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Finite elements numerical solution of a coupled profile–velocity–temperature shallow ice sheet approximation model ☆

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Abstract

This work deals with the numerical solution of a complex mathematical model arising in theoretical glaciology. The global moving boundary problem governs thermomechanical processes jointly with ice sheet hydrodynamics. One major novelty is the inclusion of the ice velocity field computation in the framework of the shallow ice model so that it can be coupled with profile and temperature equations. Moreover, the proposed basal velocity and shear stress laws allow the integration of basal sliding effects in the global model. Both features were not taking into account in a previous paper (Math. Model. Methods Appl. Sci. 12 (2) (2002) 229) and provide more realistic convective terms and more complete Signorini boundary conditions for the thermal problem. In the proposed numerical algorithm, one- and two-dimensional piecewise linear Lagrange finite elements in space and a semi-implicit upwinding scheme in time are combined with duality and Newton's methods for nonlinearities. A simulation example involving real data issued from Antarctic shows the temperature, profile and velocity qualitative behaviour as well as the free boundaries and basal effects.

Keywords: Shallow ice models; Stefan–Signorini; Moving boundary; Transport-diffusion; Duality; Finite elements

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1. Introduction

Mathematical modelling and numerical simulation of the large ice sheet mechanics and thermodynamics is not an easy task since certain open questions concerning some ice flux conditions both in the domain and on the boundary still remain. In fact, thermodynamic models, isothermal models of ice dynamics, thermomechanical models and the specific ones for the Antarctic ice sheet appearing in the literature illustrate the intrinsic difficulties associated to these physical problems [9,11]. Therefore, some studies address to certain simplified models, like the shallow ice approximation proposed in [7,9], to analyze some isolated (thermal, mechanical or dynamical) phenomena. For example, an isothermal ice profile evolution model is stated in [7] by using asymptotic techniques, temperature mathematical models for polythermal ice sheets are proposed in [7,8] and some behaviour laws for the basal thermoelastohydrodynamic phenomena are discussed in [6] according to heuristic considerations and emphazising the lack of numerical simulation studies on basal rheology.

In Huybrechts [10], an important work concerning numerical simulation of a global thermomechanical model for the behaviour of great ice masses is developed and finite differences methods for three-dimensional geometries are proposed. Nevertheless, the model equations are much simpler than those ones here proposed because, for example, the more realistic moving boundary problems governing the profile and temperature evolution have not been considered.

The shallow ice approach takes into account the orders of magnitude of the real ice sheet (i.e., width far smaller than length) for neglecting some terms in the original mass, momentum and energy conservation equations. More precisely, the shallow ice scaling of equations introduces the small parameter $\varepsilon = d_2/d_1$ where d_1 and d_2 denote the length and width orders of magnitude (typically, $d_1 = 3000$ km and $d_2 = 3$ km in Antarctic). Next, the terms of order ε^2 are neglected in the equations.

In order to simplify the ice sheet model, a bidimensional ice flux in a polar cap is commonly assumed so that the same profile is considered for the different longitudinal sections. Thus, the coordinates (x, z) take values in a section whose time dependent upper boundary is defined by $z = \eta(t, x)$ and whose flat lower boundary is given by $z = 0$. In the profile problem, the unknown ice sheet extent and profile depend on the ice accumulation-ablation ratio and the basal velocity among other factors [4]. Two different numerical approaches to the second order nonlinear parabolic model for η are described in [2,4].

Moreover, for a given profile, in the polythermal model for the ice temperature the presence of a thin basal layer at the melting temperature becomes relevant for imposing the lower boundary conditions [7]. A numerical solution technique for this shallow ice free boundary thermal problem is proposed in [3]. More recently, a first attempt to solve numerically, a semicoupled problem involving the ice sheet profile and its temperature for a prescribed velocity has been made in [5].

In this work, the main innovative goals concern to the velocity field computation in a shallow ice framework jointly with a basal sliding velocity law that depends on temperature and shear stress. Additionally, the influence of basal magnitudes on the Signorini boundary condition for the thermal problem is modelled. Finally, a complex coupled problem can be solved by means of an algorithm where the different subproblems are sequentially treated.

2. Thermoelastohydrodynamic mathematical model

In order to pose the dimensionless coupled problem a fixed rectangular domain Ω is considered. It includes not only a longitudinal section of the ice sheet but also the part of the atmosphere placed above the ice mass. Thus, the domain is defined by

$$\Omega = \{(x,z)/-1 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant z_{\max}\}, \tag{1}$$

and the ice mass domain is given, in terms of the unknown upper profile η, by

$$\Omega_{\mathrm{I}}(t) = \{(x,z)/S_-(t) \leqslant x \leqslant S_+(t),\ 0 \leqslant z \leqslant \eta(t,x)\}, \tag{2}$$

so that for any time t the inclusion $\Omega_{\mathrm{I}}(t) \subset \Omega$ holds. Therefore, we can write

$$\Omega = \Omega_{\mathrm{I}}(t) \cup \Omega_{\mathrm{A}}(t), \tag{3}$$

where $\Omega_{\mathrm{A}}(t)$ denotes the atmospheric domain. Notice that since the ice layer longitudinal extent is not fixed, the interval $(S_-(t), S_+(t)) \subset (-1,1)$ is an additional unknown.

First, the moving boundary profile problem is posed over the fixed domain $(-1,1)$. Let $t_{\max} > 0$ be a large time instant and let $a:(0,t_{\max}) \times (-1,1) \to \mathbb{R}$ and $\eta_0:(-1,1) \to \mathbb{R}$ be the given accumulation-ablation rate and initial profile, respectively. Then, for $t \in [0,t_{\max}]$, the formulation can be stated as follows [4]:

Find $\Gamma_0(t) = (S_-(t), S_+(t))$ and $\eta : \mathbf{Q} = \bigcup_{t\in[0,t_{\max}]} \Gamma_0(t) \to \mathbb{R}$ such that

$$\frac{\mathrm{D}^*\eta}{\mathrm{D}t} = \frac{\mathrm{e}^{-\gamma}}{v}\frac{\partial}{\partial x}\left(\frac{\eta^{n+2}}{n+2}\left|\frac{\partial \eta}{\partial x}\right|^{n-1}\frac{\partial \eta}{\partial x}\right) + a \quad \text{in } \mathbf{Q},$$

$$\eta \geqslant 0 \text{ in } \mathbf{Q},$$

$$\left(\frac{\mathrm{D}^*\eta}{\mathrm{D}t} - \frac{\mathrm{e}^{-\gamma}}{v}\frac{\partial}{\partial x}\left(\frac{\eta^{n+2}}{n+2}\left|\frac{\partial \eta}{\partial x}\right|^{n-1}\frac{\partial \eta}{\partial x}\right) - a\right)\eta = 0 \quad \text{in } \mathbf{Q}, \tag{4}$$

$$\eta = 0 \text{ on } \{S_-(t)\} \cup \{S_+(t)\}, t \in (0,t_{\max}),$$

$$\eta(0,x) = \eta_0(x) \text{ in } (-1,1),$$

where

$$\frac{\mathrm{D}^*\eta}{\mathrm{D}t} = \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(u_{\mathrm{b}}\eta). \tag{5}$$

u_{b} is the given basal sliding velocity and γ and v are two dimensionless parameters. Thus, the solution of (4) provides, for each time t, the ice sheet boundaries defined by

$$\Gamma_0(t) = \{(x,z)/x \in (S_-(t), S_+(t)), z = 0\}, \tag{6}$$

$$\Gamma_1(t) = \{(x,z)/x \in \Gamma_0(t), z = \eta(t,x)\}. \tag{7}$$

In the shallow ice approach, after neglecting $0(\varepsilon^2)$ in the momentum equation, we can deduce the following expression for the z-derivative of the horizontal velocity u:

$$\frac{\partial u}{\partial z} = \frac{-A(T)}{v}(\eta - z)^n \left|\frac{\partial \eta}{\partial x}\right|^{n-1} \frac{\partial \eta}{\partial x}, \tag{8}$$

where the term $A(T) = \mathrm{e}^{\gamma T}$ is associated with viscous dissipation effects, T being the temperature [3]. Next, by integrating Eq. (8) from 0 to z, we obtain the ice horizontal velocity

$$u = u_{\mathrm{b}} - \int_0^z \frac{A(T)}{v}(\eta - s)^n |\eta_x|^{n-1} \eta_x \, \mathrm{d}s. \tag{9}$$

Next, we extend to the whole domain the stream function associated to the ice velocity field in the form

$$\psi(x,z) = \begin{cases} \displaystyle\int_0^z u(x,s)\,\mathrm{d}s & \text{if } z \leqslant \eta, \\ \displaystyle\int_0^\eta u(x,s)\,\mathrm{d}s & \text{if } z > \eta, \end{cases} \tag{10}$$

so that the vertical velocity, v, can be obtained from ψ as follows:

$$v(x,z) = \begin{cases} -\dfrac{\partial}{\partial x}(\psi(x,z) & \text{if } z \leqslant \eta, \\ 0 & \text{if } z > \eta. \end{cases} \tag{11}$$

Concerning the shallow ice approximation of the energy equation, as we are interested in polythermal ice sheets, which include the presence of temperate ice (i.e. at the melting temperature $T = 0$) and possibly water regions, an appropriate two phase Stefan model is proposed. So, for each time t, we obtain the temperature by solving the following equations [3]:

$$\begin{aligned} &\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T - \beta \frac{\partial^2 T}{\partial z^2} + F = 0, \quad T < 0 \quad \text{in } \Omega_{\mathrm{C}}(t), \\ &T \geqslant 0 \text{ in } \Omega_{\mathrm{T}}(t), \\ &\beta \frac{\partial T}{\partial \vec{n}(t)} = L_{\mathrm{c}} \frac{\mathrm{d}s}{\mathrm{d}t} \text{ on } \Sigma(t), \end{aligned} \tag{12}$$

where $\vec{v} = (u, v)$ is the velocity and the nonlinear viscous dissipation term F is given by

$$F = F(T,x,z) = -\left(\frac{\alpha}{v}\right)((\eta(x) - z)\eta_x)^4 \mathrm{e}^{\gamma T}, \tag{13}$$

β, α and v being given dimensionless parameters. The subsets $\Omega_{\mathrm{C}}(t)$ and $\Omega_{\mathrm{T}}(t)$ denote the cold and the temperate ice regions, respectively. More precisely,

$$\Omega_{\mathrm{C}}(t) = \{(x,z) \in \Omega_{\mathrm{I}}(t)/T(t,x,z) < 0\},$$

$$\Omega_{\mathrm{T}}(t) = \{(x,z) \in \Omega_{\mathrm{I}}(t)/T(t,x,z) \geqslant 0\}.$$

The surface $\Sigma(t)$ is the free boundary between both subregions, $\vec{n}(t)$ notes the unitary normal vector to $\Sigma(t)$ pointing to $\Omega_{\mathrm{C}}(t)$ and $L_{\mathrm{c}} = 3.4$ is the dimensionless latent heat.

In order to complete the formulation (12) boundary and initial conditions must be prescribed. First, in terms of the dimensionless geothermic flux, g_b, the ice basal velocity and the basal stress, τ_b, the following Signorini condition is posed at the lower boundary $\Gamma_0(t)$:

$$\begin{aligned} &-\frac{\partial T}{\partial z} = g_b + \tau_b u_b \quad \text{if } T < 0, \\ &-\frac{\partial T}{\partial z} = 0 \quad \text{if } T > 0, \\ &0 < -\frac{\partial T}{\partial z} < g_b + \tau_b u_b \quad \text{if } T = 0. \end{aligned} \tag{14}$$

Modelling and numerical simulations including the term $\tau_b u_b$ is an important original aspect of this work and requires behaviour laws for both magnitudes. Theoretically, the ice begins to slide when basal temperature reaches the melting point and, consequently, basal melt water is produced. But, also sliding processes have been experimentally observed on sub-temperate regions ($T = -4.6°\text{C}$, approximately). So, taking this into account and following [6], we propose a basal velocity law in the form $u_b = f(\tau_b, T_b)$ so that u_b must be fastly decreasing when the basal temperature, T_b, decreases below and nearby the melting point ($T = 0$). More precisely, the basal shear stress and velocity, appearing in (14), are modelled by expressions

$$\tau_b = -\eta \frac{\partial \eta}{\partial x}, \quad u_b = c_b |\tau_b| \tau_b \exp(T/\delta_b), \tag{15}$$

where the parameters $c_b \in (0.1, 10)$ and $\delta_b \ll 1$. Notice that the coupling among the sliding basal velocity, the basal shear stress, the profile and the temperature involves a new advance respect to the previous models described in [3,5].

At the upper boundary $\Gamma_1(t)$ a Dirichlet condition, $T = T_A$, is considered. The choice $T_A = -1$ corresponds to 223 K mean real atmospheric temperature. Moreover, in numerical simulation the temperature T_A is also imposed in the atmospheric region $\Omega_A(t)$.

Finally, an initial temperature, T_0, on the evolution equations (12) is prescribed.

3. Discretization and variational formulation

For the time semidiscretization of the nonlinear parabolic profile problem (4) an upwind scheme of characteristics has been chosen to approximate the total derivative (5). So, we achieve a sequence of nonlinear elliptic complementarity problems that are solved by combining Lagrange finite elements approximations of degree one for Eq. (4) and duality type algorithms for the nonlinearities involved (see [2,4], for two different techniques).

Once the profile function η has been obtained at each time step, in terms of previous values of the temperature we compute the velocity field $\vec{v}$ by using numerical quadrature formulae in (9) and (11). This numerical integration procedure being highly technique due to that only temperature values at the mesh points are available.

Next, for the updated profile and velocity field, we propose a variational formulation of the Stefan–Signorini thermal problem in terms of two multivalued operators. Thus, the multivalued Heaviside operator H is introduced to rewrite the boundary condition (14):

$$\frac{\partial T}{\partial z} \in (g_b + \tau_b u_b)(H(T) - 1) \Leftrightarrow \frac{\partial T}{\partial z} + (g_b + \tau_b u_b) \in (g_b + \tau_b u_b)H(T), \tag{16}$$

and the two phase Stefan problem is posed in terms of the enthalpy operator

$$E(T) = \begin{cases} T & \text{if } T < 0, \\ [0, L_c] & \text{if } T = 0, \\ L_c & \text{if } T > 0. \end{cases}$$

So, we establish the variational formulation for the thermal problem in the form:
Find $y(t,.) \in V_A(t)$ such that

$$\int_\Omega \frac{De}{Dt}\varphi \, d\Omega + \int_\Omega \frac{\partial y}{\partial z}\frac{\partial \varphi}{\partial z} \, d\Omega + \delta \int_\Omega \frac{\partial y}{\partial x}\frac{\partial \varphi}{\partial x} \, d\Omega$$

$$+ \int_\Omega (F \circ \Lambda^{-1})(y)\varphi \, d\Omega - \int_{\Gamma_0(t)} g\varphi \, d\Gamma$$

$$+ \int_{\Gamma_0(t)} g\theta\varphi \, d\Gamma = 0, \quad \forall \varphi \in V_0(t), \tag{17}$$

$$e \in (E \circ \Lambda^{-1})(y), \tag{18}$$

$$\theta \in (H \circ \Lambda^{-1})(y), \tag{19}$$

where the classical Kirchoff change

$$y = \Lambda(T) = \int_0^T \beta \, ds = \beta T \tag{20}$$

and a small horizontal diffusion term controlled by the parameter δ have been introduced. Moreover, for simplicity we have noted $g = \beta g_b + \tau_b u_b$. In the previous formulation, the following sets have been considered:

$$V_0(t) = \{\varphi \in H^1(\Omega)/\varphi = 0 \text{ on } \Gamma_1(t) \cup \Omega_A(t)\},$$

$$V_A(t) = \{\varphi \in H^1(\Omega)/\varphi = \Lambda(T_A) \text{ on } \Gamma_1(t) \cup \Omega_A(t)\}.$$

For the time discretization of (17)–(19) an upwind characteristics scheme has been developed [3]. So, the material derivative at time $t^{m+1} = (m+1)\Delta t$ is approximated by

$$\frac{De}{Dt}((m+1)\Delta t, x, z) \approx \frac{e^{m+1} - e^m \circ \chi^m}{\Delta t}, \tag{21}$$

where

$$e^{m+1} = e((m+1)\Delta t, x, z), \tag{22}$$

χ^m is defined by

$$\chi^m(x,z) = S((m+1)\Delta t, x, z; m\Delta t),$$

S being the trajectory of the velocity field which is the solution of the final value problem

$$\begin{cases} \dfrac{\mathrm{d}S(t,x,z;s)}{\mathrm{d}s} = \vec{v}(S(t,x,z;s),s), \\ S(t,x,z;t) = (x,z). \end{cases}$$

Then, we can pose the following sequence of problems:

Find $y^{m+1} \in V_{\mathrm{A}}(t^{m+1})$ such that

$$\begin{aligned} &\frac{1}{\Delta t}\int_\Omega e^{m+1}\varphi \,\mathrm{d}\Omega - \frac{1}{\Delta t}\int_\Omega e^m \circ \chi^m \varphi \,\mathrm{d}\Omega + \delta \int_\Omega \frac{\partial y^{m+1}}{\partial x}\frac{\partial \varphi}{\partial x}\,\mathrm{d}\Omega \\ &\quad + \int_\Omega \frac{\partial y^{m+1}}{\partial z}\frac{\partial \varphi}{\partial z}\,\mathrm{d}\Omega + \int_\Omega (F\circ \Lambda^{-1})(y^{m+1})\varphi \,\mathrm{d}\Omega \\ &\quad + \int_{\Gamma_0(t^{m+1})} g\theta^{m+1}\varphi \,\mathrm{d}\Gamma - \int_{\Gamma_0(t^{m+1})} g\varphi \,\mathrm{d}\Gamma = 0 \;\; \forall \varphi \in V_0(t^{m+1}), \end{aligned} \tag{23}$$

$$e^{m+1} \in (E\circ \Lambda^{-1})(y^{m+1}), \tag{24}$$

$$\theta^{m+1} \in (H\circ \Lambda^{-1})(y^{m+1}). \tag{25}$$

Next, for the spatial discretization of Eqs. (23)–(25), we introduce a finite element triangular mesh of the domain Ω, τ_h^*. By using the classical linear piecewice Lagrange finite elements, the space V_h and its subsets V_{0h} and $V_{\mathrm{A}h}$ are defined by

$$V_h = \{\varphi_h \in C^0(\bar{\Omega})/\varphi_h|_P \in P_1, P \in \tau_h^*\}$$

$$V_{0h} = \{\varphi_h \in V_h/\varphi_h = 0 \text{ on } \Gamma_1(t^{m+1}) \cup \Omega_{\mathrm{A}}(t^{m+1})\}$$

$$V_{\mathrm{A}h} = \{\varphi_h \in V_h/\varphi_h = \Lambda(T_{\mathrm{A}}) \text{ on } \Gamma_1(t^{m+1}) \cup \Omega_{\mathrm{A}}(t^{m+1})\}.$$

Notice that for a nonconstant Dirichlet condition T_{A} an appropriate projection of T_{A} onto V_h has to be considered.

Therefore, in order to obtain the reduced temperature, y^{m+1}, we need to overcome three nonlinear aspects: a Signorini type condition on the boundary $\Gamma_0(t)$, the enthalpy operator E and the viscous dissipation function F. As the two first nonlinearities can be associated to maximal monotone operators, we treat them by means of duality methods. For this, we introduce two new unknowns, q^{m+1} and p^{m+1}, given by

$$q^{m+1} \in (H\circ \Lambda^{-1})(y^{m+1}) - \omega_1 y^{m+1}, \tag{26}$$

$$p^{m+1} \in (E\circ \Lambda^{-1})(y^{m+1}) - \omega_2 y^{m+1}, \tag{27}$$

in terms of two positive real parameters ω_1 and ω_2, respectively. Next, we use the equivalences (see [1] for details):

$$q^{m+1} \in (H \circ \Lambda^{-1} - \omega_1 I)(y^{m+1}) \Leftrightarrow q^{m+1} = (H \circ \Lambda^{-1})^{\omega_1}_{\lambda_1}(y^{m+1} + \lambda_1 q^{m+1}),$$

$$p^{m+1} \in (E \circ \Lambda^{-1} - \omega_2 I)(y^{m+1}) \Leftrightarrow p^{m+1} = (E \circ \Lambda^{-1})^{\omega_2}_{\lambda_2}(y^{m+1} + \lambda_2 p^{m+1}), \quad (28)$$

$(H \circ \Lambda^{-1})^{\omega_1}_{\lambda_1}$ being the Yosida approximation of the operator $((H \circ \Lambda^{-1}) - \omega_1 I)$ with parameter $\lambda_1 > 0$ and $(E \circ \Lambda^{-1})^{\omega_2}_{\lambda_2}$ being the Yosida approximation of the operator $((E \circ \Lambda^{-1}) - \omega_2 I)$ with positive parameter λ_2. For convergence purposes, the choice $\lambda_i \omega_i = 0.5, i = 1, 2$ has been considered. Next, we replace (26) and (27) in (23)–(25) and use the equivalences (28).

Finally, for the treatment of the nonlinear term F we follow [3] where a numerical technique based on Newton's method to linearise the problem and a finite element product approximation is developed. The linear system obtained at each Newton's iteration is solved by means of a preconditioned biconjugate gradient method because the system matrix is not necessary positive definite nor well conditioned.

4. Numerical algorithm for the coupled problem

In this section, a scheme of the numerical method to solve the coupled problem is presented. The objective is to compute the profile, the velocity and the temperature distribution of the ice sheet as well as the corresponding basal magnitudes. For this, we solve sequentially the specific equations by using the previously described numerical strategies. In fact, the pseudocode of the algorithm remains as follows:

(1) Step 0:

- Fixed domains meshing: $[-1, 1]$ for profile and Ω for velocity and temperature.
- Initialize basal velocity ($u_{\rm b}^0$), temperature ($T^0 = T_0$) and profile ($\eta^0 = \eta_0$).

(2) Step $m + 1$: Compute the unknowns at time $t^{m+1} = (m + 1)\Delta t$.

- From $u_{\rm b}^m$ and T^m, computation of $\eta^{m+1}(x), S_-(t^{m+1}), S_+(t^{m+1})$, by solving (4).
- Identification of the sets $\Omega_{\rm I}(t^{m+1}), \Omega_{\rm A}(t^{m+1}), \Gamma_0(t^{m+1})$ and $\Gamma_1(t^{m+1})$ by using (2), (3) and (6)–(7).
- Computation of the velocities, u^{m+1} and v^{m+1}, with expressions (9)–(11).
- Obtain y^{m+1} and, consequently, T^{m+1}, by solving (23)–(25).
- Update basal velocity and shear stress, $u_{\rm b}^{m+1}$ and $\tau_{\rm b}^{m+1}$, with expression (15).

For the numerical solution of Eqs. (23)–(25) we use the equivalences (28) which allow us to develop in the $m + 1$ iteration an inner iterative procedure. In such procedure, one step to update

p^{m+1} and q^{m+1} plus one step to update y^{m+1} are performed until convergence. Moreover, at each step of the y^{m+1} updating we use Newton's method to solve the nonlinear equations system.

5. Numerical example

A numerical example associated to a real data set issued from the Antarctic ice sheet is presented. Thus, the value $z_{\max} = 1.2$ guarantees that $\Omega_{\mathrm{I}}(t) \subset \Omega$ for all t. The dimensionless physical constants

$$\gamma = 11.3, \quad v = 0.01, \quad \alpha = 0.3, \quad \beta = 0.12, \quad \delta = 0,$$

$$L_{\mathrm{c}} = 3.4, \quad g = 0.18, \quad T_{\mathrm{A}}(t,x,z) = -1,$$

have been considered (see [7]). The initial profile and temperature are given by

$$\eta_0(x) = 0.5(1 - |x|^{4/3})^{3/8}, \quad x \in [-1, 1],$$

$$T(0,x,z) = \begin{cases} -1 & \text{if } z \geqslant \eta_0(x), \\ -0.5 & \text{if } z < \eta_0(x) \end{cases}$$

and the accumulation–ablation function is

$$a(t,x) = \begin{cases} \dfrac{a_0(1 - 1.5|x|^{4/3})}{(1 - |x|^{4/3})^{5/8}} & \text{if } 0 \leqslant |x| < L - \varepsilon, \\ \dfrac{a_0(1 - 1.5(L-\varepsilon)^{4/3})}{(1 - (L-\varepsilon)^{4/3})^{5/8}} & \text{if } (L - \varepsilon) \leqslant |x| \leqslant L, \end{cases}$$

where $L = 1, \varepsilon = 10^{-4}$ and $a_0 = 1.24 \times 10^{-5}$. In the basal velocity law (15) the constants $c_{\mathrm{b}} = 0.1$ and $\delta_{\mathrm{b}} = 0.001$ have been taken.

For the numerical algorithm, a uniform finite element mesh with 2001 nodes for the interval $[-1, 1]$ and an unstructured and locally refined triangular finite element mesh with 4977 nodes for the domain $\Omega = [-1, 1] \times [0, z_{\max}]$ have been used. The time step $\Delta t = 10^{-1}$ (which represents 10^4 years) and the parameters $\omega_1 = 0.005, \omega_2 = 0.05$ in the duality method have been considered.

Figs. 1 and 2 show the evolution of basal magnitudes T_{b} and u_{b}, respectively. First, notice the expansion process of the ice base from $t = 25$ to $t = 75$, the temperate basal ice being located at the

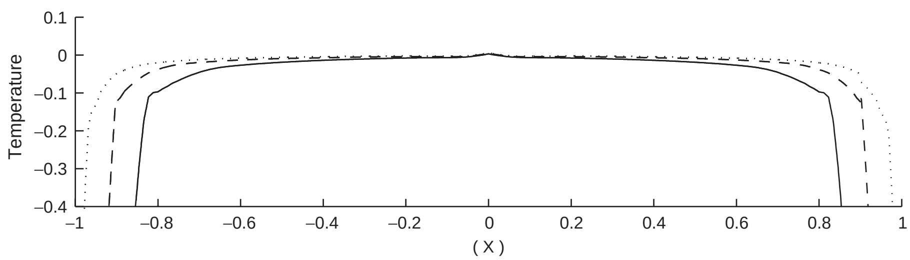

Fig. 1. Basal temperature: $t = 25$ $(-)$, $t = 50(--)$, $t = 75$ $(\cdots)$.

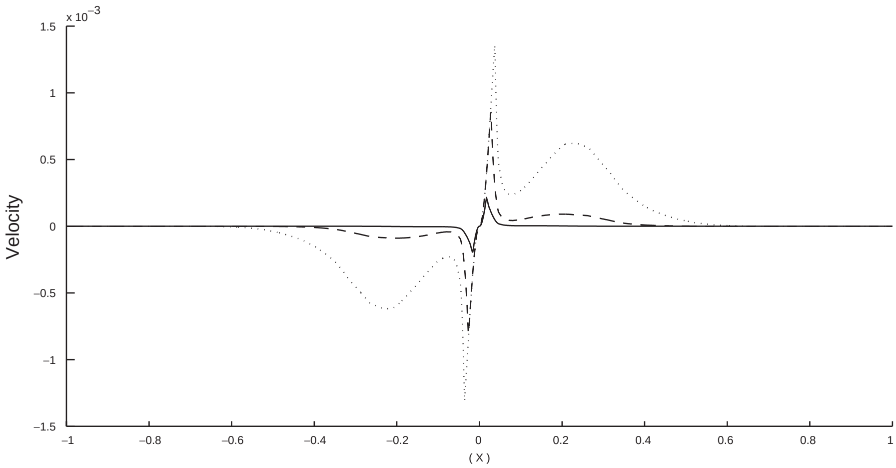

Fig. 2. Basal velocity: $t = 25$ (−), $t = 50$ (− −), $t = 75$ (· · ·).

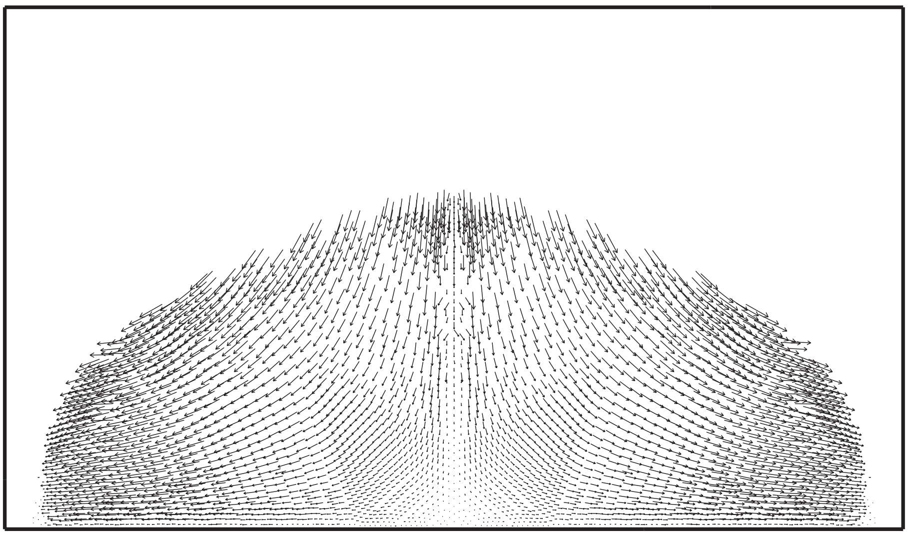

Fig. 3. Velocity field at time $t = 50$.

center. In Fig. 2, basal sliding from the center to the margins even occurs in small cold ice regions as stated by the model.

For $t = 50$, Figs. 3 and 4 show the computed velocity field and the isotherms, respectively. Notice the qualitative aspect of the velocity and the behaviour of the moving boundaries appearing in the problem: the ice sheet base free boundary $S_{-}(t)$ and $S_{+}(t)$, the ice–atmosphere interface $\eta(t)$ and the cold ice–temperate region interface (melting point isotherm adhered at the ice-sheet bottom).

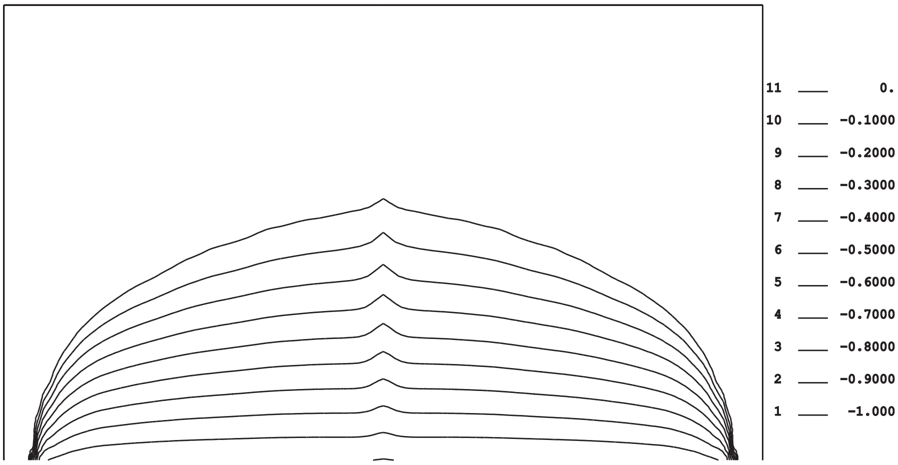

Fig. 4. Isotherms at time $t = 50$.

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