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How to Fuzzify a Closure Space

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It is shown here how the Galois connections theory provides a canonical procedure to fuzzify a crisp closure space. Roughly speaking, if S is a space equipped with an $\cap$-closed set $\mathscr{F}$ of subsets (the closed subsets), the closed fuzzy subsets of S ≪are≫ the Galois connections between $\mathscr{F}$ and the valuation set. The closed crisp subsets are particular closed fuzzy subsets.

PRELIMINARIES

Let us recall what a Galois connection is. A Galois connection between the ordered sets P and Q is a couple of mappings $P \rightleftarrows^f_g Q$ (or $(f, g) \in Q^P \times P^Q$) such that f and g are decreasing, $1_P \leqslant g \circ f$ and $1_Q \leqslant f \circ g$. Let us denote by $\boldsymbol{\Gamma(P, Q)}$ the set of such Galois connections. This set can clearly be cartesianily ordered.

When P and Q are complete lattices, it is known that $\Gamma(P, Q)$ is itself a complete lattice. In this case, it is also known that, whenever $(f, g) \in \Gamma(P, Q)$ and $A \subset P$, $f(\bigvee A) = \bigwedge_{a \in A} f(a)$. But we have better. Let us define

$$\boldsymbol{\Delta(P, Q)} = \left\{ h \in Q^P / \forall A \subset P, h\left(\bigvee A\right) = \bigwedge_{a \in A} h(a) \right\}.$$

The mapping $(f, g) \mapsto f$ from $\Gamma(P, Q)$ toward $\Delta(P, Q)$ is an isomorphism (of complete lattices). (See [1 or 3]).

RESULTS

In the whole following text, $\mathbf{S}$ is a set, equipped with a closure structure: *$\mathscr{F}$ is an $\bigcap$-closed set of subsets of S.* The associated closure $F: 2^S \to \mathscr{F}$ is defined by $F(X) = \bigcap \{ Y \in \mathscr{F} / X \subset Y \} (= \bar{X})$.

Let $\mathbf{L}$ be a *complete lattice* (with $0 < 1$). The fuzzy subsets of S are (by definition) the members of L^S.

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THEOREM 1. *The complete lattice* $\Gamma(L, \mathscr{F})$ *is isomorphic to*:

$$\mathscr{G} = \left\{ \alpha \in L^S / \forall X \subset S, \bigwedge_{x \in X} \alpha(x) = \bigwedge_{x \in \bar{X}} \alpha(x) \right\}.$$

More precisely, we have the following reciprocal isomorphisms (of complete lattices):

$$\Gamma(L, \mathscr{F}) \underset{v}{\overset{u}{\rightleftarrows}} \mathscr{G}$$

with

$$u_{f,g}(x) = g(F(\{x\}))$$

and

$$v_\alpha = (i, j), \qquad i(t) = \alpha^{-1}(\uparrow t), \qquad j(X) = \bigwedge_{x \in X} \alpha(x).$$

Proof. Let us first prove that u and v are well defined. We shall denote by $\bigvee$ the supremum in the complete lattice $\mathscr{F}$. Assume first that (f, g) is in $\Gamma(L, F)$. We have to prove that $u_{f,g}(=\varphi)$ is in $\mathscr{G}$. For each subset X of S we have

$$\bar{X} = F\left(\bigcup_{x \in X} \{x\} \right) = F\left(\bigcup F(\{x\}) \right) = \bigvee F(\{x\}).$$

We deduce $g(\bar{X}) = g(\bigvee F(\{x\})) = \bigwedge g(F(\{x\})) = \bigwedge \varphi(x)$. But then also $g(\bar{X}) = g(\overline{F(X)}) = \bigwedge_{x \in F(X)} \varphi(x)$. Hence $u_{f,g}$ is in $\mathscr{G}$. Assume now that α is in $\mathscr{G}$. We have to prove that $i(t)$ is a member of $\mathscr{F}$ and that (i, j) is Galois. For the first point, let z be a member of $\overline{i(t)}$: since

$$i(t) = \{x \in E / \alpha(x) \geqslant t\} \qquad (\text{with } t \in L),$$

we get

$$\alpha(z) \geqslant \bigwedge_{x \in \overline{i(t)}} \alpha(x) = \bigwedge_{x \in i(t)} \alpha(x) \geqslant t$$

and z is a member of $i(t)$. The second point is easily checked.

Since u and v are obviously increasing, we just have to prove now that u and v are reciprocal bijections. For each member x of S,

$$u_{v_\alpha}(x) = u_{i,j}(x) = j(F(\{x\}))$$

$$= \bigwedge_{y \in F(\{x\})} \alpha(y) = \bigwedge_{y \in \{x\}} \alpha(y) = \alpha(x).$$

In order to finish, let us prove that, for each member (f, g) of $\Gamma(L, \mathscr{F})$, $v_{u_{f,g}} = (f, g)$. Let us put for the moment $v_{u_{f,g}} = (i, j)$. We get (by using successively the definitions of v and u):

$$\begin{aligned} i(t) &= (u_{f,g})^{-1}(\uparrow t) = \{x \in S / t \leqslant u_{f,g}(x)\} \\ &= \{x \in S / t \leqslant g(F(\{x\}))\} \\ &= \{x \in S / F(\{x\}) \subset f(t)\} \\ &= \{x \in S / x \in f(t)\} = f(t). \end{aligned}$$

Then $f = i$. Now, for $X \in \mathscr{F}$,

$$\begin{aligned} j(X) &= \bigwedge_{x \in X} u_{f,g}(x) = \bigwedge_{x \in X} g(F(\{x\})) \\ &= g\left(\bigvee_{x \in X} F(\{x\})\right) = g\left(F\left(\bigcup_{x \in X} F(\{x\})\right)\right) \\ &= g\left(F\left(\bigcup_{x \in X} \{x\}\right)\right) = g(F(X)) = g(X). \quad \blacksquare \end{aligned}$$

Note that, whenever α is a member of $\mathscr{G}$, $\alpha^{-1}(\uparrow t)$ is a member of $\mathscr{F}$ for every t in L. Thus, each level of α (i.e., each $\alpha^{-1}(\uparrow t)$) is closed. We now extend the notion of closed subset by the following definition.

DEFINITION 2. A fuzzy subset α of S is said $\mathscr{F}$-closed (or closed) if and only if

$$\forall X \subset S \qquad \bigwedge_{x \in X} \alpha(x) = \bigwedge_{x \in \bar{X}} \alpha(x).$$

So, the fuzzy subsets of S are the members of $\mathscr{G}$. In order to justify the terminology of Definition 2, we have to prove that $\mathscr{G}$ is Λ-closed. Then, we shall make the link precise between the closed subsets and the fuzzy closed ones and we shall prove that the closure associated to $\mathscr{G}$ is a prolongation of F.

THEOREM 3. (1) *The set $\mathscr{G}$ is an Λ-closed subset of L^S. Let us denote by $G: L^S \to \mathscr{G}$ the associated closure.*

(2) $\mathscr{G} \cap 2^S = \mathscr{F}$.

(3) *For every member α of 2^S, $F(\alpha) = G(\alpha)$.*

Proof. (1) We have to prove that, for each $\mathfrak{m} \subset \mathscr{G}$, the fuzzy subset $\gamma = \bigwedge \mathfrak{m}$ is a member of $\mathscr{G}$. For a given subset X of S,

$$\begin{aligned}\bigwedge_{x \in X} \gamma(x) &= \bigwedge_{x \in X} \left(\bigwedge_{\mu \in \mathfrak{m}} \mu(x) \right) \\ &= \bigwedge_{\mu \in \mathfrak{m}} \left(\bigwedge_{x \in X} \mu(x) \right) = \bigwedge_{\mu \in \mathfrak{m}} \left(\bigwedge_{x \in \bar{X}} \mu(x) \right) \\ &= \bigwedge_{x \in \bar{X}} \left(\bigwedge_{\mu \in \mathfrak{m}} \mu(x) \right) = \bigwedge_{x \in \bar{X}} \gamma(x).\end{aligned}$$

(2) We identify a member α of 2^S to the crisp $A = \alpha^{-1}(1)$. Such a crisp subset α belongs to $\mathscr{G}$ if and only if, for each $X \subset S$,

$$\bigwedge_{x \in X} \alpha(x) = 1 \Rightarrow \bigwedge_{x \in \bar{X}} \alpha(x) = 1.$$

In other words, α is in $\mathscr{G}$ if and only if $X \subset A \Rightarrow \bar{X} \subset A$. And this is equivalent to $A \in \mathscr{F}$.

(3) We still identify α to A. Let us identify $\bar{A} = B$ to the associated characteristic function β ($\geqslant \alpha$). By (2), we know that β is in $\mathscr{G}$. By the definition of G,

$$G(\alpha) = \bigwedge \{\gamma \in \mathscr{G} / \alpha \leqslant \gamma\}.$$

Hence

$$\alpha \leqslant G(\alpha) \qquad (= \varphi) \leqslant \beta.$$

For each $a \in A$, $1 = \alpha(a) \leqslant \varphi(a)$, then $\varphi(a) = 1$. For each $x \in S \backslash \bar{A}$, $\varphi(x) \leqslant \beta(x) = 0$, then $\varphi(x) = 0$. Finally $1 = \bigwedge_{a \in A} \varphi(a) = \bigwedge_{a \in \bar{A}} \varphi(a)$, and $\varphi(x) = 1$ for each $x \in \bar{A}$. Then $\varphi = \beta$. ∎

EXAMPLE 1. *Initial subsets of an ordered set.* A subset X of the ordered set S is said to be initial if and only if

$$x \leqslant y \in X \Rightarrow x \in X.$$

The set $\mathscr{F}$ of the initial subsets of S is obviously $\bigcap$-closed. Which are the fuzzy closed subsets of S? Assume α is such a subset. Then

$$\bigwedge_{y \in \{x\}} \alpha(y) = \bigwedge_{y \in \overline{\{x\}}} \alpha(y) = \bigwedge_{y \leqslant x} \alpha(y).$$

Then $y \leqslant x$ implies $\alpha(x) \leqslant \alpha(y)$ and α is decreasing.

Conversely, assume that α is a decreasing mapping from S toward L. Whenever $y \leqslant z \in X$, we can deduce $\bigwedge_{x \in X} \alpha(x) \leqslant \alpha(z) \leqslant \alpha(y)$ and then $\bigwedge_{x \in X} \alpha(x) \leqslant \bigwedge_{y \in \bar{X}} \alpha(y)$. Finally α is in $\mathscr{G}$.

So the fuzzy closed subsets are exactly the decreasing mappings.

EXAMPLE 2. *Subgroupoids of a groupoid.* A groupoid is a set uniquely equipped with a binary operation. A subgroupoid of a given groupoid is, by definition, a subset which is closed by that operation.

THEOREM 4. *Let $(S, \mathscr{F})$ be the closure space constituted by the complete lattice of the subgroupoids of a groupoid S. A fuzzy subset $\varphi(\varepsilon L^S)$ is closed if and only if, whenever x and y are in S:*

$$(\varphi(x)) \wedge (\varphi(y)) \leqslant \varphi(xy).$$

Proof. Assume that φ is closed. Let P be the subset (of S) the only elements of which are x and y. By Definition 2, we can write

$$(\varphi(x)) \wedge (\varphi(y)) = \bigwedge_{z \in \bar{P}} \varphi(z).$$

Since xy is in $\bar{P}$, we get $(\varphi(x)) \wedge (\varphi(y)) \leqslant \varphi(xy)$.

Conversely, assume that φ satisfies (for all x, y) the inequality of the theorem. Let X be an arbitrary (crisp) subset of S. If z is in $\bar{X}$, it must be (at least in one manner) composed of a finite (nonnull) number of elements of X: let $A(\subset X)$ be the considered finite set. By using finitely many of the inequality of the text of the theorem we get

$$\varphi(z) \geqslant \bigwedge_{a \in A} \varphi(a) \geqslant \bigwedge_{x \in X} \varphi(x).$$

We deduce $\bigwedge_{z \in \bar{X}} \varphi(z) \geqslant \bigwedge_{x \in X} \varphi(x)$. Hence φ is closed. ∎

These fuzzy closed subsets can be named the fuzzy subgroupoids of S. By the same procedure, we can define, for instance, the fuzzy subgroups of a group, and we find again the definitions of [4].

EXAMPLE 3. $\mathscr{F} = 2^S$. It is clear that $\mathscr{G} = L^S$: F and G are the "identity" closures. Then, we have the reciprocal isomorphisms p and q and the reciprocal isomorphisms u and v:

$$\Delta(L, 2^S) \underset{q}{\overset{p}{\rightleftarrows}} \Gamma(L, 2^S) \underset{v}{\overset{u}{\rightleftarrows}} L^S,$$

where u and v are given in Theorem 1 and where p and q are deduced (for instance) of [1]:

$$p(f) = (f, g) \qquad \text{with} \quad g(X) = \bigvee_{X \subset f(t)} t$$

$$q(f, g) = f.$$

Let us compute the reciprocal isomorphisms up and qv:

$$u_{pf}(x) = gF(\{x\}) = \bigvee_{F(\{x\}) \subset f(t)} t = \bigvee_{x \in f(t)} t$$

$$q_{v_\alpha}(t) = i(t) = \alpha^{-1}(\uparrow t).$$

So, we find again the representation theorem of Negoita and Ralescu [2].

EXAMPLE 4. *$\mathscr{F} = 2^S$ and $L = 2^T$ (T is a set).* By defining again u and v as in Theorem 1 and by defining the reciprocal isomorphisms a and b by $a(\gamma) = \{(x, y) \in S \times T / y \in \gamma(x)\}$ we get the compositions:

$$\Gamma(2^T, 2^S) \underset{v}{\overset{u}{\rightleftarrows}} (2^T)^S \underset{b}{\overset{a}{\rightleftarrows}} 2^{S \times T}.$$

Then, whenever ρ is a member of $2^{S \times T}$, $v_{b_\rho} = (i, j)$ is the Galois connection $2^T \rightleftarrows_j^i 2^S$ defined as in Theorem 1. More precisely, we get

$$\begin{aligned} i(Y) &= b_\rho^{-1}(\uparrow Y) = \{x \in S / Y \subset b_\rho(x)\} \\ &= \{x \in S / \forall y \in Y, y \in b_\rho(x)\} \\ &= \{x \in S / \forall y \in Y, (x, y) \in \rho\}; \\ j(X) &= \bigcap_{x \in X} b_\rho(x) = \{y \in T / \forall x \in X, y \in b_\rho(x)\} \\ &= \{y \in T / \forall x \in X, (x, y) \in \rho\}. \end{aligned}$$

Then v_{b_ρ} is the classical Galois connection of the binary relation ρ. We find again the fact that each binary relation can be identified with a Galois connection (see [3]).

OPEN PROBLEMS

(1) For a given $\mathscr{F}$, what are the m ($\subset L^E$) verifying the conditions (1), (2), and (3) of Theorem 3 (where $\mathscr{G}$ would be replaced by $\mathfrak{m}$)?

(2) Let us define:

$$\mathbb{F} = \{\mathscr{F} \subset 2^S / \mathscr{F} \bigcap\text{-closed}\}$$
$$\mathbb{C} = \{\mathscr{C} \subset L^S / \mathscr{C} \bigwedge\text{-closed}\}.$$

Study the mapping

$$\mathbb{F} \to \mathbb{C} \qquad \text{defined by } \mathscr{F} \mapsto \mathscr{G}.$$

Conclusion

It seems that the proposed procedure is the best one to fuzzify a mathematical structure. It can easily be used to "soften" such things as convexity structures or topological spaces.

References

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