, this result does not prove that any moments of X_n are finite, contrary to Corollary 2.

(ii) It might also be of interest to note that the Lyapunov exponent of the FIGARCH process as defined in [6] is zero. So our result proves that such a feature is not in contradiction with strict stationarity.

(iii) In the specific case of IGARCH processes, which are particular parametric subclasses of IARCH(∞) processes, [3] have a different set of assumptions on the distribution of z_0: they assume that $\mathbb{P}(z_0^2 = 0) = 0$ and that the support of the distribution of z_0^2 is unbounded.

(iv) The moment $\mathbb{E}[z_0^2 \log(z_0^2)]$ can be arbitrarily large (possibly infinite) if the distribution of z_0^2 has a sufficiently heavy tail. It is infinite for instance if the distribution of z_0^2 is absolutely continuous with a density bounded from below by $1/(x^2 \log^2(x))$ for x large enough. In that case, condition (15) holds for any sequence $\{a_j\}$ such that $A_{p^*} < \infty$ for some $p^* < 1$. This conditions allows for a power-law decay of the coefficients a_j, for instance $a_j \sim cj^{-\delta}$, for some $\delta > 1$.

Corollary 2 can be used to prove the existence of a causal strictly stationary solution to some FIGARCH(p, d, q) equations. Let us illustrate this in the case of the FIGARCH($0, d, 0$) equation, that is (5) and (6) with $d \in (0, 1)$, $a_0 > 0$ and $a_j = \pi_j(d)$ for all $j \geq 1$, where

$$\pi_1(d) = d, \qquad \pi_j(d) = \frac{d(1-d)\cdots(j-1-d)}{j!}, \quad j \geq 2.$$

Corollary 3. *Assume that* $\{z_k\}_{k\in\mathbb{Z}}$ *is a sequence of i.i.d. random variables such that* $\mathbb{E}[z_0^2] = 1$ *and* $\mathbb{P}\{|z_0| = 1\} < 1$. *Then there exists* $d^* \in [0, 1)$ *such that, for all* $d \in (d^*, 1)$, *the*

FIGARCH$(0, d, 0)$ *equation has a unique causal stationary solution satisfying* $\mathbb{E}[|X_n|^{2p}] < \infty$ *for all* $p < 1$.

Proof. For $d \in (0, 1]$ and $p \in (1/(d + 1), 1]$, define

$$H(p, d) = \log \sum_{j=1}^{\infty} \pi_j^p(d), \qquad L(d) = \sum_{j=1}^{\infty} \pi_j(d) \log(\pi_j(d)).$$

For $d \in (0, 1)$, $\pi_j(d) \sim cj^{-d-1}$, so that $H(p, d)$ is defined on $(1/(d + 1), 1]$. Moreover, it is decreasing and convex with respect to p, $H(1, d) = 0$ and $\partial_p H(1, d) = L(d)$. Also, $\pi_j(d)/d$ is a decreasing function of d and $\lim_{d\to 1} \pi_j(d) = 0$ for all $j \geq 2$. Thus, by bounded (and monotone) convergence, for all $p \in (1/2, 1)$, it holds that $\lim_{d\to 1, d<1} H(p, d) = 0$. By convexity of H with respect to p, the following bound holds:

$$0 \leq -L(d) \leq \frac{H(p, d)}{1 - p}.$$

Hence $\lim_{d\to 1} L(d) = 0$. By assumption, we have $\mathbb{E}[z_0^2 \log(z_0^2)] > 0$. This implies that there exists $d^* \in (0, 1)$ such that $L(d)+\mathbb{E}[z_0^2 \log(z_0^2)] > 0$ (i.e. (15) holds) if $d > d^*$. Thus Corollary 2 proves the existence of the corresponding FIGARCH$(0, d, 0)$ processes. □

Remark. It easily seen that $L(d) \leq \log(d)$ so that $\lim_{d\to 0} L(d) = -\infty$, i.e. (15) does not hold for small d. We conjecture, but could not prove, that $L(d)$ is increasing, so that (15) holds if *and only if* $d > d^*$ (with $d^* = 0$ if $\mathbb{E}[\xi_0 \log(\xi_0)] = \infty$). But this does not prove that the FIGARCH$(0, d, 0)$ does not exist for $d \leq d^*$.

4. Open problems

Now that a proof of existence of some FIGARCH and related processes is obtained under certain conditions, there still remain some open questions. We state a few of them here.

(i) Condition (15) is not necessary for the existence of a stationary causal solution, but it implies finiteness of all moments up to 1 of X_n^2 (with of course $\mathbb{E}[X_n^2] = \infty$). The problem remains open as regards knowing whether there exists a stationary solution under a mild assumption on z_0, such as (13) for instance. If a solution exists, say $\{X_n\}$, then, as seen in the proof of Theorem 1, the sequence $\{S_n\}$ defined in (8) is well defined and $Y_n = S_n^{1/2} z_n$ is also a stationary causal solution which satisfies moreover $Y_n^2 \leq X_n^2$. But we cannot prove without more assumptions that these solutions are equal.

(ii) Tail behaviours of the marginal distribution of GARCH processes have been investigated by Basrak et al. [2], following [8], but there are no such results in the ARCH(∞) case. Under suitable conditions, we have shown that the squares of the FIGARCH process X_n^2 have finite moments of all orders $p < 1$, but necessarily, $\mathbb{E}[X_n^2] = \infty$. Thus, it is natural to conjecture that perhaps under additional conditions on the distribution of z_0, the function $x \to \mathbb{P}(X_n^2 > x)$ is regularly varying with index -1.

(iii) The memory properties of the FIGARCH process are of course of great interest. The sequence $\{X_n\}$ is a strictly stationary martingale increment sequence, but $\mathbb{E}[X_n^2] = \infty$. So does it hold that the partial sum process $n^{-1/2} \sum_{k=1}^{[nt]} X_k$ converges weakly to the Brownian motion? For $p \in [1, 2)$, do the sequences $\{|X_n|^p\}$ have distributional long memory in the sense that $n^{-H} \sum_{k=1}^{[nt]} \{|X_k|^p - \mathbb{E}[|X_k|^p]\}$ converges to the fractional Brownian motion with Hurst index H for a suitable $H > 1/2$?

(iv) Statistical inference. The FIGARCH(p, d, q) is a parametric model, so the issue of estimation of its parameter is naturally raised. Also, if d is linked to some memory property of the process, semi-parametric estimation of d would be of interest.

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