ELSEVIER
Journal of Pure and Applied Algebra 177 (2003) 49–59
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

On rack cohomology

P. Etingof[a,*], M. Graña[b]

[a] Math Department, Office 2-176 77 Mass. Ave., MIT, Cambridge, MA 02319, USA
[b] Math Department, Office 2-155 77 Mass. Ave., MIT, Cambridge, MA 02319, USA

Received 30 January 2002; received in revised form 1 May 2002
Communicated by C. Kassel

Abstract

We prove that the lower bounds for Betti numbers of the rack, quandle and degeneracy cohomology given in Carter et al. (J. Pure Appl. Algebra, 157 (2001) 135) are in fact equalities. We compute as well the Betti numbers of the twisted cohomology introduced in Carter et al. (Twisted quandle cohomology theory and cocycle knot invariants, math. GT/0108051). We also give a group-theoretical interpretation of the second cohomology group for racks.

MSC: 16W30; 57M27

1. Introduction

A *rack* is a pair $(X,\triangleright)$ where X is a set and $\triangleright : X\times X\to X$ is a binary operation such that:
(1) The map $\phi_x : X\to X$, $\phi_x(y)=x\triangleright y$, is a bijection for all $x\in X$, and
(2) $x\triangleright(y\triangleright z)=(x\triangleright y)\triangleright(x\triangleright z)\ \forall x,y,z\in X$.

It is easy to show that $(X,\triangleright)$ is a rack if and only if the map $R : X^2\to X^2$ given by $R(x,y)=(x,x\triangleright y)$ is an invertible solution of the quantum Yang–Baxter equation $R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$.

Racks have been studied by knot theorists in order to construct invariants of knots and links and their higher dimensional analogs (see [4] and references therein). A basic example of a rack is a group with the operation $x\triangleright y=xyx^{-1}$ (or, more generally, a conjugation invariant subset of a group).

* Corresponding author.
E-mail addresses: etingof@math.mit.edu (P. Etingof), matiasg@math.mit.edu (M. Graña).

PII: S0022-4049(02)00159-7

Several years ago, Fenn et al. [6] proposed a cohomology theory of racks. Namely, for each rack X and an abelian group A, they defined cohomology groups $H^n(X,A)$. This cohomology is useful for knot theory and also, as was recently found, for the theory of pointed Hopf algebras [7]. There have been a number of results about this cohomology [9,11,3], in particular it was shown in [3] that for a finite rack X and a field k of characteristic zero, the Betti numbers $\dim H^n(X,k)$ are bounded below by $|X/{\sim}|^n$, where $\sim$ is the equivalence relation on X generated by the relation $z \triangleright y \sim y\ \forall y,z \in X$. The equality was anticipated in [3], and proved in a number of cases [9,11], but not in general.

The main result of this paper implies that the Betti numbers of a finite rack are always equal to $|X/\sim|^n$. The proof is based on a group-theoretical approach to racks, originating from the works [10,12] on set-theoretical solutions of the quantum Yang–Baxter equation. Namely, we use the structure group G_X and the reduced structure group G_X^0 of a rack X considered in [10,12].

We also give a group-theoretic interpretation of the second cohomology group $H^2(X,A)$, which is used in the theory of Hopf algebras. Namely, we show that this group is isomorphic to the group cohomology $H^1(G_X, \mathrm{Fun}(X,A))$, where $\mathrm{Fun}(X,A)$ is the group of functions from X to A. This is a relatively explicit description, since it is shown by Soloviev [12] that for a finite rack X, the group G_X is a central extension of the finite group G_X^0 by a finitely generated abelian group. Thus the cohomology of G_X can be studied using the Hochschild–Serre sequence.

2. Definitions and notation

Definition 2.1. The *structure group* of a rack X is the group G_X with generators being the elements of X and relations $x \cdot y = (x \triangleright y) \cdot x\ \forall x, y \in X$. [1]

The group G_X acts on X from the left by $\triangleright$. Consider the quotient G_X^0 of G_X by the kernel of this action, i.e. the group of transformations of X generated by $x\triangleright$. This group is called the reduced structure group of X.

Remark 2.2. The groups G_X, G_X^0 were studied by Soloviev [12] (we note that in his work, racks are called "derived solutions"). In particular, he showed that the category of racks is equivalent to the category of quadruples (G, X, ρ, π), where G is a group, X a set, $\rho : G \times X \to X$ a left action, and $\pi : X \to G$ an equivariant mapping (where G acts on itself by conjugation), such that $\pi(X)$ generates G and the G-action on X is faithful. Namely, the quadruple corresponding to X is simply (G_X^0, X, ρ, π), where ρ and π are obvious.

Now let us define rack cohomology. Let X be a rack. Let G_X be its structure group. Let M be a right G_X-module. We define a cochain complex $(C^\bullet(X,M), d)$, where

[1] This group appears already in the work of Joyce [8], who pointed out that the functor $X \to G_X$ is adjoint to the functor assigning to a group the underlying rack (with the conjugation operation). Thus the group G_X can be viewed as the "enveloping group" of X.

$C^n(X,M) = \mathrm{Fun}(X^n,M)$, $n \geqslant 0$, with differential

$$df(x_1,\dots,x_{n+1}) = \sum_{i=1}^{n} (-1)^{i-1}(f(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1}) - f(x_1,\dots,x_{i-1},x_i \triangleright x_{i+1},\dots,x_i \triangleright x_{n+1})\cdot x_i)$$

(here X^0 is a set of one element, and $\mathrm{Fun}(Y,Z)$ is the set of functions from Y to Z for any sets Y,Z).

Definition 2.3. The cohomology of $C^\bullet(X,M)$ is called the rack cohomology of X with coefficients in M.

This includes the ordinary rack cohomology with coefficients in an abelian group A, introduced in [6] (this corresponds to taking $M = A$ with the trivial action of G_X), as well as the twisted rack cohomology introduced in [2] (in this case one needs to take a $\mathbb{Z}[T,T^{-1}]$ module M, and define a right action of G_X on it by $vx = Tv$, $x \in X$).

Remark 2.4. One can also define the dual notion of rack homology. As usual, it is completely analogous to cohomology, so we will not consider it.

Remark 2.5. In [1] there is a more general definition of cohomology, with coefficients in objects of a wider category than that of G_X-modules. When restricted to G_X-modules, the definition there takes as differential the map d', defined by

$$d'f(x_1,\dots,x_{n+1}) = \sum_{i=1}^{n} (-1)^{i-1}(f(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})(x_1 \triangleright (x_2 \triangleright (\cdots x_i)))^{-1} - f(x_1,\dots,x_{i-1},x_i \triangleright x_{i+1},\dots,x_i \triangleright x_{n+1})).$$

This complex is isomorphic to the one we consider here, by means of the map

$$T : (C^\bullet(X,M),d) \to (C^\bullet(X,M),d'),$$

defined by $(Tf)\,(x_1,\dots,x_n) = f(x_1,\dots,x_n)(x_1\cdots x_n)^{-1}$.

3. The structure of rack cohomology

Let M be a right G_X-module. Then $C^n(X,M) = \mathrm{Fun}(X^n,M)$ is also a right G_X-module, with the action defined on the generators by

$$(f\cdot y)(x_1,\dots,x_n) = f(y \triangleright x_1,\dots,y \triangleright x_n)\cdot y.$$

Lemma 3.1. (1) *The coboundary operator $d : C^n(X,M) \to C^{n+1}(X,M)$ is a map of G_X-modules. In particular, there is a natural right action of G_X on the groups of cocycles $Z^n(X,M)$, coboundaries $B^n(X,M)$, and cohomology $H^n(X,M)$.*

(2) *$H^n(X,M)$ is a trivial G_X-module.*

Proof.
(1) Straightforward.
(2) Let $f \in Z^n(X,M)$ and consider $f_y \in C^{n-1}(X,M)$, defined by the formula

$$f_y(x_2,\dots,x_n) = f(y,x_2,\dots,x_n).$$

Note that

$$\begin{aligned} d(f_y)\,(x_1,\dots,x_n) &= (f - f\cdot y)(x_1,\dots,x_n) - (df)(y,x_1,\dots,x_n) \\ &= (f - f\cdot y)(x_1,\dots,x_n). \end{aligned} \tag{3.2}$$

Then $f\cdot y = f$ in $H^n(X,M)$. □

Remark 3.3. The action $f\cdot y$ and the assignments $f \mapsto f_y$, as well as (3.2), appear in [9].

By Lemma 3.1 we can consider the subcomplex $C^\bullet_{\mathrm{inv}}(X,M) = C^\bullet(X,M)^{G_X}$. We define the *invariant rack cohomology* $H^\bullet_{\mathrm{inv}}(X,M) = H^\bullet(C^\bullet_{\mathrm{inv}}(X,M))$. Clearly, we have a natural map

$$\xi : H^\bullet_{\mathrm{inv}}(X,M) \to H^\bullet(X,M),$$

induced by the inclusion of complexes.

Remark 3.4. If $f \in Z^n_{\mathrm{inv}}(X,M)$, by the proof of Lemma 3.1 part 2, it is clear that $f_y \in Z^{n-1}(X,M)\ \forall y \in X$.

For M,N right G_X-modules, consider the natural multiplication map

$$C^a(X,M) \times C^b(X,N) \to C^{a+b}(X,M\otimes N).$$

This map will be denoted by $f,g \mapsto f\otimes g$.

Lemma 3.5. *Suppose that A is a trivial G_X-module. Then for any $f \in C^i(X,A)$, $g \in C^j_{\mathrm{inv}}(X,N)$, one has*

$$d(f\otimes g) = df\otimes g + (-1)^i f\otimes dg.$$

Proof. The proof is straightforward. We note that the statement becomes false if A is nontrivial as a G_X-module or g is not invariant. □

Lemma 3.5 shows that if $f \in Z^i(X,A)$ and $g \in Z^j_{\mathrm{inv}}(X,N)$ then $f\otimes g \in Z^{i+j}(X,A\otimes N)$. Furthermore, by the same lemma, the cohomology class of $f\otimes g$ depends only of the cohomology classes of f and g. Thus, we have a product

$$H^\bullet(X,A) \times H^\bullet_{\mathrm{inv}}(X,N) \to H^\bullet(X,A\otimes N).$$

In particular, if R is a (unital) ring with the trivial G_X-action, then $H^\bullet_{\mathrm{inv}}(X,R)$ is a graded algebra, and for any left R-module M with a compatible G_X-action, $H^\bullet_{\mathrm{inv}}(X,M)$ is a graded left $H^\bullet_{\mathrm{inv}}(X,R)$-module.

4. Cohomology of finite racks

In this section we will assume that X is a finite rack.

Let M be a right G_X-module, such that the kernel K of the action of G_X on M has finite index. Let L be the intersection of K with the kernel Γ of the action of G_X on X, and let $G = G_X/L$ (notice that G is finite). Assume that the multiplication by $|G|$ is an isomorphism $M \to M$.

Lemma 4.1. *Under these conditions the map $\xi : H^\bullet_{\mathrm{inv}}(X,M) \to H^\bullet(X,M)$ is an isomorphism.*

Proof. The complex $C^\bullet(X,M)$ is a complex of G-modules. On each term of this complex we have a projector given by $P = |G|^{-1} \sum_{g\in G} g$, which projects to G_X-invariants. This projector commutes with the differential, so the complex $C^\bullet(X,M)$ is representable as a direct sum of complexes:

$$C^\bullet(X,M) = C^\bullet_{\mathrm{inv}}(X,M) \oplus C^\bullet(X,M)(1-P).$$

By Lemma 3.1, the second summand is acyclic: indeed, any cohomology class in it satisfies $cP = 0$, while the lemma says that $cP = c$, hence $c = 0$. This implies the desired statement. □

In particular, for any ring R with trivial G_X-action, such that $N = |G_X^0|$ is invertible in R (for example, $R = \mathbb{Z}[1/N]$ or $R = \mathbb{Q}$), the cohomology $H^\bullet(X,R)$ is an algebra, and if M is an R-module with a compatible G_X action then $H^\bullet(X,M)$ is a left module over this algebra.

Let $\mathrm{Orb}(X) = X/G_X$ be the set of G_X-orbits on X, and $m = |\mathrm{Orb}(X)|$. The main result in this section is

Theorem 4.2. *Under the conditions of Lemma* 4.1, *we have*

$$H^\bullet(X,R) \simeq T_R^\bullet(H^1(X,R)) \simeq T_R^\bullet(\mathrm{Fun}(\mathrm{Orb}(X),R)) \simeq \mathrm{Fun}(\mathrm{Orb}(X)^\bullet,R)$$

as an algebra (*where $T_R^\bullet(B)$ denotes the tensor algebra of an R-bimodule B*), *and if M is an R-module with a compatible G_X action then*

$$H^\bullet(X,M) \simeq T_R^\bullet(H^1(X,R)) \otimes_R M^{G_X} \simeq T_R^\bullet(\mathrm{Fun}(\mathrm{Orb}(X),R)) \otimes_R M^{G_X}$$
$$\simeq \mathrm{Fun}(\mathrm{Orb}(X)^\bullet, M^{G_X})$$

as a left module over the algebra $H^\bullet(X,R)$.

Before proving the theorem, we will derive a corollary.

Corollary 4.3. *The Betti numbers of X are $\dim H^i(X,\mathbb{Q}) = m^i$. Furthermore, the only primes which can appear in the torsion of $H^\bullet(X,\mathbb{Z})$ are those dividing N.*

Proof. The first assertion is clear taking $R = \mathbb{Q}$. For the second one, take $R = \mathbb{Z}[1/N]$ (or $R = \mathbb{Z}/p$, $p \nmid N$) and apply the universal coefficient theorem. □

Remark 4.4. This, together with the lower bounds for the Betti numbers of the quandle and degeneracy cohomology in [3] and the splitting result of [9], implies that those lower bounds are in fact equalities.

Proof of Theorem 4.2. Since $M^{G_X} = H^0(X, M)$, for any M we have an obvious multiplication mapping $\mu : T^\bullet(H^1(X,R)) \otimes_R M^{G_X} \to H^\bullet(X,M)$, which is compatible with the algebra and module structures. Thus, all we have to show is that μ is an isomorphism.

Let us first show that μ is injective. This is in fact the lower bound of [3], but we will give a different proof. The proof is by induction in degree. The base of induction is clear. Assume the statement is known in degrees $< n$, and $c \in \mathrm{Fun}(\mathrm{Orb}(X)^n, M^{G_X})$ is such that $\mu(c) = 0$. This means that the pullback $f : X^n \to M$ of the function c is a coboundary: $f = dg$. Because f is invariant (under the diagonal action of G_X), and $C^\bullet = C^\bullet_{\mathrm{inv}} \oplus C^\bullet(1-P)$, we can assume that g is invariant. This means that for any $y \in X$, we have $(dg)_y = d(g_y)$ (we recall that $g_y(x_1,\dots,x_l) := g(y,x_1,\dots,x_l)$). Thus, $f_y = dg_y$. But f_y is a pullback of a function $c_y \in \mathrm{Fun}(\mathrm{Orb}(X)^{n-1}, M^{G_X})$, so by the induction assumption $c_y = 0$. Hence $c = 0$.

Now let us prove that μ is surjective. For this it suffices to show that $H^n(X,M) \subset H^1(X,R)H^{n-1}(X,M)$. Let $c \in H^n(X,M)$. By Lemma 4.1, the element c can be represented by an invariant cycle, $f \in Z^n_{\mathrm{inv}}(X,M)$. By Remark 3.4, $f_y \in Z^{n-1}(X,M)$ for all $y \in X$. For each $y \in X$, decompose f_y as $f_y = (f_y)^+ + (f_y)^-$, where

$$(f_y)^+ = f_y \cdot P \in Z^{n-1}_{\mathrm{inv}}(X,M) \quad \text{and} \quad (f_y)^- = f_y \cdot (1-P) \in Z^{n-1}(X,M).$$

These functions give rise to unique functions $f^+, f^- \in C^n(X,M)$ such that $(f^\pm)_y = (f_y)^\pm \ \forall y \in X$. Moreover, it is clear that $f = f^+ + f^-$. Since $(f^+)_y \in Z^{n-1}_{\mathrm{inv}}(X,M)\ \forall y$, it is easy to see that $f^+ \in Z^n(X,M)$. Thus also, $f^- \in Z^n(X,M)$. Let us see now that $f^\pm$ are invariant: for any $h \in C^n(X,M), g \in G_X$, we have the equality $h_y \cdot g = (h \cdot g)_{g^{-1}y}$, which implies that

$$f^+_{gy} = f_{gy} \cdot P = f_{gy} \cdot g^{-1}P = (f \cdot g^{-1})_y \cdot P = f^+_y,$$

and thus $(f^+ \cdot g)_y = (f^+_{gy}) \cdot g = (f^+)_y$. Since this equality holds $\forall y \in X$, we have $f^+ \in Z^n_{\mathrm{inv}}(X,M)$ as claimed. Since $f \in Z^n_{\mathrm{inv}}(X,M)$, we also have $f^- \in Z^n_{\mathrm{inv}}(X,M)$. Now, as G_X acts trivially on cohomology, there exists $h \in C^{n-1}(X,M)$ such that $d(h_y) = f^-_y$ for each $y \in X$. Take $\tilde{h} = hP$. We have

$$d((h \cdot g)_y) = d(h_{gy} \cdot g) = d(h_{gy}) \cdot g = f^-_{gy} \cdot g = (f^- \cdot g)_y = f^-_y,$$

and thus, by (3.2), $(d\tilde{h})_y = d(\tilde{h}_y) = f^-_y$, whence $d\tilde{h} = f^-$. Thus, f^- is a coboundary, and we can assume that $f = f^+$. In other words, $f \in \mathrm{Fun}(\mathrm{Orb}(X), Z^{n-1}(X,M)^{G_X})$. This means that $f = \sum_{s \in \mathrm{Orb}(X)} 1_s \otimes f(s)$, where 1_s is the characteristic function of s with values in R. Since 1_s is a cocycle, we have proved that $c \in H^1(X,R)H^{n-1}(X,M)$, as desired. $\square$

Now let M be a semisimple finite dimensional G_X-module over a field k of characteristic zero (but we do not require the image of G_X to be finite). In this case, we have

Theorem 4.5. *Lemma* 4.1 *and Theorem* 4.2 *are true for such* M.

Proof. By a Chevalley's theorem [5], the representations $C^n(X,M) = \mathrm{Fun}(X,k)^{\otimes n} \otimes M$ are semisimple (as tensor products of semisimple representations). Therefore, there exists an invariant projector $P: C^\bullet \to (C^\bullet)^{G_X}$. The rest of the proof is the same as in the previous case. $\square$

Recall [12] that G_X is a central extension of the finite group G_X^0 with kernel being the finitely generated abelian group Γ.

Corollary 4.6. *If M is a finite dimensional $\mathbb{Q}[G_X]$-module and $M(1)$ the generalized eigenspace for the trivial character of Γ, then $H^\bullet(X,M) = H^\bullet(X,M(1))$.*

Proof. Write $M = \bigoplus_\chi M(\chi)$, where χ runs over the characters of Γ. We have $H^\bullet(X,M) = \bigoplus_\chi H^\bullet(X,M(\chi))$. Now, we prove by induction on the dimension of $M(\chi)$ that if χ is nontrivial then $H^\bullet(X,M(\chi)) = 0$. If $\dim M(\chi) = 0$, the cohomology clearly vanishes. Suppose now that $\dim M(\chi) = n > 0$ and for smaller dimensions the statement is known. Let M_0 be a simple submodule of $M(\chi)$. We have then the short exact sequence of complexes

$$0 \to C^\bullet(X,M_0) \to C^\bullet(X,M(\chi)) \to C^\bullet(X,M(\chi)/M_0) \to 0.$$

The first complex is acyclic by Theorem 4.5, the third one is acyclic by the induction assumption, so by the long exact sequence in cohomology, the complex in the middle is also acyclic. The induction step and the corollary are proved. $\square$

Corollary 4.7. *Let M be a finite dimensional $\mathbb{Q}[T^{\pm 1}]$-module. Then the twisted rack cohomology $H^i_T(X,M)$ equals the twisted rack cohomology $H^i_T(X,M(1))$, where $M(1)$ is the generalized eigenspace of T in M with eigenvalue* 1.

To compute the Betti numbers of twisted cohomology, the only lacking case is that in which the elements of the rack X act on M by a Jordan block with 1 on the diagonal.

Proposition 4.8. *Let M be a $\mathbb{Q}G_X$-module with basis $\{v_1,\dots,v_k\}$ on which the elements of X act by $v_i \mapsto v_{i-1} + v_i$ ($v_0 := 0$). Then $\dim H^n(X,M) = m^n$, where $m = |\mathrm{Orb}(X)|$.*

Before proving the proposition we state two easy lemmas:

Lemma 4.9. *Let $(C^\bullet, d)$, be a complex and suppose that $C^\bullet = C_1^\bullet \oplus C_2^\bullet$ and that the differential d has the form $\begin{pmatrix} d_1 & \alpha \\ 0 & d_2 \end{pmatrix}$ for this decomposition. Then α induces a map $\alpha_*^n: H^{n-1}(C_2^\bullet) \to H^n(C_1^\bullet)$. Consider then the short exact sequence of complexes*

$$0 \to C_1^\bullet \xrightarrow{i} C^\bullet \xrightarrow{p} C_2^\bullet \to 0$$

and let $\beta^n: H^{n-1}(C_2^\bullet) \to H^n(C_1^\bullet)$ be the connecting homomorphism. Then $\beta^n = \alpha_*^n$.

Proof. Since $d^2 = 0$, we have $d_1\alpha = -\alpha d_2$, whence it induces a map in cohomology. The second assertion follows in a straightforward way from the definition of the connecting homomorphism. $\square$

Lemma 4.10. *Let $C^\bullet = C_1^\bullet \oplus C_2^\bullet$ be as in Lemma 4.9. Suppose that $(C_2^{\bullet'}, d_2')$ is a complex and that $f : C_2^{\bullet'} \to C_2^\bullet$ is a quasi-isomorphism. Then $(\mathrm{id} \oplus f) : C_1^\bullet \oplus C_2^{\bullet'} \to C^\bullet$ is a quasi-isomorphism, where the first complex has differential given by $\begin{pmatrix} d_1 & \alpha f \\ 0 & d_2' \end{pmatrix}$.*

Proof. This follows easily from the 5-lemma. $\square$

Proof of Proposition 4.8. The proof is by induction on k. If $k = 1$ the assertion is Corollary 4.3. Assume that the result is true for dimensions $< k$. Let us decompose $C^\bullet = C^\bullet(X, M_1) \oplus C^\bullet(X, M_2)$, where M_1 is generated by $v_1, \dots, v_{k-1}$ and M_2 is generated by v_k. Notice that the differential d in $C^\bullet$ can be written as $\begin{pmatrix} d_1 & \alpha \\ 0 & d_2 \end{pmatrix}$, where $d_i : C^\bullet(X, M_i) \to C^\bullet(X, M_i)$ are the differentials of the same complex we are considering for M of dimension $k - 1$ and 1 respectively.

Let us take $C_2^{\bullet'} = T^\bullet(\mathrm{Fun}(\mathrm{Orb}(X), \mathbb{Q}))$. By Theorem 4.2, the inclusion $i : C_2^{\bullet'} \to C_2^\bullet$ is a quasi-isomorphism, and thus by Lemma 4.10 we can work with $C^\bullet(X, M_1) \oplus T^\bullet(\mathrm{Fun}(\mathrm{Orb}(X), \mathbb{Q}))$. We consider the long exact sequence

$$\to H^{n-1}(C_2^{\bullet'}) \xrightarrow{\beta^n} H^n(C_1^\bullet) \xrightarrow{i^n} H^n(C_1^\bullet \oplus C_2^{\bullet'}) \xrightarrow{p^n} H^n(C_2^{\bullet'}) \xrightarrow{\beta^{n+1}} H^{n+1}(C_2^{\bullet'}) \to \tag{4.11}$$

Let $\bar{\alpha} = \alpha|_{C_2^{\bullet'}}$ and consider the induced map in cohomology $\bar{\alpha}_*$, i.e.,

$$\bar{\alpha}_*^n : H^{n-1}(C_2^{\bullet'}) = T^{n-1}(\mathrm{Fun}(\mathrm{Orb}(X), \mathbb{Q})) \to H^n(C_1^\bullet) = H^n(X, M_1).$$

By Lemma 4.9, $\beta^n = \bar{\alpha}_*^n$. We claim that $\mathrm{rk}\ \bar{\alpha}_* = \mathrm{rk}\ \bar{\alpha}$. To see this, it suffices to prove that $\mathrm{Im}\ \bar{\alpha}^n \cap B^n(C_1^\bullet) = 0$. Suppose that $\bar{\alpha}^n(f) \in B^n(C_1^\bullet)$, then it has the form $\bar{\alpha}^n(f) = \sum_{i=1}^{k-1} b_i v_i$, where $b_i \in C^n(X, \mathbb{Q})$. Furthermore, it is clear that $b_{k-1} \in B^n(X, \mathbb{Q})$. On the other hand, if $\pi : X \to \mathrm{Orb}(X)$ is the canonical projection, we have

$$\alpha^n(f)\ (x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^i f(\pi\ (x_1), \dots, \pi(x_{i-1}), \pi(x_{i+1}), \dots, \pi(x_n)) v_{k-1},$$

which shows that $b_{k-1} \in T^n(\mathrm{Fun}(\mathrm{Orb}(X), \mathbb{Q}))$. But it is shown in the injectivity part of the proof of Theorem 4.2 that $T^n(\mathrm{Fun}(\mathrm{Orb}(X), \mathbb{Q})) \cap B^n(X, \mathbb{Q}) = 0$, and the claim is proved.

Then, $\mathrm{rk}\ \beta^n = \mathrm{rk}\ \bar{\alpha}^n$. But the latter is not difficult to compute: if we consider the complex $(D^\bullet, \hat{d})$, where $D^n = \mathrm{Fun}((\mathrm{Orb}(X))^n, \mathbb{Q})$ and $\hat{d}$ is given by

$$\hat{d}(f)\ (a_1, \dots, a_n) = \sum_{i=1}^{n} (-1)^i f(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n),$$

then it is clear that $\bar{\alpha}^n$ and $\hat{d}^n$ have the same rank. Furthermore, it is well known that $D^\bullet$ is acyclic (it gives the reduced cohomology of a simplex of dimension $m - 1$). It is easy then to compute the rank of $\hat{d}$; we have $\mathrm{rk}\ \hat{d}^n = m^{n-1} - m^{n-2} + m^{n-3} - \dots \pm 1$.

We add this computation to the long exact sequence (4.11) and we are done: we have $\operatorname{rk}\beta^n = m^{n-1} - m^{n-2} + \cdots \pm 1$, and since by the inductive assumption $\dim H^n(C_1^\bullet) = m^n$, then $\operatorname{rk} i^n = m^n - m^{n-1} + \cdots \pm 1$. Also, we have $\operatorname{rk}\beta^{n+1} = m^n - m^{n-1} + \cdots \pm 1$ and since $\dim H^n(C_2^{\bullet\prime}) = m^n$, we get $\operatorname{rk} p^n = m^{n-1} - m^{n-2} + \cdots \pm 1$. Thus, $\dim H^n(C^\bullet) = \operatorname{rk} i^n + \operatorname{rk} p^n = m^n$, proving the inductive step. □

Since for M as above we have $\dim M^{G_X} = 1$, we have proved:

Corollary 4.12. *Let M be a right $\mathbb{Q}G_X$-module on which all the elements of X act by the same operator. Then* $\dim H^n(X,M) = m^n \times \dim M^{G_X}$.

Remark 4.13. It is interesting to study the graded algebra $H^\bullet_{\mathrm{inv}}(X,\mathbf{k})$, where $\mathbf{k}$ is a field of characteristic p dividing $|G_X^0|$, to which Theorem 4.2 does not apply. One may ask the following questions about this ring:

- Is it finitely generated?
- What is its Poincaré series? Is it a rational function?

5. A relation with group cohomology

In this section, for any rack, X, we want to give a group theoretical interpretation of the group $H^2(X,A)$ (where A is a trivial G_X-module). This group is useful in the theory of pointed Hopf algebras [7].

We start with the following obvious, but useful proposition.

Proposition 5.1. *Let A be a trivial G_X-module. Then one has a natural isomorphism of complexes $J : C^n(X,A) \to C^{n-1}(X,\operatorname{Fun}(X,A))$, $n \geqslant 1$, where we consider the action of G_X on $\operatorname{Fun}(X,A)$ given by $(hy)\,(x) = h(y \triangleright x)$. It is given by $(Jf)\,(x_1,\ldots,x_{n-1})(x_n) = f(x_1,\ldots,x_n)$. In particular, it induces an isomorphism*

$$H^n(X,A) \to H^{n-1}(X,\operatorname{Fun}(X,A)).$$

Remark 5.2. We note that this proposition becomes false if the action of G_X on A is not trivial.

Now we give the main result of this section. Let M be a right G_X-module.

Proposition 5.3. $H^1(X,M) \simeq H^1(G_X,M)$.

Propositions 5.1 and 5.3 imply

Corollary 5.4. *If A is a trivial G_X-module, then $H^2(X,A) \simeq H^1(G_X,\operatorname{Fun}(X,A))$.*

Proof of Proposition 5.3. Let $C^\bullet(G,M)$ be the standard complex of a group G with coefficient in a right G-module M. Let $\eta : C^1(G_X,M) \to C^1(X,M)$ be the homomorphism induced by the natural map $X \to G_X$. It is easy to show that this homomorphism maps cocycles to cocycles and coboundaries to coboundaries. Thus, it induces a homomorphism $\eta : H^1(G_X, M) \to H^1(X, M)$. Thus, our job is to show that any $f \in Z^1(X, M)$ lifts uniquely to a 1-cocycle on G_X.

To do this, recall that a map $\pi : G_X \to M$ is a 1-cocycle if the map $\hat{\pi} : G_X \to G_X \ltimes M$ given by $g \mapsto (g, \pi(g))$ is a homomorphism. On the other hand, we have a map $\xi_f : X \to G_X \ltimes M$ given by $\xi_f(x) = (x, f(x))$. So we need to show that ξ_f extends to a homomorphism $G_X \to G_X \ltimes M$. But the group G_X is generated by X with relations $xy = (x \triangleright y)x$. Thus, we only need to check that $\xi_f(x)$, $\xi_f(y)$ satisfy the same relations. But it is easy to check that this is exactly the condition that $df = 0$. We are done. $\square$

Another, more conceptual, proof runs as follows: let N be a right X-module (i.e., a right G_X-module) and consider on $X \times N$ the following structure:

$$(x, n) \triangleright (y, m) = (x \triangleright y, n(1 - (x \triangleright y)^{-1}) + mx^{-1}).$$

It is easy to verify that this is a rack structure on the product; we shall denote it by $(X \ltimes N, \triangleright)$ (it is actually the same structure as in [1] for the left X-module N with $x \cdot n = nx^{-1}$). We have then, with a straightforward proof,

Lemma 5.5. *Let $\omega : X \to N$ and define $\hat{\omega} : X \to X \ltimes N$ by $\hat{\omega}(x) = (x, \omega(x)x^{-1})$. Then $\hat{\omega}$ is a rack homomorphism if and only if $\omega \in Z^1(X, N)$.*

Take $\alpha : X \ltimes N \to G_X \ltimes N, \alpha(x, n) = (x, nx)$. One can check that in the square

$$\begin{tikzcd} X \arrow[r, "\hat{\omega}"] \arrow[d] & X \ltimes N \arrow[d, "\alpha"] \\ G_X \arrow[r, "\hat{\pi}"] & G_X \ltimes N \end{tikzcd}$$

each of ω, π determines uniquely the other in such a way that the diagram is commutative.

Remark 5.6. Corollary 5.4 holds also when A is nonabelian. In this case $H^2(X, A)$ is the quotient of the set $Z^2(X, A) = \{f : X \times X \to A \mid f(x \triangleright y, x \triangleright z) f(x, z) = f(x, y \triangleright z) f(y, z)\}$ by the equivalence relation $f \sim f'$ if there is a $\gamma : X \to A$ such that $f'(x, y) = \gamma(x \triangleright y) f(x, y) \gamma(y)^{-1}$. The proof is the same as in the abelian case.

Acknowledgements

The work of P.E. was supported by the NSF grant DMS-9988796. The work of M.G. was supported by Conicet.

References

[1] N. Andruskiewitsch, M. Graña, From racks to pointed Hopf algebras, to appear. math.QA/0202084.
[2] J.S. Carter, M. Elhamdadi, M. Saito, Twisted quandle cohomology theory and cocycle knot invariants, math.GT/0108051.
[3] J.S. Carter, D. Jelsovsky, S. Kamada, M. Saito, Quandle homology groups, their Betti numbers, and virtual knots, J. Pure Appl. Algebra 157 (2001) 135–155.
[4] J.S. Carter, M. Saito, Quandle homology theory and cocycle knot invariants, math.GT/0112026.
[5] C. Chevalley, Théorie des groupes de Lie. Tome III. Théorèmes généraux sur les algèbres de Lie, Hermann & Cie, Paris, 1955.
[6] R. Fenn, C. Rourke, B. Sanderson, James bundles and applications, preprint available at http://www.maths.warwick.ac.uk/∼bjs.
[7] M. Graña, On Nichols algebras of low dimension, in: N. Andruskiewitsch, W. Ferrer Santos, H.-J. Schneider (Eds.), New Trends in Hopf Algebra Theory, Contemp. Math. 267 (2000) 111–136.
[8] D. Joyce, A classifying invariant of knots, the knot quandle, J. Pure Appl. Algebra 23 (1982) 37–65.
[9] R.A. Litherland, S. Nelson, The Betti numbers of some finite racks, math.GT/0106165.
[10] J.-H. Lu, M. Yan, Y.-C. Zhu, On set-theoretical Yang–Baxter equation, Duke Math. J. 104 (2000) 1–18.
[11] T. Mochizuki, Some calculations of cohomology groups of finite Alexander quandles, preprint available at http://math01.sci.osaka-cu.ac.jp/∼takuro.
[12] A. Soloviev, Non-unitary set-theoretical solutions to the quantum Yang–Baxter equation, Math. Res. Lett. 7 (5–6) (2000) 577–596.