An efficient solution methodology to study the response of a beam on viscoelastic and nonlinear unilateral foundation: Static response

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A R T I C L E   I N F O

Article history:
Received 26 September 2012
Received in revised form 7 February 2013
Available online 4 April 2013

Keywords:
Euler Bernoulli beam
Nonlinear analysis
Unilateral contact
Tensionless foundation
Galerkin method
Convergence

A B S T R A C T

Many engineering materials and foundations such as soils demonstrate nonlinear and viscoelastic behaviour. Yet, it is challenging to develop static and dynamic models of systems that include these materials and are able to predict the behaviour over a wide range of loading conditions. This research is focused on a specific example: a pinned–pinned beam interacting with polyurethane foam foundation. Two cases, when the foundation can react in tension and compression as well as only in compression, are considered. The model developed here is capable of predicting the response to static as well as dynamic forces, whether concentrated or distributed. Galerkin’s method is used to derive modal amplitude equations. In the tensionless foundation case, the contact region changes with beam motion and the estimation of the co-ordinates of the lift-off points is embedded into the solution procedure. An efficient solution technique is proposed that is capable of handling cases where there are multiple contact and non-contact regions. Depending on the loading profiles a high number of modes may need to be included in the solution and to speed up computation time, a convolution method is used to evaluate the integral terms in the model. The adaptability of the solution scheme to complicated loading patterns is demonstrated via examples. The solution approach proposed is applicable to dynamic loadings as well and in these cases the automated treatment of complicated response patterns makes the convolution approach particularly attractive. The influence of various parameters on the static response is discussed.

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1. Introduction

Materials such as flexible polyurethane foam used for cushioning in the furniture and automotive industries (White et al., 2000), building soils (Gajan et al., 2010) and biological materials such as muscle tissue (Pioletti and Rakotomanana, 2000), all exhibit highly nonlinear and viscoelastic behaviour. To design systems that incorporate these materials as support structures, it is necessary to be able to understand and predict the static and dynamic behaviour of these systems. If the viscoelastic material and supported structure have significantly different stiffness and inertial characteristics, one may model the combined system as a structure on a viscoelastic foundation. Thus the focus of this work is on understanding the response of a beam on a viscoelastic foundation. While this is one of the simplest type of systems of interest, loss of contact between the structure and the foundation makes this a challenging problem to solve.

Previous work in this area can be classified into three broad categories based on: (I) beam models (e.g., Euler–Bernoulli beam or Timoshenko beam, (Ruge and Birk, 2007)) and foundation models (e.g., single or two-parameter models, (Dutta and Roy, 2002)), (II) beam-foundation interaction models (e.g., bilateral or unilateral) and (III) the type of response studied (e.g., response to static or dynamic or moving loads and different boundary conditions).

A Winkler foundation, in which the supporting medium is taken into account as a system of infinitesimally close springs, producing forces that are functions of the beam displacement, is the simplest and most often adopted model. It assumes that the foundation applies only a reaction force normal to the beam’s undeformed position and that the reaction force is proportional to the beam deflection (Hetenyi, 1946; Timoshenko and Gere, 1961). These types of models are often referred to as one-parameter models. However, several two parameter models like those developed by Filonenko–Borodich, Pasternak, Kerr, Vlasov and Leontev have also been studied (Kerr, 1964; Dutta and Roy, 2002). The choice of beam model depends on the problem being studied. For example, while at low frequencies both the Euler–Bernoulli and Timoshenko beam models on a Winkler type elastic foundation give similar results, at higher frequencies the latter model tends to be more accurate (Ruge and Birk, 2007). Because the beam being considered in the present work is slender and because the applications of our interest fall into the low frequency region (0–100 Hz), Euler–Bernoulli beam model is chosen.
The most common beam-foundation model (regardless of being Winkler or Pasternak variety for the foundation, or being Euler–Bernoulli or Timoshenko model for the beam) allows for both compressive and tensile stresses to exist across the interface between the beam and foundation (bilateral foundation). If a downward transverse load is applied to a beam resting on such foundation, the beam will be compressed into the foundation. If the direction of the load is reversed, the beam and the foundation are pulled up, creating a tension in the foundation. With railroad tracks on the soil foundation being one of the primary motivations, the static and dynamic behaviour of infinite beams on elastic and viscoelastic foundations has been studied by a number of investigators. Kargarnovin et al. (2005) studied the response of infinite beams supported by nonlinear viscoelastic foundation subjected to harmonic moving loads, by solving the governing equations using perturbation method in conjunction with complex Fourier transformation. Younesian et al. (2006) studied the vibration response of a Timoshenko beam supported by a viscoelastic foundation with randomly distributed parameters along the beam length and subjected to a harmonic moving load by employing appropriate Green’s functions. In most of these works the beam was assumed to be infinite and so any applied point load is centered on the beam and all the results emanate from the inherent symmetry of the problem.

Vibration of a finite Euler–Bernoulli beam, supported by non-linear viscoelastic foundation traversed by a moving load was studied and frequency responses of different harmonics, local stability and internal-external resonance conditions were examined by Ansari et al. (2010). Chaotic dynamics of a finite beam on Winkler type soil (Lenci and Tarantino, 1996) and the nonlinear dynamic behaviour and instabilities of a beam under harmonic forcing (Santee and Goncalves, 2006) are some examples of the large body of research on finite beams on bilateral foundations reported in the literature. However, in many of applications adhesion between the beam and the foundation is not assured and so an assumption of bilateral behaviour (compression and stretching of the foundation) is not appropriate. For example, in cushioning applications (seat-occupant systems) or even in railroad structures for that matter, the foundation cannot really react in tension. It has been shown that the phenomenon of lift-off is important as a triggering mechanism for railroad track buckling due to constrained thermal expansion (Kish et al., 1984). Therefore, for such applications, a more appropriate model would include a foundation which reacts to compressive forces but cannot react in tension. Such a foundation, described as one-way, tensionless or a unilateral, is the primary focus of the present work.

The study of response of beams supported by a tensionless foundation is complicated by the need to determine the contact region. Perhaps, it is because of this mathematical difficulty that the static and dynamic responses of a beam on a tensionless foundation have received only limited attention. In the earliest work reported in the literature on this class of problems, the static behaviour of infinite beams resting on tensionless foundations was studied (Tsai and Westmann, 1967; Weitsman, 1970). Investigations of an infinite beam on a tensionless foundation under a moving load were carried out in order to determine the conditions under which separation would occur (Weitsman, 1971) and also the location, magnitude and extent of the lift-off regions (Choros and Adams, 1979). Again, as mentioned before, the concept of having to deal with infinite beams introduces unavoidable symmetry in the problem, which actually makes the solution relatively simpler.

Studies on the behaviour of finite and semi-infinite beams on tensionless foundations do appear in the literature. For finite beams, one of two approaches are typically adopted in these studies. In the first approach, referred to here as Method A, the local boundary conditions are applied while ensuring continuity in the regions of contact and non-contact between the beam and the foundation. Equations are solved giving exact solution when the foundation is assumed to have linear elastic properties. In the second approach, referred to here as Method B, the global boundary conditions are applied and contact functions are defined based on the solution. In this case the solution is approximated by using Galerkin/Ritz methods.

In Method A, there are multiple governing equations defining the behaviour of each section of the beam and the form of these equations change depending on whether the beam is in contact or not. In addition, the equations for each section are constrained to meet the section-specific boundary and continuity conditions. Using this method, Zhang and Murphy (2004) studied the static response under symmetric and asymmetric loads. Silveira et al. (2008) developed a semi-analytical methodology, using a Ritz-type approach, to study the elastic equilibria and instability in beams, columns and arches resting on a tensionless Winkler-type elastic foundation. Though static and dynamic behaviour of finite or semi-infinite beams resting on a tensionless Winkler foundation have been studied under a variety of loading and boundary conditions (Celep et al., 1989; Raju and Rao, 1993; Coskun and Engin, 1999; Lancioni and Lenci, 2007, 2010), there has been little research done on the nonlinear behaviour of beams or plates on nonlinear elastic foundations. Most of these studies are focused on applications like that of railroad tracks on soil foundation, and damping is typically given little or no consideration. In the applications of interest here, damping cannot be neglected since we are dealing with structures interacting with viscoelastic materials. While in future, hereditary-type viscoelastic models will be included in the analysis, as a first step, here the viscoelasticity of the foundation is modelled as velocity proportional damping.

In spite of this simple model of damping, this is a challenging problem to study because of the potential of having only partial contact and the contact region changing as the beam vibrates. Even with linear foundation models, when there is partial contact the problem is nonlinear. The four cases illustrated in Fig. 1 are for:

\[ F_f(w) = \begin{cases} \frac{K_1}{w} & \text{if } w \leq 0 \\ \frac{K_1}{w^3} & \text{if } w > 0 \end{cases} \]

Fig. 1. Illustration of force in the foundation \( F_f(w) \), where \( w \) is the deflection for: (a) linear bilateral foundation, (b) linear unilateral foundation, (c) nonlinear bilateral foundation, and (d) nonlinear unilateral foundation. Foundation force \( F_f(w) = K_1 w + K_2 w^3 \), where \( K_1 \) and \( K_2 \) are the foundation stiffness parameters. In cases (a) and (b) \( K_1 = 0 \), and in cases (b) and (d) \( F_f(w) = 0 \) for \( w > 0 \).
(1) a bilateral linear foundation with the same constant stiffness both in tension and compression (see Fig. 1(a)), (2) a unilateral linear foundation with a constant stiffness in compression, which can be modelled as a bilateral foundation with the nonlinear stiffness characteristic illustrated in Fig. 1(b), (3) a similar case to (1), but with a cubic nonlinear stiffness in tension and compression as shown in Fig. 1(c) and lastly, (4) a similar case to (2), but with a cubic nonlinear stiffness in compression as shown in Fig. 1(d). From Fig. 1(b) it can be seen that the problem is nonlinear even though the material is linear. Also, when the beam is oscillating the contact regions of the beam with the foundation for the unilateral cases (2) and (4) are continuously changing. In the next section, this unilateral behaviour is modelled through use of an auxiliary contact function \( H(w) \), where \( w(x,t) \) is the beam deflection, \( x \) is the position on the beam along its length direction, and \( t \) is the time.

Investigators that used the Method A solution procedure (Zhang and Murphy, 2004; Coskun and Engin, 1999; Silveira et al., 2008) dealt with simple loads giving rise to one or two contact regions at the most. With Method A, as the number of contact and non-contact regions increase, the number of governing differential equations increase each with their own set of local boundary conditions and continuity equations. If the loads are dynamic, the contact region will change differently in every cycle making Method A very difficult to apply. Thus, Method B was chosen as the solution procedure in this research. The technique is enhanced so that it can automatically handle multiple changing contact regions and complicated static loads. The response to static loading is described in this paper, the approach can also be used to find solutions when dynamic loads are applied.

The example under consideration in this work is a pin-ended, pinned beam resting on viscoelastic, nonlinear and tensionless foundation. The problem is formulated to study the response behaviour of the beam when subject to axial static load and transverse static and dynamic loads. Galerkin’s method using linear modal basis functions is employed to derive the modal amplitude equations. These equations also involve the co-ordinates of the unknown lift-off points which are to be determined as a part of the solution procedure. The modal amplitude equations are solved by using a Newton–Raphson technique and lift-off points are obtained from the mode shape predicted at each stage of the iteration. The zero locations of the mode shape are determined by using a bisection method which was found to be more reliable than other root solving methods. Method B has some limitations, but, they are controllable to some extent. The primary limitation of Method B comes from having to deal with the high number of modes that need to be included in the solution to reach the required accuracy, for some types of loading. As the number of modes increase, determining the results of the integration in the intermediate steps of the solution procedure become computationally expensive. This issue is circumvented by expressing the mode shapes in the complex exponential form and expanding products in the integrands as a sum of complex exponentials. The results of the multiplication are evaluated by convolving the coefficient vectors associated with the terms being multiplied. Another challenge is defining the contact function dynamically as the estimates of the modal amplitudes increase each with their own set of local boundary conditions and continuity equations. Towards the end of this region and the beginning of region II, the strain rate or elastic stress (Azizi et al. (2000) and others have included a viscoelastic term to account for the memory of the foam. It is typically a convolution of hereditary kernel and the strain, strain rate or elastic stress (Azizi et al., 2012). It is possible to incorporate these more complex models into the modelling and solution technique described in this paper. Even though damping does not play any role in studying the static response, the modelling approach described below sets the stage for a more general problem when loads can be dynamic.

A homogeneous pinned–pinched beam of length 2\( L \) on a Winkler type non-linear, viscoelastic and unilateral foundation is considered as illustrated in Fig. 3. While in the illustration concentrated loads are applied at \( x = 0 \) (\( F_0 + F(t) \)), \( x = -L/2 \) (\( F_1 \)) and \( x = +L/2 \) (\( F_2 \)), and there is an axial load (\( p \)), the model can account for the modelling and solution approach are described in detail and some illustrative examples for various static loading conditions are given.

2. Modelling of the beam on a viscoelastic foundation

For the flexible polyurethane foam used for cushioning in car seats, the elastic modulus changes with the amount of compression in the material (White et al., 2000). In Fig. 2 is shown a typical measurement of car seat foam when it is quasi-statically loaded in compression and then unloaded. Nonlinearity and hysteresis are the two characteristics of foam immediately evident from the force–deflection curve. The three distinct compression regions identified in Fig. 2 correspond to three different compression mechanisms (White et al., 2000). Readers are referred to Leenslag et al. (1997) and Cavender (1993) for more details on foam behaviour. In region I, the struts that comprise of the foam cell walls bend elastically under the action of compressive load and the static stiffness (or the tangent to the curve) remains approximately constant. Towards the end of this region and the beginning of region II, the struts enter a buckling phase. Therefore in region II, they easily bend with a very slight increase in load. In this region, the material is softer as indicated by the tangent stiffness. Lastly in region III, a sharp increase in local stiffness is observed as a result of compaction or densification of the material.

In this paper, the initial compression behaviour (0–20%) of foam in region I and beginning of region II is modelled as a cubic polynomial. However, note that the modelling approach or the solution technique proposed can be extended for higher order polynomial nonlinearities, which also model the foam behaviour at higher compression rates. Here, damping is assumed to be composed of two parts: the damping force due to the velocity proportional dissipative elements, and the damping force due to foundation. White et al. (2000) and others have included a viscoelastic term to account for the memory of the foam. It is typically a convolution of a hereditary kernel and the strain, strain rate or elastic stress (Azizi et al., 2012). It is possible to incorporate these more complex models into the modelling and solution technique described in this paper. Even though damping does not play any role in studying the static response, the modelling approach described below sets the stage for a more general problem when loads can be dynamic.
concentrated \((F_i \delta(x-x_i))\) or distributed \((G(x,t))\), transverse or axial \((p)\) and static or dynamic loads. \(x_i\) denotes the location of a concentrated load. The foundation is assumed to be non-linear with force in the form: \(F_i(x,t) = K_i \dot{w}(x,t) + K_3 \ddot{w}(x,t)\), where \(w(x,t)\) is the deflection of the beam and \(K_1\) and \(K_2\) are the foundation stiffness parameters. \(C_w\) is the damping force due to the velocity proportional dissipative elements, and \(C_f\) describes the damping force due to foundation. \(w_t\) is the time derivative of \(w(x,t)\). The damping force due to foundation, \(C_w\), and \(F_i(x)\) act only where the beam and foundation interact with each other. By using Hamilton’s principle and employing Euler–Bernoulli beam theory, the equation of motion can be derived as:

\[
EIw_{xxx} + Pw_{xx} + \rho A w_{tt} + C_w w_t + C_f w + \left[ \mathcal{C}_1 \dot{w}^2 + k_1 w + k_3 w^3 \right] \mathcal{H}(w) = g(\xi, \tau) + \mathcal{F}_1(t) + \mathcal{F}_0 \delta(\xi) + \sum_i f_i \delta(\xi - \xi_i).
\]

For convenience, the hat on \(w\) is dropped in the above equation and in the subsequent discussion. The corresponding contact function is defined as:

\[
H(w) = \begin{cases} 
1 & \text{when } w(\xi, \tau) \leq 0, \\
0 & \text{when } w(\xi, \tau) > 0.
\end{cases}
\]

\(\Lambda_n^l\) are the lift-off points defined by \(w(x,t) = 0\) corresponds to lift-off points of the beam at any time \(t\). There can be multiple lift-off points along the length based on the deflection shape. If \(\Lambda_n^l\) and \(\Lambda_n^u\) are the co-ordinates of the lift-off points of \(n\)th contact region \(|CR_n : w(\xi,t) \leq 0, \Lambda_n^l < \xi < \Lambda_n^u|\), then the contact length of \(CR_n\) is defined as \(L_{CR_n} = |\Lambda_n^u - \Lambda_n^l|\). Introducing the non-dimensional parameters,

\[
w = \frac{w}{T}, \quad \xi = \frac{x}{L}.
\]

\[
k_1 = \frac{K_1 L^4}{EI}, \quad k_3 = \frac{K_3 L^6}{EI}.
\]

\[
c_0 = \frac{C_0 \left( L^4/\rho A \right)^{\frac{1}{2}}}{\mathcal{C}_1}, \quad \mathcal{C}_f = \frac{C_f \left( L^5/\rho A \right)^{\frac{1}{2}}}{\mathcal{C}_1},
\]

\[
\tau = \frac{\rho A L^4}{EI}, \quad \omega_2 = \omega_0 \left( \frac{\rho A L^4}{EI} \right)^{\frac{1}{2}},
\]

\[
f_i(t) = \frac{F_i(t)L^3}{EI}, \quad f_i = \frac{F_i L^3}{EI},
\]

\[
g(\xi, \tau) = \frac{G(x,t)L^3}{EI}, \quad p = \frac{PL^2}{EI}.
\]

Eq. (1) can be reduced to:

\[
\begin{align*}
\ddot{w} + \frac{p}{m} w_{xx} + \frac{w_{xt}}{c_t} + \frac{\dot{w}}{c_f} + \left[ \mathcal{C}_1 \frac{\dot{w}^2}{2} + k_1 w + k_3 w^3 \right] \mathcal{H}(w) &= g(\xi, \tau) + \mathcal{F}_1(t) + \mathcal{F}_0 \delta(\xi) + \sum_i f_i \delta(\xi - \xi_i).
\end{align*}
\]

For convenience, the hat on \(w\) is dropped in the above equation and in the subsequent discussion. The corresponding contact function is defined as:

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\end{cases}
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k_1 = \frac{K_1 L^4}{EI}, \quad k_3 = \frac{K_3 L^6}{EI}.
\]

\[
c_0 = \frac{C_0 \left( L^4/\rho A \right)^{\frac{1}{2}}}{\mathcal{C}_1}, \quad \mathcal{C}_f = \frac{C_f \left( L^5/\rho A \right)^{\frac{1}{2}}}{\mathcal{C}_1},
\]

\[
\tau = \frac{\rho A L^4}{EI}, \quad \omega_2 = \omega_0 \left( \frac{\rho A L^4}{EI} \right)^{\frac{1}{2}},
\]

\[
f_i(t) = \frac{F_i(t)L^3}{EI}, \quad f_i = \frac{F_i L^3}{EI},
\]

\[
g(\xi, \tau) = \frac{G(x,t)L^3}{EI}, \quad p = \frac{PL^2}{EI}.
\]
\[ F_s = f(\tau) \Xi_s(0) + f_1 \Xi_s(-0.5) + f_2 \Xi_s(0.5) + f_0 \Xi_s(0) + \int_{-1}^{1} g(\zeta, \tau) \Xi_s(\zeta) d\zeta, \]

where \( \delta_{i,m} \) is the Kronecker delta, where \( m = 1, 2, \ldots, N \), \( s = 1, 2, \ldots, N \), and \( EI \) is the flexural rigidity of the beam. Note that the vector \( \mathbf{Q} = \mathbf{Q}(T, \tau) \) is a function of the modal amplitudes as well. This function along with the matrix \( \mathbf{B} \) depends on the contact and non-contact regions of the foundation as specified by the function \( H(w) \) in Eq. (5).

3. Solution procedure for the static response

For the static case with a constant load, \( T \) is constant in Eq. (7) and the steady state solution can be obtained from the expression:

\[ T = \mathbf{B}^{-1}(\mathbf{F} - \mathbf{Q}). \]  

(9)

However, note that the right hand side is also unknown due to the vector \( \mathbf{Q} \) which depends on nonlinearity in the foundation as well as on the unknown lift-off points and the corresponding contact lengths. Matrix \( \mathbf{B} \) similarly, is unknown because of the unknown contact regions.

The solution methodology developed in this section is applicable for both linear or nonlinear foundation models. Because the contact length is not known a priori and because the foundation reacts only in compression and not in tension, even for the case when the foundation is linear, the problem is nonlinear and the solution is sought by using an iterative scheme. The initial guess to this scheme is an initial deflection shape, defined by \( T_0 \) which by using Eq. (6) gives a corresponding approximation to the deflection of the beam, \( \psi_0(\zeta) \). \( \psi(\zeta) \) denotes an approximation to the solution. The lift-off points for such initial guess are obtained by solving the nonlinear transcendental equation \( \psi(\zeta) = 0 \). First, \( \psi_0(\zeta) \) is evaluated for a range of values of \( \zeta \) to find approximate locations of the lift-off points based on the change in sign in \( \psi(\zeta) \). The estimates are then refined employing the bisection method. The lift-off points \( \zeta_0 = \{\zeta_0^1, \zeta_0^2, \zeta_0^3, \ldots\} \) obtained are used to evaluate the matrices \( \mathbf{B} \) and \( \mathbf{Q} \) (Eq. (8)). With the updated matrices and the initial guess vector \( T_0 \), the nonlinear equations for the modal amplitudes in Eq. (9) are solved using a Newton–Raphson iterative technique. The new estimates of the modal amplitudes \( T \) are used to find and refine the locations of the lift-off points by the bisection method. These iterations are carried out until the prescribed tolerance is achieved on the function value \( BT - F + Q \) which should be \( \mathbf{0} \) for the exact solution \((T, \zeta_0)\).

In the Newton–Raphson iterations, a numerical Jacobian computed by finite differences is used to converge to the solution as opposed to a closed-form Jacobian. Finding the slope of Eq. (9) in closed-form is not possible because of the lack of knowledge in the contact behaviour, unless some information about the number of contact and non-contact regions is deduced from loadings. For simple loading conditions reasoning a possible deflection shape and there by the number of contact and non-contact regions is achievable. In such cases, a closed-form Jacobian can be realised for computational efficiency. However, the current implementation of a numerical Jacobian is not to be perceived as a shortcoming if the step-size between two consecutive iterations is as low as \( 10^{-3} \), the results obtained a numerical Jacobian are accurate. Besides, for complex loading conditions that arise in a broader spectrum of practical applications, especially when the loads are dynamic, lack of initial knowledge of the contact behaviour is inevitable. The objective of this paper is to present a solution method precisely to such complex situations. In the current scheme, there is no restriction on the type or number of loads or on the number of contact regions unlike the previous works in this area adapting a case-by-case basis according to the possible deflection shape (for e.g., see Silveira et al., 2008; Zhang and Murphy, 2004; Coskun and Engin, 1999; Celep et al., 1989). The approach introduced in this paper is more general and is valid for more complicated cases of loading with multiple contact regions.

As will be seen in the subsequent sections, the number of contact regions, the relative foundation-beam stiffness and the type of loading largely effect the convergence of the solution. In such situations, it is desirable to increase the number of modes included in the solution, and study and compare the differences arising in convergence. Also, the solution technique must be capable of automatically accommodating for multiple and changing lift-off regions. Note that, if one wants to understand the response to dynamic loading, all the issues mentioned before have to be addressed in every time-step, while simultaneously reducing the computational time until the response reaches steady state. Evaluating the integrals in Eq. (8) is by far the most computationally involving steps of this approach with respect to the challenges described. To address this, a technique is proposed to compute these integrals by expanding the integrand as a product of sum of complex exponentials (see Appendix A). The coefficients of the resulting product can be determined by convolving the coefficients of the two individual complex harmonic series being multiplied together. Multiple successive convolutions can be used for higher order terms. In contrast to restricting to a particular deflection shape, conventionally simplifying the integrands and calculating the integrals, the proposed approach cannot only handle a high number of modes and multiple contact regions, but also results in a huge reduction in computational time, thus setting a stage to study the response to dynamic loads as well.

An example shown in Fig. 4 illustrates the manner how a deflection shape given as an initial guess (blue-thick or green-dashed lines) evolves to the final solution. The inputs to the solution procedure are an initial guess, the number of modes to be included in the solution, the system parameters and the loading. The solution scheme updates the number of lift-off points and hence the contact regions in every iteration while converging to the final equilibrium shape. If the norm of the difference between the modal amplitudes \((T_i)\) in two consecutive iterations is less than \( 10^{-6} \) mm, then the
solution is deemed to be converged. The initial guess for $T_0$ in most of the examples in this paper is a vector of normally distributed random sequence with zero mean and unit standard deviation. Within fifty simulations of each case considered, it is observed that the solution converges from such an initial guess, if the linear stiffness $K_1$ of the unilateral foundation is below $10^4$ N m$^{-2}$. For a nonlinear foundation, the solution did not always converge from such an initial guess if $K_3 > 10^4K_1$, where $K_1 > 40,000$ N m$^{-2}$. However, the solution converged seven out of ten times when the linear solution with $K_3 = 0$ was an initial guess to the nonlinear problem ($K_3 \neq 0$).

4. Simulation and discussion

While the subsequent examples (also the one in Fig. 4), consider a 300 mm × 25.4 mm × 1 mm pinned–pinned aluminium ($E = 70$ GPa) beam, the modelling presented in Section 2 is not limited to these examples but is more general and widely applicable. The stiffness of the foundation in most of the examples is obtained from 2.53 min quasi-static compression test data of 76.2 mm polyurethane foam cube (Deshmukh, 2010). A linear approximation of the elastic curve at various pre-compression levels is used. For example, $K_1 = 42,500$ N m$^{-2}$ corresponds to the stiffness of some polyurethane foams close to zero percent compression. While these stiffness values are typical of car seat foams at low compressions, the CONFOR foams primarily used in impact absorption can be 5–10 times stiffer and the stiffness of both types of foams at high compressions can be significantly bigger than that at low compressions. Also, if not mentioned specifically, the foundation is assumed to be tensionless.

4.1. Effect of number of modes $N$

An exact solution is obtained by considering a very large number (theoretically infinite!) of modes. However, practically, as shown in Eq. (6), the solution is approximated as a sum of finite number of modes. But how many number of modes can be considered enough? In this section, an attempt is made to answer this question by studying the effect of number of modes on the convergence of the static solution.

For example, in Fig. 5, the static deflection shapes of a beam on a unilateral foundation of stiffness $K_1 = 42,500$ N m$^{-2}$ are shown. A concentrated load $F_1 = -15$ N is applied at $x_1 = -0.9L$. First, five modes are considered in the solution and the solution shown as the lightest-blue curve. Gradually, the number of modes considered is increased through twenty (seen as darkest-blue) and all the solutions are superposed on the same figure.

Note that while the solution seemed to converge as the number of modes is increased, there are significant disparities in the way the solution converged at different locations along the length of the beam (see the inset figures). Hence, instead of choosing the deflection at an arbitrary point as a norm to decide about convergence, a supremum norm between two consecutive modal approximations is used. That is, the maximum difference between an $N$th mode and $(N - 1)$th mode approximations is used as a norm. When this difference is at least less than three orders of magnitude of maximum deflection in the static response, the solution is said to be converged with respect to the number of modes.

It is also important to investigate how the solution is affected by the type of loading. For this reason, the load $F_1 = -15$ N applied at $x_1 = -0.9L$ is now gradually moved in steps of 0.1L towards the middle of the beam, thereby changing the asymmetric loading to a symmetric one. The convergence of solution as a function of number of modes in the solution is illustrated in Fig. 6 for two different foundation stiffness values, and the converged deflection shapes are shown in Fig. 7. Clearly, from Fig. 6 and the insets, twenty modes are sufficient for all the cases of loading considered. Note that there is no particular trend to compare one curve with another because depending on the location of the applied load, some modes participate and some modes do not. But, overall one

Figure 5. Static deflection shape for a load $F_1 = -15$ N applied at $x_1 = -0.9L$. 5-mode approximation (——) through 20-mode approximation (——–) are compared. Inset figures show the contrasting ways in which the modal approximations converge at two different locations along the length of the beam. Zero load equilibrium position is shown by (——–).

Figure 6. Illustrations of the convergence behaviour of the solution with increasing number of modes as the point of application of load $F_1 = -15$ N is changed between $x_1 = -0.9L$ (——–) to 0L (——) for a foundation stiffness of (a) $K_1 = 42,500$ N m$^{-2}$, (b) $K_1 = 212,500$ N m$^{-2}$. Inset figures show a zoomed-in view of the plot between $N = 15$ to 20 modes.
can see the convergence as the number of modes considered is >15 high. The maximum change is 0.04% of the maximum deflection for the softer foundation and is 0.07% for the stiffer foundation. The static deflection shapes plotted in Fig. 7 are therefore twenty-mode approximations. In Fig. 7(a), one lift-off point is seen if the load is applied anywhere in the first quarter of the beam’s length from the left end. However, if −0.5L < x1 < 0, there is no lift-off of the beam. Also, the highly asymmetric nature of the deflection shape when the load is nearer to the left-end becomes symmetric when the load is at the center, as expected. This can also be seen from Fig. 7(a), the even modes do not participate for the darkest blue curve corresponding to the load being applied at the center of the beam.

A similar example for higher foundation stiffness is shown in Figs. 6(b) and 7(b). The foundation stiffness considered is five times higher than the previous value (K1 = 212.500 N m−2) to approximately represent the behaviour of stiffer foams used in impact absorption applications. Even in this case, twenty modes are sufficient for convergence. Because of higher stiffness, however, not only are the deflections lower in magnitude, but also the ratio of positive deflections to negative is relatively higher than in the previous example. This is due to the fact that the foundation offers a greater resistance and beam is more free to move upwards than downwards. Another noticeable difference is in the lift-off behaviour. As the location where the load is applied is changed from x1 = −0.9L to x1 = 0L, the number of lift-off points is changed from one to two. The two lift-off points are symmetric with respect to zero when x1 = 0L. This phenomenon is shown in the inset in Fig. 7(b).

Thus to summarise, while in both the above examples, considering twenty modes gave consistent and credible equilibrium shapes, one must note that the convergence is highly dependent on the type of loading, relative foundation-beam stiffness (k1 = K1/(EI/L3)) and the number of lift-off regions. Complex loading situations might give rise to more lift-off regions and hence would require a higher number of modes to be considered in the solution. Even though the evaluation of the integrals in Eq. (8) appears to be difficult with high number of modes and/or lift-off regions, especially if k1 ≠ 0, the current solution technique is quite efficient and accommodating even in such situations.

### 4.2. Effect of nonlinearity

As described in Section 1, Fig. 1, even for a linear unilateral foundation the problem is nonlinear. This is illustrated in more detail in Fig. 8, where N = 20-mode solutions are plotted. In this example, a concentrated load F0 is applied at the center of the beam, i.e., at x0 = 0. On a bilateral foundation, as the compressive load is doubled from F0 = −10 N to F0 = −20 N, the mid-point deflection of the beam is doubled as well. Similarly, if the magnitude of the load remained the same but the sign is reversed, so did the mid-point deflection, showing the linear character of the response on a linear bilateral foundation. However, on a unilateral foundation, although the deflection is doubled when the compressive load is doubled, the change in deflection is not proportional to the change in load when load reverses sign with no change in magnitude. In fact, the static response of the beam on a unilateral foundation in the tensile load case, corresponds to that of a pinned–pinned beam with no foundation. Because of this, even though the foundation is linear, the response on a unilateral foundation is piecewise linear, or nonlinear.

Consider the response when bilateral foundation is nonlinear, i.e., when k1 ≠ 0. Example responses using a 20-mode solution for this case are shown in Fig. 9. The parameter values for k1 and K1 are chosen such that the force–deflection curve in Fig. 9 approximates the softening type behaviour of the foam shown in Fig. 2.

For a linear foundation, as expected, the mid-point deflection is proportional to the applied force. The nonlinear foundation
exhibits a softening behaviour under the action of compressive load. For low compressive loads the midpoint deflection is proportional to applied load similar to the linear foundation case. But after \( F_0 = 6 \, \text{N} \), the foundation gets softer and therefore the beam deflects more even with a slight increase in load. In the nonlinear foundation case, the deflections are higher because of softening of the foundation.

### 4.3. Effect of relative foundation-beam stiffness, \( k_1 \)

In Section 4.1, the relative foundation-beam stiffness \( (k_1 = k_1/(EI/L^4)) \) was increased by increasing the foundation stiffness \( (K_1) \). A similar effect can be achieved by increasing the beam length. In this section, the effects of increasing the beam length \( L \) and the foundation stiffness \( (K_1) \) on the deflection and contact behaviour are investigated in different loading scenarios.

As the beam length \( 2L \) is increased, the relative foundation-beam stiffness \( k_1 \) increases as the fourth power of \( L \). This indicates physically, that the beam is getting more and more flexible relative to the foundation of constant stiffness \( (K_1 = 42500 \, \text{N} \cdot \text{m}^{-2}) \). Fig. 10 shows the contact length and midpoint deflection as functions of beam length. The abscissa can be broadly divided into 3 regions. In region I, as the beam length increases, the midpoint deflection increases (gets more negative!). This continues until \( L = 230 \, \text{mm} \) starting the region II. Until this length, the deflection shape is first-mode predominant.

From \( L = 230 \, \text{mm} \) onwards however, the mid-point deflection decreases and increasing curvature of the deflection shape indicates participation of higher modes. This phenomenon is illustrated in the inset figures corresponding to regions I and II. In both the regions, the contact length grows linearly with slope one. This indicates that the entire beam is in contact with the foundation. This persists until \( L = 442 \, \text{mm} \). Quickly, as the beam length is increased further, the contact length begins to shrink. This beam length separating the growing versus shrinking contact lengths is called critical length (Zhang and Murphy, 2004). In region III, the contact length and midpoint deflection both decrease. For considerably longer beams, the contact length reaches the contact length for the infinite beam case (Weitsman, 1970; Zhang and Murphy, 2004). It should be noted that the results are consistent with Zhang and Murphy (2004) and Weitsman (1970) even though Galerkin method has been used in the current solution procedure as opposed to solving the boundary value problems in every region of contact and no-contact for exact solutions. It is also verified that the contact length is independent of the load while the deflection shape is directly proportional to the load when the foundation is linear.

The effect of foundation stiffness was briefly seen in Section 4.1 when illustrating the effect of number of modes included in the solution. In this section, the effect is illustrated with a different example in a more complicated loading situation. The static response of the unilateral case is compared in Fig. 11, with the bilateral and no-foundation cases for two values of foundation stiffness \( (K_1 = 5000, \, 50,000 \, \text{N} \cdot \text{m}^{-2}) \). Consider three loads \( F_0 = 70 \, \text{N}, \ \text{and} \ F_0 = 500 \, \text{N} \) applied symmetrically along the length at \( x_1 = -0.5L, \ \text{and} \ x_2 = 0.5L \). Just as with all the above examples, \( K_1 \) is set to zero, and so the foundation is linear. In this case also, following the same routine described in the Section 4.1, it is observed that 20 modes are enough for the solution to converge with regards to the number of modes. The static response of the beam when \( K_1 = 5000 \, \text{N} \cdot \text{m}^{-2} \) is shown in Fig. 11(a). Three cases: (1) when there is no-foundation, (2) when the foundation reacts both in tension and compression (bilateral), and (3) when the foundation reacts only in tension (unilateral) have been compared here. Clearly, for the bilateral case, the positive deflections are lower and the negative deflections are higher when compared to the beam with no-foundation beneath. This indicates the tensile reaction of foundation on the beam. On the other hand, for the unilateral case, the beam is free to move upwards than downwards. Since the foundation reacts only in compression, it does not allow the beam to penetrate more and the beam is free to deflect in the other direction.

Now consider the same beam with identical loading conditions, but on a foundation with 10 times higher stiffness \( (K_1 = 50,000 \, \text{N} \cdot \text{m}^{-2}) \). As shown in Fig. 11(b), when there is no-foundation, the negative deflections of the beam are higher than the other two cases. However, for the bilateral foundation case, since the beam has to overcome both tension and compression, the deflections are the lowest. If the foundation is tensionless, the positive deflections are higher than the negative as observed earlier. Note that while midpoint deflection is highest for the no-foundation case and the least for unilateral foundation case, this trend is not the same everywhere along the length. For example at \( x = \pm 0.8L \), the deflection is least in bilateral foundation case and highest in unilateral. The midpoint deflection as a function of foundation stiffness is plotted in the Fig. 12.

The behaviour observed here is qualitatively consistent with the results presented by Coskun and Engin (1999) for a free-free beam. Also, from the slope of both curves it can be seen that the rate at which the deflections increase with increasing stiffness is higher for softer foundations than for stiffer foundations. The green arrows in the inset figure show the direction of beam deflection. As the stiffness is increased, the beam moves towards the zero load equilibrium in the bilateral case as opposed to moving upwards in the unilateral case.

### 4.4. Effect of distributed load

The effect of distributed load \( G(x) \) on the static response is studied. Note that the term \( G(x) \) in Eq. (1) can be used to also define the uniformly distributed weight of the beam. The static response and the lift-off behaviour of weightless beam is completely different from that of a beam with uniformly distributed weight (Weitsman, 1970). In this section a weightless beam is subjected to two types of distributed loads: antisymmetric sinusoidal and rectangular loads.

Fig. 13 shows the static response of the beam subjected to rectangular and sinusoidal loads for one spatial period. The deflections in Fig. 13(a), corresponding to the rectangular load case, are higher
than those for the sinusoidal case in Fig. 13(b) as expected. Further, due to the anti-symmetric nature of loading in both cases, there is a zero deflection at $x = 0$ for the bilateral and no-foundation cases. However, due to the tensionless character of the foundation, the deflections in the unilateral case are more positive compared to the other two cases.

In the bilateral foundation case of Fig. 13(a), the first five non-zero Fourier coefficients of the rectangular load $G_n$ shown in Fig. 13(a) are:

- $b_2 = \frac{40}{\pi}$
- $b_6 = \frac{40}{3\pi}$
- $b_{10} = \frac{40}{5\pi}$
- $b_{14} = \frac{40}{7\pi}$
- $b_{18} = \frac{40}{9\pi}$

The first five non-zero modal amplitudes of the response for this case are:

- $T_2 = 1.19538$ mm
- $T_6 = 0.01203$ mm
- $T_{10} = 0.00095$ mm
- $T_{14} = 0.00018$ mm
- $T_{18} = 0.00005$ mm

The response does not change much when higher modes are included. Even though the 35 modal amplitudes were non-zero, most of them were very small compared to $T_2$. For the case shown in Fig. 13(b), only a harmonic load with amplitude equal to the non-zero Fourier coefficient, $b_2 = \frac{40}{\pi}$ is applied, and the
corresponding modal amplitude response for the bilateral case is found to be \( T_2 = 1.19538 \text{ mm} \) with \( T_j = 0, \ j \neq 2 \). This case is different from the former in that there are many non-zero modal amplitudes in the response to a rectangular load, although they are small. Interestingly, as the amplitude of the sinusoidal load is made equal to the first non-zero harmonic \( (b_2) \) of the rectangular load, the response to both the loads become very similar as seen in Figs. 13(a) and (b). Also, the contribution from the higher harmonics of the rectangular load results in slightly higher deflections compared to that for the sinusoidal load.

These characteristics described for the linear no-foundation and bilateral cases are of course not valid for the unilateral case. Even for the linear foundation, the unilateral problem is nonlinear. This phenomenon is evident from the modal amplitudes. For example, modal amplitudes of the responses are plotted in Fig. 14 for the three cases when beam is subjected to a sinusoidal load shown inset of Fig. 13(b). In the bilateral and no-foundation cases, the input force being purely sinusoidal, excites only one mode corresponding to the loading distribution (i.e., 2nd mode, meaning \( T_j = 0, \ j \neq 2 \)). However, for the unilateral case, all the modes are excited. The first few modes are predominant while the contributions from the higher modes start to die out.

The same solution algorithm used for the above examples can as well be used for more complicated loading situations involving multiple contact regions as shown in Fig. 15. Here the static response of the beam subject to distributed loads varying for three cycles along the length is illustrated. In contrast to the previous case in Fig. 13, the deflections in Fig. 15 are much lower. Specifically, the differences in the deflection between bilateral foundation and no-foundation case of Fig. 13 become very small for the case shown in Fig. 15, indicating a negligible effect of foundation. Note from Eq. (8) that the stiffness of the \( mth \) mode for a pinned–pinned beam with no axial load \( (p) \) is \( B_{mm} = m^4 \pi^4/16 \). The foundation stiffness is kept constant for the two cases, but the modal stiffness of the beam has increased from \( B_{22} = 2^4 \pi^4/16 \) to \( B_{66} = 6^4 \pi^4/16 \), by a factor of 81. Therefore, the total response is dominated by the beam alone, making the bilateral and no-foundation cases look very similar. In order to observe the pronounced difference between the bilateral and no-foundation cases, seen in Fig. 13, the stiffness of the foundation was increased 81 times the previous value of \( K_1 \) (42500 N m \(^{-2}\)). The results are shown in Fig. 16.

5. Conclusions

The static responses of a beam on a nonlinear tensionless foundation subjected to concentrated and distributed static loads have been studied. To account for tensionless character of the foundation, an auxiliary contact function is introduced and the solution of the model is sought in the form of a sum of modes of the linear problem. The modal amplitude equations derived involve the coordinates of the unknown lift-off points. Hence, even for a linear foundation the problem is nonlinear and the solution is obtained by an iterative technique. The unknown lift-off points, forming a part of the solution scheme, are obtained by employing bisection method in every iteration. Also, the solution algorithm is modified so that the technique automatically allows for a wider range of static or dynamic excitations involving multiple contact regions. Further, the complicated integrals required are solved by expressing the mode shapes in the complex exponential form and expanding products in the integrands as a sum of complex exponentials. The results of the multiplication are evaluated by convolving the coefficient vectors associated with the terms being multiplied. Within each iteration the current estimate of the beam deflection is used to determine the number of contact and non-contact regions and the approximate location of lift-off points that define the edges of those regions. The bisection method was found to be the most reliable method for determining the location of the lift-off points to a required accuracy with in each iteration. This solution
The effect of various system parameters viz., beam length, linear foundation stiffness and loads on the static response is presented. It is observed that as the relative foundation-beam stiffness (\(k_1\)) is increased, there are significant differences in the lift-off behaviour. At the same time, the number of modes required for convergence also increases with increasing \(k_1\). The response of the unilateral case is compared to the bilateral and no-foundation cases at every stage. It is observed that as the foundation stiffness is increased, the deflections in the bilateral case tend to move towards the zero load equilibrium, while that of the unilateral tend to get more positive due to the increased foundation reaction.

Acknowledgements

The authors gratefully acknowledge support from the National Science Foundation, Dynamic Systems Program, through the Grant CMMI-0728101. Dr. Eduardo Misawa is the Program Director.

Appendix A. Evaluating integrals in Eq. (8)

In this appendix, a sample calculation is shown to demonstrate how convolution can be used to evaluate the integrals of the type that appear in Eq. (8).

Consider a two mode approximation with one contact region, similar to the one shown in Fig. 13 and denote the integral limits in Eq. (8) by \(\zeta_0\) (lift-off point) and 1 (end of the beam). For example, just consider a two-mode solution in which terms like, 

\[ Q_1 = k_1 \int_{\zeta_0}^{1} \left[ T_1 \sin \frac{\pi}{2} (1 + \zeta) + T_2 \sin \frac{2\pi}{2} (1 + \zeta) \right]^3 \sin \frac{\pi}{2} (1 + \zeta) d\zeta \]

of \(Q\) vector and elements of the \(B\) and \(C\) matrices need to be evaluated. The integrand has to be expressed as a sum of sines and/or cosines before the integration can be done. Here this exercise in trigonometry/algebra is relatively simple. However, it would be desirable to set a stage to incorporate an arbitrary number of modes and be able to cope with multiple contact regions that change as the estimates of \(T\) change in each iteration. A method has been devised to deal automatically with an arbitrary number of modes and contact regions.

The approach is illustrated here for the simple two-mode case and the term \(Q_1\) given in Eq. (A.1):

1. The expression for \(w(\zeta)\) is first converted into a sum of complex exponentials form and the coefficients of the resulting expression are collected in a vector as shown below:

\[ w(\zeta) = T_1 \sin \frac{\pi}{2} (1 + \zeta) + T_2 \sin \frac{2\pi}{2} (1 + \zeta) + T_3 \sin \frac{3\pi}{2} (1 + \zeta) \]

\[ = -T_2 z^{-2} - T_1 z^{-1} + T_0 + T_1 z + T_2 z^2 \]

Where, \(f = \sqrt{-1}\), \(z = \exp(\pi\zeta/2)\) and \(D_0 = w(0) = T_1\). The coefficient vector, \(w_{\text{coef}}\) can now be written as:

\[ w_{\text{coef}} = \left[ -T_2 z^{-2} - T_1 z^{-1} + T_0 + T_1 z + T_2 z^2 \right] \]  

(A.3)

In general for an \(N\)-mode approximation, the coefficient vector \(w_{\text{coef}}\) consists of \(2N + 1\) elements from \(-T_{2N}/2\) to \(+T_{2N}/2\).

2. The coefficients of the cubic, \(w_{\text{coef}}^3\), in the integrand of Eq. (A.1) are same as the elements of the vector obtained by convolving \(w_{\text{coef}}\) in Eq. (A.3) with itself and the result with \(w_{\text{coef}}\) again:

\[ w_{\text{coef}}^3 = (w_{\text{coef}} * w_{\text{coef}}) * w_{\text{coef}}. \]

(A.4)

Here, \(*\) represents convolution. Consider the sequences \(y_i[n]\) of length \(2Y_1 + 1\) (from \(-Y_1\) to \(+Y_1\)) and \(y_j[n]\) of length \(2Y_2 + 1\) (from
\( y_j[n] = y_1[n] + y_2[n] = \sum_{m=-\infty}^{n} y_1[m] y_2[n-m], \) \( (A.5) \)

where \( n = -(Y_1 + Y_2) \) to \((Y_1 + Y_2)\). Convolution of causal and acausal finite sequences is widely used in discrete-time signal processing (Oppenheim et al., 1999).

Note that the resulting vector of Eq. (A.4) has 7 elements. In general, for an N-mode approximation, the vector \( \mathbf{w}^{\text{N-pdf}} \) has \( 6\times N + 1 \) elements, whereas for a \( \mathbf{w}^{\text{M-pdf}} \) it has only \( N + 1 \) elements. The cubic term comes from a cubic stiffness. For higher orders of stiffnesses, e.g., \( N \)th order, the result of the \( M-1 \) convolutions would have \( 2\times MN + 1 \) elements from \(-MN \to MN\).

The further product in the integrand appearing due to the projection onto the first mode, can also be evaluated along similar lines. If the resulting vector is called \( \mathbf{w}^{\text{M-pdf}} \), superscript \( p_1 \) indicates the projection of the cubic on the 1st mode,

\[ \mathbf{w}^{\text{M-pdf}}_{\text{proj}} = \mathbf{w}^{\text{M-pdf}} * \left[ -\frac{1}{2} \cdot \frac{1}{2} \right] = \mathbf{w}^{\text{M-pdf}}_{\text{proj}} * \mathbf{\Xi}^{\text{M-pdf}}. \]

\( \mathbf{\Xi}^{\text{M-pdf}} \) represents the coefficient vector formed from the first mode function. It must be noted that in general, when the cubic is projected onto the \( s \)th mode of an \( N \)-mode approximation, \( \mathbf{\Xi}^{\text{M-pdf}} \) will have \( 2s + 1 \) elements from \(-s \to s \) (with the first and last being the only non-zero elements), and \( \mathbf{w}^{\text{M-pdf}}_{\text{proj}} \) consists of \( 6N + 2s + 1 \) elements in total from \(-3N \to 3N + s \). Also, for \( M \)th degree stiffness polynomial, there are \( 2(\times MN + s) + 1 \) elements from \(-MN \to MN + s \).

4. Once the coefficients of all the terms in the sum are calculated as in steps (1)–(3), the integration is straightforward because after \( (N-1) \) elements from the \( k \)th term is the exponential for e.g., the integral of the \( k \)th term in the sum of \( Q_i \) in Eq. (A.1) can be evaluated as:

\[ \mathbf{w}^{\text{M-pdf}}_{\text{proj}}(k) \int_{k}^{\infty} \exp \left[ k \pi \xi \right] d\xi = \mathbf{w}^{\text{M-pdf}}_{\text{proj}}(k) \frac{\exp \left[ k \pi \xi \right] \pi}{k^{\pi}} \bigg|_{k}^{\infty}. \]

Note that, \( \mathbf{w}^{\text{M-pdf}}_{\text{proj}}(k) \) is the \( k \)th element of the coefficient vector \( \mathbf{w}^{\text{M-pdf}}_{\text{proj}} \), and \( k = -(3N - s) \to 3N + s \). \( Q_i \) is thus the sum of such \( 6N + 2s + 1 \) integrals.

This algorithm for calculating the integrals, as opposed to determining the trigonometric functions and integrating, is very flexible and also results in considerable reduction in computational time. For instance, a 20-mode approximation utilising this algorithm takes less than 0.83 of the time taken otherwise, evaluating the integrals conventionally.

References


