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The number of convex polyominoes reconstructible from their orthogonal projections

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Abstract

Many problems of computer-aided tomography, pattern recognition, image processing and data compression involve a reconstruction of bidimensional discrete sets from their projections. [3–5, 10, 12, 16, 17]. The main difficulty involved in reconstructing a set Λ starting out from its orthogonal projections (V, H) is the 'ambiguity' arising from the fact that, in some cases, many different sets have the same projections (V, H). In this paper, we study this problem of ambiguity with respect to convex polyominoes, a class of bidimensional discrete sets that satisfy some connection properties similar to those used by some reconstruction algorithms. We determine an upper and lower bound to the maximum number of convex polyominoes having the same orthogonal projections (V, H), with $V \in \mathbb{N}^n$ and $H \in \mathbb{N}^m$. We prove that under these connection conditions, the ambiguity is sometimes exponential. We also define a construction in order to obtain some convex polyominoes having the same orthogonal projections.

Résumé

La reconstruction des ensembles discrets à deux dimensions par leurs projections est utilisée en plusieurs problèmes de tomographie assistée par ordinateur, reconnaissance de formes, elaboration d'images et compression de données [3–5, 10, 12, 16, 17]. La difficulté principale de la reconstruction d'un ensemble Λ à partir de ses projections orthogonales (V, H) est donnée par l'ambiguité qui vient de l'existence, en certains cas, de plusieurs ensembles ayant les mêmes projections (V, H). Dans ce papier, nous étudions ce problème d'ambiguité dans l'ensemble des polyominos convexes, une classe d'ensembles discrets à deux dimensions qui ont des propriétés de connession semblables à celles utilisées par quelques algorithmes de reconstruction. Nous déterminons une limitation supérieure et une limitation inférieure pour le nombre maximum de polyominos convexes qui satisfont les couples de vecteurs (V, H), avec $V \in \mathbb{N}^n$, $H \in \mathbb{N}^m$. Nous suggérons aussi une construction qui permet d'obtenir des polyominos convexes ayant les mêmes projections orthogonales.

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1. Introduction

A *cell* is a unitary square $[i, i+1] \times [j, j+1]$, in which $i, j \in \mathbb{N}_0$. The ith *column projection* and the jth *row projection* of a finite set of cells Λ are the number of cells in the ith row and jth column of Λ, respectively. The reconstruction of a set Λ of cells from its *orthogonal projections* (V,H) consists in determining a set Λ whose ith column projection and jth row projection are equal to v_i and h_j, respectively, in which $V = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$ and $H = (h_1, h_2, \ldots, h_m) \in \mathbb{N}^m$ are two assigned vectors. Several authors [1,3–5,7,13–15,19] have studied the theory of reconstructing objects from their projections and have developed various algorithms for determining Λ starting out from (V,H). The main difficulty involved in reconstructing a set Λ starting out from its orthogonal projections (V,H) is the 'ambiguity' arising from the fact that, in some cases, many different sets have the same projections (V,H). For example, if $V = (v_1, v_2, \ldots, v_n) = (1, 1, \ldots, 1)$ and $H = (h_1, h_2, \ldots, h_m) = (1, 1, \ldots, 1)$, $n!$ different sets have the same projections. In an effort to reduce this ambiguity and facilitate reconstruction, many authors have suggested giving 'a priori' some of set Λ's properties (such as convexity, connection and symmetries) and using this information in the algorithms that reconstruct Λ (see [4,7,13,14]). Chang and Chow [4] define an algorithm that reconstructs sets Λ so that they are convex and symmetrical with respect to two orthogonal axis, while Kuba's [13] heuristic algorithm reconstructs sets Λ with connected rows and columns. However, we have found that while some properties imposed on the sets completely eliminate ambiguity and make it possible to obtain efficient reconstruction algorithms [7] others only partially reduce ambiguity, thus making the set's reconstruction very difficult [1].

In this paper, we study the ambiguity problem with respect to a certain class of sets, called *convex polyominoes*, on which some connection constraints have been imposed. A *polyomino* is a connected finite set of adjacent cells lying two by two along a side. A polyomino is defined up to a translation. A convex polyomino has rows and columns connected. Polyominoes have long been studied by mathematicians (see, for example, [2,6,8,9,11,18]) and convex polyominoes satisfy some connection properties similar to those used by some reconstruction algorithms [4,14]. We determine an upper and lower bound for the maximum number of convex polyominoes having the same orthogonal projections (V,H), with $V \in \mathbb{N}^n$, and $H \in \mathbb{N}^m$ and prove that, under these connection conditions, ambiguity is still very high in some cases (2^{h-1} with $h = \min\{n,m\}$). We also define a construction for obtaining convex polyominoes having the same orthogonal projections. In another paper of ours [1], we studied the reconstruction of sets that satisfy some connection constraints and showed that with some of these constraints, it is possible to verify the existence of a set Λ having (V,H) projections in polynomial time, while, with other constraints the problem is NP-complete. Judging by both these results, and the ones in our present paper, we can deduce that it is not always possible to reduce ambiguity and reconstruction is not always facilitated by making sets Λ satisfy some connection properties.

2. Preliminaries

We define the *column* (*row*) of a polyomino as the intersection between the polyomino and an infinite vertical strip $[i,i+1]\times\mathbb{R}$ (horizontal $\mathbb{R}\times[i,i+1]$), where $i\in\mathbb{N}_0$. A polyomino is *convex* if all its columns and rows are connected. Let P be a convex polyomino and R the smallest rectangle containing it. Let $[N,N']$ ($[W,W']$, $[S,S']$, $[E,E']$) be the intersection of P's border with the upper side (left, lower, right) of R (see Fig. 1).A *stack* polyomino is a convex polyomino such that S coincides with W' and S' coincides with E. A *directed convex* polyomino is a convex polyomino such that S coincides with W'.

Notations. By $\mathscr{C}$ ($\mathscr{DC},\mathscr{S}$) we indicate the class of convex polyominoes (directed convex and stack).

We introduce the concept of satisfiability, which allows us to formulate the problem of reconstructing sets of cells from their projections.

Definition 1.1. Let $V=(v_1,v_2,\ldots,v_n)\in\mathbb{N}^n$ and $H=(h_1,h_2,\ldots,h_m)\in\mathbb{N}^m$ be two assigned vectors such that $\sum_{i=1}^n v_i=\sum_{j=1}^m h_j$. The pair (V,H) is said to be satisfiable in the class of cell sets $\mathscr{P}$ if there is at least one set $A\in\mathscr{P}$, such that its ith column projection and its jth row projection are equal to v_i and h_j, respectively, for $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$. We also say that A satisfies (V,H) in $\mathscr{P}$.

Problem 2.2. *Two vectors $V\in\mathbb{N}^n$ and $H\in\mathbb{N}^m$ having been assigned, find out if (V,H) is satisfiable in a class of cell sets $\mathscr{P}$, and determine the number of sets in $\mathscr{P}$ that satisfy it.*

Example 1. $V=(4,3,4,3,2)$, $H=(1,4,5,5,1)$. The polyominoes in Fig. 2 satisfy (V,H) in $\mathscr{C}$.

Definition 1.3. Let $V\in\mathbb{N}^n$ and $H\in\mathbb{N}^m$. By $\delta_p(V,H)$ we indicate the number of $\mathscr{P}$'s polyominoes that satisfy (V,H). We denote

$$\Delta_p(n,m)\stackrel{\text{def}}{=}\max\{\delta_p(V,H)\colon V\in\mathbb{N}^n,\ H\in\mathbb{N}^m\}.$$

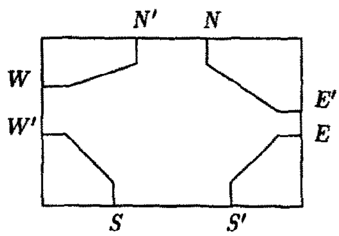


Fig. 1. Convex polyomino.

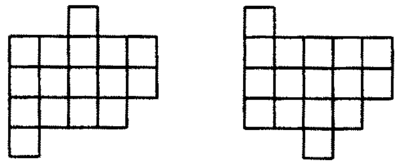


Fig. 2. Convex polyominoes.

2. Maximum number of polyominoes having the same orthogonal projections

In [7], it is shown that:

Proposition 3.1. *Let $V \in \mathbb{N}^n$ and $H \in \mathbb{N}^m$. There is at most one polyomino satisfying (V,H) in $\mathscr{DC}$.*

Therefore, we have $\Delta_{\rm dc}(n,m)=1$. Moreover, since stack polyominoes are special directed convex polyominoes, we obtain $\Delta_{\rm s}(n,m)=1$. Example 1 shows that the number of convex polyominoes satisfying a pair of vectors (V,H) can be greater than one and so usually $\Delta_{\rm c}(n,m)\neq 1$. Let us now define a lower bound to $\Delta_{\rm c}(n,m)$.

Proposition 3.2. *Let $n,m \in \mathbb{N}$. $\Delta_{\rm c}(n,m)\geqslant 2^k$, in which $k=\lfloor \frac{1}{2}\min\{n,m\}\rfloor$.*

Proof. Let $m=n=2k$. We consider the pair (V_1,H_1) such that $V_1=H_1$ and :

$$v_i=\begin{cases} 2i-1, & 1\leqslant i\leqslant k,\\ 4k-2i+1, & k+1\leqslant i\leqslant 2k.\end{cases}$$

Let P_k be the stack polyomino in Fig. 3, in which k is the number of P_k's rows. If $1\leqslant j\leqslant k$, then $P_j\subseteq P_k$. Let P be the convex polyomino obtained by joining polyominoes P_k and $P_k^{'}$ as shown in Fig. 3 ($P_k^{'}$ is obtained by turning P_k over axis x and by shifting one step right). Polyomino P satisfies (V_1,H_1). We now consider polyominoes $P_i,P_i^{'}$, such that $P_i\subseteq P_k$ and $P_i^{'}\subseteq P_k^{'}$, that is, with $i\leqslant k$. If P_i and $P_i^{'}$ are moved one step to the right and to the left, respectively, a convex polyomino $\bar{P}$ satisfying (V_1,H_1) is again obtained. We now take into consideration polyominoes P_j and $P_j^{'}$, such that $P_j\subseteq P_i$ and $P_j^{'}\subseteq P_i^{'}$ ($j<i$), with respect to polyomino $\bar{P}$. If we move P_j and $P_j^{'}$ one step to the left and to the right, respectively, we obtain a convex polyomino $\tilde{P}$ satisfying (V_1,H_1) (see Fig. 4). We therefore deduce that $\delta_c(V_1,H_1)\geqslant 2^k$.

Let $m=n=2k+1$. Let (V_2,H_2) be a pair of vectors such that $V_2=H_2$ and

$$v_i=2i,\qquad 1\leqslant i\leqslant k,$$

$$v_{k+1}=2k+1,$$

$$v_i=4k-2i+4,\quad k+2\leqslant i\leqslant 2k+1.$$

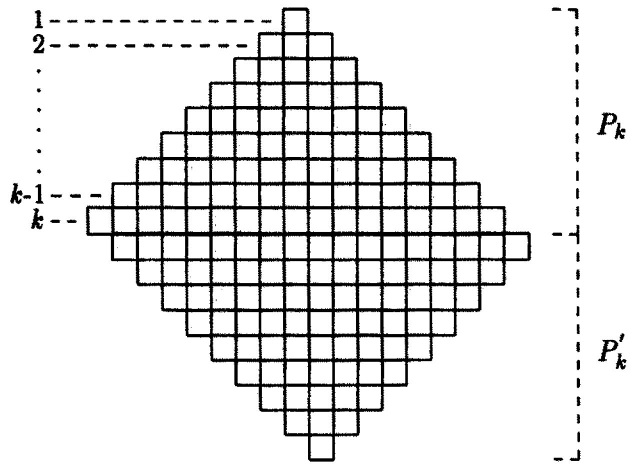


Fig. 3. Convex polyomino $P = P_k \cup P'_k$.

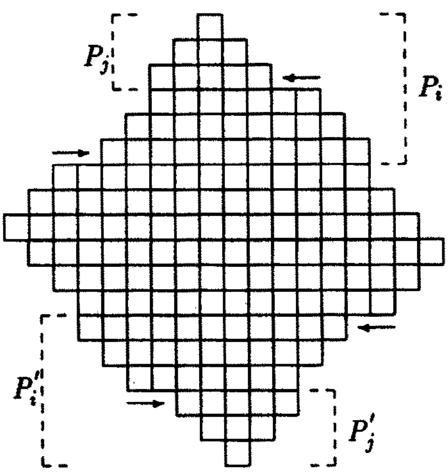


Fig. 4. Convex polyomino $\tilde{P}$.

Let Q_k be the stack polyomino in Fig. 5, where k is the number of Q_k's rows. If $1 \leqslant j \leqslant k$, then $Q_j \subseteq Q_k$. Let Q be the convex polyomino obtained by joining polyominoes Q_k and Q'_k (where Q'_k is obtained by turning Q_k over axis x and by shifting one step right) to a row $2k+1$ long as shown in Fig. 5. Polyomino Q satisfies (V_2, H_2). If we move Q_i, Q'_i $(i \leqslant k)$ one step to the right and to the left, respectively, we again obtain a convex polyomino $\bar{Q}$ that satisfies (V_2, H_2). We therefore deduce that $\delta_c(V_2, H_2) \geqslant 2^k$.

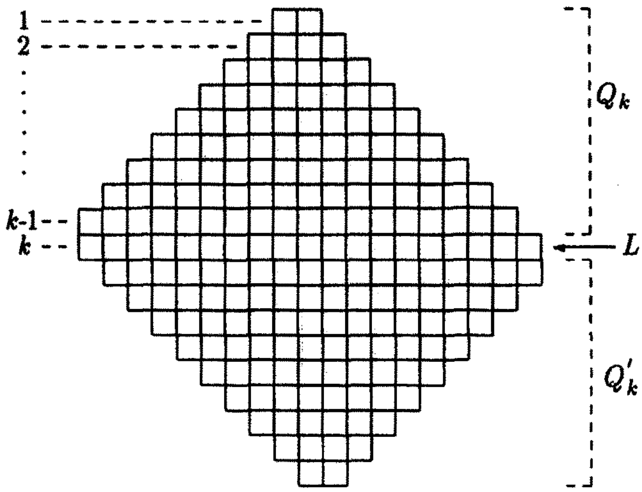


Fig. 5. Convex polyomino $Q = Q_k \cup Q'_k$.

Let $n < m$ and $n = 2k$. Let (V_3, H_3) be a pair of vectors such that:

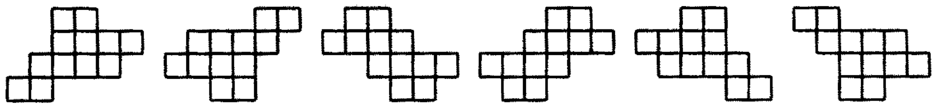
$$v_i = \begin{cases} 2i - 1 + m - n, & 1 \leqslant i \leqslant k, \\ 4k - 2i + 1 + m - n, & k + 1 \leqslant i \leqslant 2k. \end{cases}$$

$$h_j = \begin{cases} 2j - 1, & 1 \leqslant j \leqslant k, \\ 2k, & k + 1 \leqslant j \leqslant k + m - n, \\ 4k - 2j + 2(m - n) + 1, & k + m - n + 1 \leqslant j \leqslant m. \end{cases}$$

Let P be the convex polyomino obtained by joining polyomino P_k, $2k$ long $m - n$ rows and polyomino P'_k. Polyomino P satisfies (V_3, H_3). If we move polyominoes P_i and P'_i (with $i < k$) as in the previous cases, we obtain polyominoes that satisfy (V_3, H_3). Consequently, $\delta_c(V_3, H_3) \geqslant 2^k$. By symmetry, case $n > m$ and $m = 2k$ is analogous to the preceding one. In the same way, we show that if $n < m$ and $n = 2k + 1$ (or, $n > m$ and $m = 2k + 1$), there is a pair (V_4, H_4) such that $\delta_c(V_4, H_4) \geqslant 2^k$. We therefore deduce that $\Delta_c(n, m) \geqslant 2^k$, where $k = \left\lfloor \frac{1}{2} \min\{n, m\} \right\rfloor$. □

Remark 1. If we have an algorithm that constructs all the convex polyominoes having orthogonal projections (V, H), with $V \in \mathbb{N}^n$ and $H \in \mathbb{N}^m$, since $\Delta_c(n, m) \geqslant 2^k$, where $k = \left\lfloor \frac{1}{2} \min\{n, m\} \right\rfloor$, then there are some pairs (V, H) for which the algorithm's complexity is not polynomial.

Remark 2. There are some pairs (V, H), with $V \in \mathbb{N}^n, H \in \mathbb{N}^m$, such that $\delta_c(V, H) > 2^k$. For example, if $V = (1, 2, 3, 3, 2, 1)$ and $H = (2, 4, 4, 2)$, then the following polyominoes satisfy (V, H) and, therefore, $\delta_c(V, H) = 6 > 4$.



We now determine an upper bound for $\Delta_c(n,m)$.

Definition 3.3. Let $V=(v_1,v_2,\dots,v_n)\in\mathbb{N}^n$, $H=(h_1,h_2,\dots,h_m)\in\mathbb{N}^m$ and $v_1=m-k$ with $0\leqslant k\leqslant m-1$. We denote by $\Omega_i(V,H)$ the set of convex polyominoes satisfying (V,H) and having the minimum ordinate of the first column equal to i $(0\leqslant i\leqslant k)$ (see Fig. 6). We denote

$$\alpha_{k,i}^{(n,m)} \stackrel{\text{def}}{=} \max\{\,|\Omega_i(V,H)|:\ V\in\mathbb{N}^n,\ m\in\mathbb{N},\ H\in\mathbb{N}^m \textit{ and } v_1=m-k\}.$$

We will prove that $\alpha_{k,i}^{(n,m)}\leqslant\binom{k}{i}$. Since we proceed by induction on n with m fixed, from now on we denote $\alpha_{k,i}^{(n,m)}$ by $\alpha_{k,i}^{(n)}$ for the sake of simplicity.

Lemma 3.4. *Let $n,m\in\mathbb{N}$. If $0\leqslant k\leqslant m-1$ and $0\leqslant i\leqslant k$ then $\alpha_{k,i}^{(n)}\geqslant\alpha_{k,i}^{(n-1)}$.*

Proof. There are $V\in\mathbb{N}^{n-1}$, $H\in\mathbb{N}^m$ such that $v_1=m-k$ and $\alpha_{k,i}^{(n-1)}=|\Omega_i(V,H)|$. Let $P\in\Omega_i(V,H)$. We consider polyomino P' obtained by adding an $m-k$ long column and having minimum ordinate i to the left of P's first column (see Fig. 7). Polyomino P' is convex and satisfies (V',H'), where $V'\in\mathbb{N}^n$, $H'\in\mathbb{N}^m$. Therefore, we obtain

$$\alpha_{k,i}^{(n)}\geqslant|\Omega_i(V',H')|\geqslant|\Omega_i(V,H)|=\alpha_{k,i}^{(n-1)}.\qquad\square$$

Lemma 3.5. *Let $V\in\mathbb{N}^n$, $H\in\mathbb{N}^m$. If $v_1>v_2$ and P is a convex polyomino that satisfies (V,H), then the minimum ordinate i of the first column is such that $i=0$ or $i=m-v_1$.*

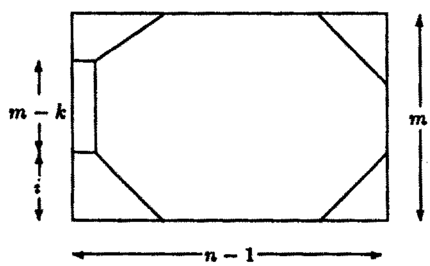


Fig. 6. Polyomino whose first column is $m-k$ long and has minimum ordinate i.

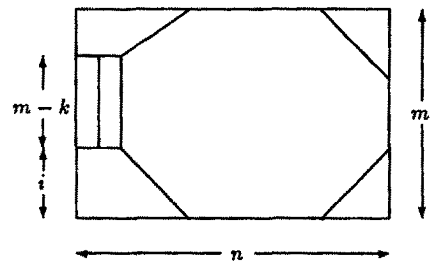


Fig. 7. Polyomino whose first two columns are $m-k$ long and have minimum ordinate i.

Proof. If $v_1 > v_2$ and P is the convex polyomino that satisfies (V,H), from [7] it follows that P is a convex polyomino directed towards northeast or southeast. Consequently, the first column's minimum ordinate is $i=0$ or $i=m-v_1$. $\square$

Lemmas 3.4 and 3.5 are used for proving the following theorem:

Theorem 3.6. *Let $n,m \in \mathbb{N}$. If $0 \leqslant k \leqslant m-1$ and $0 \leqslant i \leqslant k-1$ then $\alpha_{k,i}^{(n)} \leqslant \binom{k}{i}$.*

Proof. By the convex polyominoes' symmetry, we have

$$\alpha_{k,i}^{(n)} = \alpha_{k,k-i}^{(n)}, \quad 0 \leqslant i \leqslant k,$$

and so we take into consideration only the $\alpha_{k,i}^{(n)}$ with $0 \leqslant i \leqslant \lceil \frac{1}{2}(k-1) \rceil$.

If $i=0$ (the minimum ordinate of the convex polyominoes' first column is equal to 0), then the polyominoes are directed and it follows from Proposition 3.1 that there is at most one polyomino that satisfies a pair (V,H) with $V \in \mathbb{N}^n$, $H \in \mathbb{N}^m$ and $v_1 = m-k$. Consequently, we obtain $\alpha_{k,0}^{(n)} = 1$, for $0 \leqslant k \leqslant m-1$, and therefore the theorem is proved for $i=0$.

We assume that $1 \leqslant i \leqslant \lceil \frac{1}{2}(k-1) \rceil$ and examine the polyominoes' second column which is $v_2 = m-j$ long. From Lemma 3.5 we deduce that $v_1 \leqslant v_2$, that is $0 \leqslant j \leqslant k$. Let h be the second column's minimum ordinate. There are three possible cases:

(i) if $0 \leqslant j \leqslant i-1$, then $0 \leqslant h \leqslant j$ and so the second column can assume $j+1$ positions (see Fig. 8),

(ii) if $i \leqslant j \leqslant k-i$, then $0 \leqslant h \leqslant i$ and so the second column can assume $i+1$ positions (see Fig. 9),

(iii) if $k-i+1 \leqslant j \leqslant k$, then $j-k+i \leqslant h \leqslant i$ and so the second column can assume $k-j+1$ positions (see Fig. 10).

As a consequence, if the minimum ordinate i in the first column is such that $1 \leqslant i \leqslant \lceil \frac{1}{2}(k-1) \rceil$, and the second column is $m-j$ long, with $0 \leqslant j \leqslant k$, then the second column can assume various positions linked to j. For each position, there are $\alpha_{j,h}^{(n-1)}$ convex polyominoes satisfying a pair (V',H') with $V' \in \mathbb{N}^{n-1}, H' \in \mathbb{N}^m$ and $v_1' =$

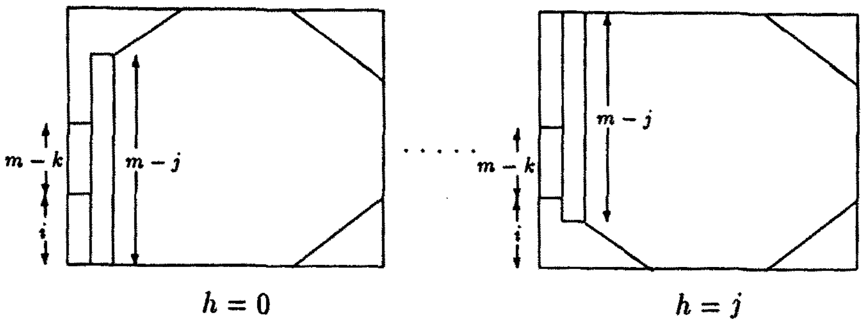


Fig. 8. Convex polyominoes whose second column is $m-j$ long, where $0 \leqslant j \leqslant i-1$.

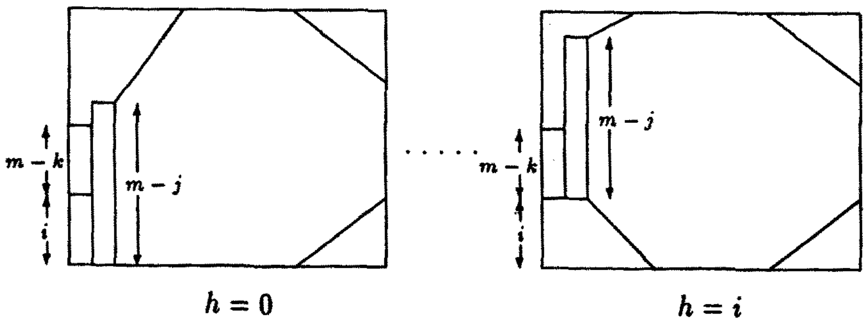


Fig. 9. Convex polyominoes whose second column is $m-j$ long, where $i \leqslant j \leqslant k-i$.

$m-j$. We therefore deduce that

$$\alpha_{k,i}^{(n)} \leqslant \max_{0 \leqslant j \leqslant k} \left\{ \sum_{h=a}^{b} \alpha_{j,h}^{(n-1)} \right\},$$

in which

$$a=0, \quad b=j \qquad \text{for } 0 \leqslant j \leqslant i-1,$$

$$a=0, \quad b=i \qquad \text{for } i \leqslant j \leqslant k-i,$$

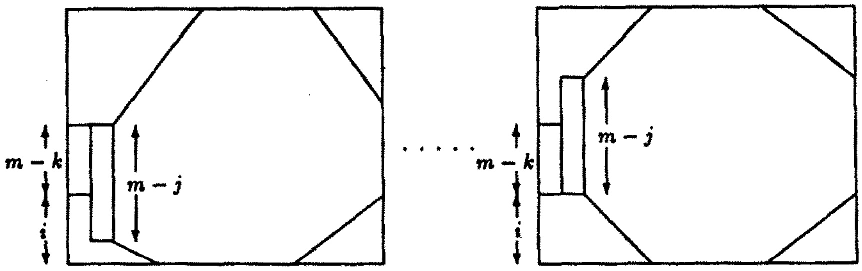
$$a=j-k+i \quad b=i \qquad \text{for } k-i+1 \leqslant j \leqslant k.$$

By means of Lemma 3.4, we obtain

$$\alpha_{k,i}^{(n)} \leqslant \max_{0 \leqslant j \leqslant k-1} \left\{ \sum_{h=a}^{b} \alpha_{j,h}^{(n)}, \ \alpha_{k,i}^{(n-1)} \right\}, \tag{1}$$

$$\alpha_{k,0}^{(n)} = 1, \tag{2}$$

$$\alpha_{j,j}^{(n)} = 1. \tag{3}$$



$h = k - j + 1$ $\qquad\qquad$ $h = i$

Fig. 10. Convex polyominoes whose second column is $m - j$ long, where $k - i + 1 \leqslant j \leqslant k$.

We now show that $\alpha_{k,1}^{(n)} \leqslant \binom{k}{1}$. By means of Eqs. (1) and (2), we get

$$\alpha_{k,1}^{(n)} \leqslant \max_{1 \leqslant j \leqslant k-1} \left\{ 1, 1 + \alpha_{j,1}^{(n)}, \alpha_{k,1}^{(n-1)} \right\}.$$

However, from Eq. (3) we obtain $\alpha_{1,1}^{(n)} = 1$ and so, if $k > 1$, by induction on k we obtain

$$\alpha_{k,1}^{(n)} \leqslant \max_{1 \leqslant j \leqslant k-1} \left\{ 1, 1 + \binom{j}{1}, \alpha_{k,1}^{(n-1)} \right\} = \max \left\{ \binom{k}{1}, \alpha_{k,1}^{(n-1)} \right\}.$$

By following the same procedure and using Lemma 3.4 and the induction hypothesis $\alpha_{j,1}^{(n)} \leqslant \binom{j}{1}$ with $j < k$, we obtain

$$\alpha_{k,1}^{(n-1)} \leqslant \max \left\{ \binom{k}{1}, \alpha_{k,1}^{(n-2)} \right\},$$

and consequently

$$\alpha_{k,1}^{(n)} \leqslant \max \left\{ \binom{k}{1}, \alpha_{k,1}^{(n-2)} \right\}.$$

By recurrence, we thus get

$$\alpha_{k,1}^{(n)} \leqslant \max \left\{ \binom{k}{1}, \alpha_{k,1}^{(1)} \right\} = \binom{k}{1}.$$

In the same way, we obtain

$$\alpha_{j,h}^{(n)} \leqslant \binom{j}{h}, \tag{4}$$

for $0 \leqslant j \leqslant k-1$ and $0 \leqslant h \leqslant i-1$. We now verify that $\alpha_{k,i}^{(n)} \leqslant \binom{k}{i}$.

From relations (1) and (4) we deduce that

$$\alpha_{k,i}^{(n)} \leqslant \max \left\{ \sum_{\substack{h=0 \\ 0 \leqslant j \leqslant i-1}}^{j} \binom{j}{h}, \sum_{\substack{h=0 \\ i \leqslant j \leqslant k-i}}^{i-1} \binom{j}{h} + \alpha_{j,i}^{(n)}, \sum_{\substack{h=j-k+i \\ k-i+1 \leqslant j \leqslant k}}^{i-1} \binom{j}{h} + \alpha_{j,i}^{(n)}, \alpha_{k,i}^{(n-1)} \right\}.$$

From eq. (3) we have $\alpha_{i,i}^{(n)} = 1$. We therefore assume $\alpha_{j,i}^{(n)} \leqslant \binom{j}{i}$ for $i \leqslant j \leqslant k-1$. Consequently

$$\alpha_{k,i}^{(n)} \leqslant \max\left\{ \sum_{\substack{h=0\\0\leqslant j\leqslant i-1}}^{j} \binom{j}{h},\ \sum_{\substack{h=0\\i\leqslant j\leqslant k-i}}^{i} \binom{j}{h},\ \sum_{\substack{h=j-k+i\\k-i+1\leqslant j\leqslant k}}^{i} \binom{j}{h},\ \alpha_{k,i}^{(n-1)} \right\},$$

and it follows that

$$\alpha_{k,i}^{(n)} \leqslant \max\left\{ \binom{k}{i},\ \alpha_{k,i}^{(n-1)} \right\}.$$

By following the same procedure and using Lemma 3.4 and induction hypothesis $\alpha_{j,i}^{(n)} \leqslant \binom{j}{i}$ with $j < k$, we obtain

$$\alpha_{k,i}^{(n-1)} \leqslant \max\left\{ \binom{k}{i}, \alpha_{k,i}^{(n-2)} \right\},$$

and finally

$$\alpha_{k,i}^{(n)} \leqslant \max\left\{ \binom{k}{i}, \alpha_{k,i}^{(1)} \right\} = \binom{k}{i}. \qquad \square$$

From Theorem 3.6, we deduce:

Corollary 3.7. *Let $n,m \in \mathbb{N}$. $\Delta_c(n,m) \leqslant 2^{h-1}$, where $h = \min\{n,m\}$.*

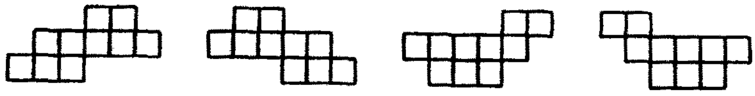
Proof. The number of convex polyominoes satisfying a pair (V,H) with $V \in \mathbb{N}^n, H \in \mathbb{N}^m$ and $v_1 = m-k$ with $0 \leqslant k \leqslant m-1$, is $\sum_{i=0}^{k} \alpha_{k,i}^{(n)}$, and so, from Theorem 3.6 we obtain

$$\sum_{i=0}^{k} \alpha_{k,i}^{(n)} \leqslant \sum_{i=0}^{k} \binom{k}{i} = 2^k.$$

However, $0 \leqslant k \leqslant m-1$ and therefore $\Delta_c(n,m) \leqslant 2^{m-1}$. In the same way, we obtain $\Delta_c(n,m) \leqslant 2^{n-1}$, and then the corollary. $\square$

Let us take an example in which $\Delta_c(n,m) = 2^{h-1}$.

Example 2. Let $V = (1,2,2,2,2,1)$, $H = (3,5,2)$. The following convex polyominoes satisfy (V,H) in $\mathscr{C}$. From Corollary 3.7, we obtain $\Delta_c(3,6) \leqslant 4$ and, consequently, $\Delta_c(3,6) = 4$.



By means of Proposition 3.1 and Corollary 3.7, we deduce the following theorem:

Theorem 3.8. *Let $n, m \in \mathbb{N}$. The number $\Delta_c(n,m)$ (the maximum number of convex polyominoes having the same orthogonal projections (V,H), with $V \in \mathbb{N}^n$ and $H \in \mathbb{N}^m$) is such that*

$$2^{\lfloor h/2 \rfloor} \leqslant \Delta_c(n,m) \leqslant 2^{h-1},$$

where $h = \min\{n,m\}$.

4. A construction for obtaining convex polyominoes having the same orthogonal projections

Let us examine the convex polyomino P_1 in Fig. 11, in which region A belongs to P_1 and region B does not belong to P_1. Let P_1' be the polyomino obtained by eliminating region A from P_1 and inserting in it the region B. The convex polyominoes P_1 and P_1' satisfy the same pair (V_1, H_1). Let P_2 be another polyomino having a shape similar to P_1 and let Q be the convex polyomino obtained by superposing P_1 and P_2 as shown in Fig. 12. The cells labelled "+" are added in order to maintain Q's convexity. If we change P_2 in the same way as we did for P_1 (by eliminating region C and inserting the region D), we obtain a polyomino P_2' that satisfies the same pair of vectors that P_2 does. In polyomino Q, it is possible to change P_1 and P_2 separately and so we obtain 4 convex polyominoes having the same projections (V_2, H_2).In general, if we use i convex polyominoes $P_1, P_2, \ldots, P_i$ having similar shapes, it is possible to construct 2^i convex polyominoes that satisfy the same pair (V_i, H_i).

Remark 3. By means of this construction, we obtain the polyominoes needed for proving Proposition 3.2. For example, polyomino P in Fig. 3 is obtained by superposing k convex polyominoes and so there are 2^k convex polyominoes that satisfy (V_k, H_k).

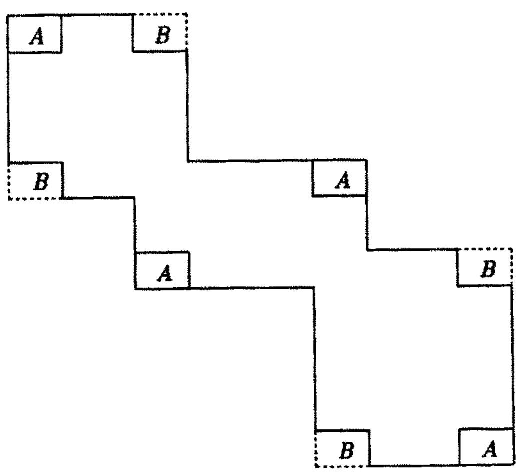


Fig. 11. Polyomino P_1.

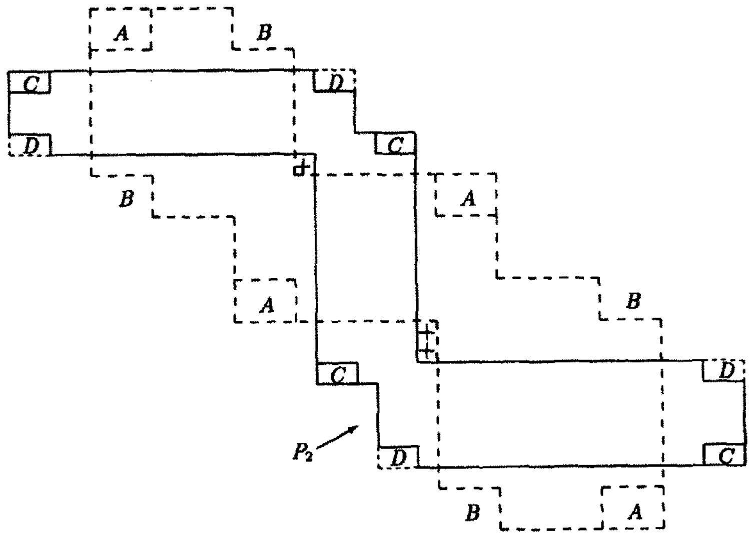


Fig. 12. Convex polyomino Q.

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