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On the completeness of order-theoretic models of the λ-calculus

Furio Honsell [a], Gordon Plotkin [b,*,1]

[a] *Dipartimento di Matematica e Informatica, Unversità de Udine, Italy*

[b] *Laboratory for Foundations of Computer Science, School of Informatics, University of Edinburgh, JCMB, King's Bldgs., Mayfield Rd., Edinburgh EH9 3JZ, UK*

A R T I C L E I N F O

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A B S T R A C T

Scott discovered his domain-theoretic models of the λ-calculus, isomorphic to their function space, in 1969. A natural completeness problem then arises: whether any two terms equal in all Scott models are convertible. There is also an analogous consistency problem: whether every equation between two terms, consistent with the λ-calculus, has a Scott model. We consider such questions for wider sets of sentences and wider classes of models, the pointed (completely) partially ordered ones. A negative result for a set of sentences shows the impossibility of finding Scott models for that class; a positive result gives evidence that there might be enough Scott models. We find, for example, that the order-extensional pointed ω-cpo models are complete for Π_1-sentences with positive matrices, whereas the consistency question for Σ_1-sentences with equational matrices depends on the consistency of certain critical sentences asserting the existence of certain functions analogous to the generalized Mal'cev operators first considered in the context of the λ-calculus by Selinger.

1. Introduction

Scott discovered his remarkable domain-theoretic models of the untyped λ-calculus, more precisely the $\lambda\beta\eta$-calculus, in 1969: see the remarks in [10] and see [12] for further references. Having a class of mathematically interesting models available, a natural *completeness* problem arises: if two terms are equal in the class, here that of all Scott's models, are they then equal in all models, equivalently, are they $\beta\eta$-convertible? As stated so far, the problem is not quite precise: one should refer to a specific class of models such as all pointed ω-cpos isomorphic to their own function space. This equational completeness problem seems to have appeared first in the literature in [5]; a positive solution was given in [2], but with models taken in the category with objects the pointed partial orders with lubs of both increasing ω_0- and ω_1-chains, and with morphisms the monotonic functions preserving just the lubs of the increasing ω_1-chains.

A related problem exists for the $\lambda\beta$-calculus and, e.g., pointed ω-cpos having their own function space as a retract. There is also a natural *consistency* problem: if an equation is consistent, meaning one cannot use it to derive the equation $x = y$, does it have a model, e.g., a non-trivial pointed ω-cpo isomorphic to its own function space?

In this paper, we consider problems of these kinds for the λ-calculus, various classes of models and various sets of sentences (i.e., closed formulae), not only equations. Here is our general framework. Let $\mathscr{T}$ be a first-order theory, e.g., that for combinatory logic, for the $\lambda\beta$-calculus or for the $\lambda\beta\eta$-calculus; let $\mathscr{C}$ be a class of models of $\mathscr{T}$; and let $\mathscr{F}$ be a set of sentences. We say that $\mathscr{C}$ is $\mathscr{F}$*-complete* if the following holds:

$$\forall \varphi \in \mathscr{F}.(\forall \mathscr{M} \in \mathscr{C}.\, \mathscr{M} \models \varphi) \Rightarrow (\forall \mathscr{M} \models \mathscr{T}.\, \mathscr{M} \models \varphi)$$

* Corresponding author. Fax: +44 131 667 7209.
E-mail address: gdp@inf.ed.ac.uk (G. Plotkin).

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and we say that $\mathscr{C}$ is $\mathscr{F}$*-consistent* if the following holds:

$$\forall \varphi \in \mathscr{F}.(\exists \mathscr{M} \models \mathscr{T}.\, \mathscr{M} \models \varphi) \Rightarrow (\exists \mathscr{M} \in \mathscr{C}.\, \mathscr{M} \models \varphi)$$

As Scott's methods yield models of the $\lambda\beta\eta$-calculus and the $\lambda\beta$-calculus, we are mainly interested in those calculi, but we also discuss combinatory logic. As regards sentences, expanding our interest from equations to first-order sentences is mathematically natural and enables us to take some 'distance' from the hard, unsolved, equational problems but nevertheless try to cast some light on them. As a strategy it has pluses and minuses, making counterexamples easier to find but positive results harder to prove. In this regard we take a standard approach, considering various prenex sets of sentences, and seeking counterexamples of minimal logical complexity and positive results for sentences of maximal logical complexity.

As regards models, we are mainly interested in very general classes, such as those admitting pointed partial orders, or pointed ω-cpos, with corresponding results generally going through for pointed dcpos. Despite their generality, results for these classes do have some bearing on Scott models. On the one hand, negative results apply also to Scott models, as $\mathscr{F}$-completeness or $\mathscr{F}$-consistency for a class of models implies the same for any subclass. On the other hand, positive results can be taken as providing some evidence that Scott models are also complete or consistent in the relevant sense.

Following [6,1], one may also look for categorical rather than partially ordered models; it would be interesting to try to extend the results of this paper to such models, cf. [8, Section 3]. Related work appears in [9,4]. Much of the emphasis there is on a somewhat different question, also generalising the equational completeness question: whether a λ-theory is the theory of a model in a given class of models.

After some technical preliminaries in Section 2, we consider questions of completeness in Section 3, in particular giving Theorem 4, establishing completeness for Π_1-sentences with positive matrix. Section 4 considers questions of consistency, in particular presenting a reduction of Σ_2-consistency to the consistency with the λ-calculus of certain nonlinear equations analogous to those holding for generalized Mal'cev operators [11]. The paper concludes with some final remarks in Section 5.

2. Preliminaries

We refer the reader to Barendregt's book [3] for background information on the λ-calculus and combinatory logic. We generally follow his notation, noting any significant differences where they occur. We consider three first-order theories with equality, $\mathscr{T}_{\mathrm{CL}}$, $\mathscr{T}_{\beta}$ and $\mathscr{T}_{\beta\eta}$, over the signature with two constants K and S and a binary 'application' operation $\cdot$; the application operation is written as an infix, or even omitted entirely, and, whether written explicitly or implicitly, applications are associated to the left. The equational theory of combinatory logic is given by the following two equations:

$$\mathrm{K}xy = x$$

$$\mathrm{S}xyz = xz(yz)$$

We take $\mathscr{T}_{\mathrm{CL}}$ to be given by these two equations together with the sentence $\mathrm{K} \neq \mathrm{S}$, in order to exclude trivial models. Second, $\mathscr{T}_{\beta}$ is the extension of $\mathscr{T}_{\mathrm{CL}}$ by the following three axioms:

$$(\forall z.\, xz = yz) \Rightarrow \mathbf{1}x = \mathbf{1}y$$

$$\mathbf{1}_2\mathrm{K} = \mathrm{K}$$

$$\mathbf{1}_3\mathrm{S} = \mathrm{S}$$

where $\mathbf{1}_1 =_{\mathrm{def}} \mathbf{1} =_{\mathrm{def}} \mathrm{S}(\mathrm{KI})$ and $\mathbf{1}_{n+1} =_{\mathrm{def}} \mathrm{S}(\mathrm{K}\mathbf{1})(\mathrm{S}(\mathrm{K}\mathbf{1}_n))$, and where I is the identity combinator SKK. Third, $\mathscr{T}_{\beta\eta}$ is the extension of $\mathscr{T}_{\mathrm{CL}}$ by the following axiom of extensionality:

$$(\forall z.\, xz = yz) \Rightarrow x = y$$

Models are, as usual, interpretations $\mathscr{M} = (X, \mathrm{K}, \mathrm{S}, \cdot)$ satisfying the axioms. The theory $\mathscr{T}_{\mathrm{CL}}$ axiomatises the non-trivial combinatory algebras [3, 5.1.8]; the theory $\mathscr{T}_{\beta}$ axiomatises the non-trivial λ-models [3, 5.6.3]; and the theory $\mathscr{T}_{\beta\eta}$ axiomatises the non-trivial extensional λ-models.

We now consider the relation between our theories and combinatory logic and the λ-calculus. It is convenient to allow extra constants. For the theories, one extends the signature with a given set of extra constants, but no further axioms; for the calculi one extends the terms (and, consequently, substitution and conversion) by a given set of constants, in the evident way.

In one direction, we translate terms t of $\mathscr{T}_{\mathrm{CL}}$ (respectively, $\mathscr{T}_{\beta}$, $\mathscr{T}_{\beta\eta}$) to terms M_t of combinatory logic (respectively, the $\lambda\beta$-calculus, the $\lambda\beta\eta$-calculus). In the case of combinatory logic, M_t is just t. In the other two cases, application is read as application, K and S as $\lambda x.\lambda y.x$ and $\lambda f.\lambda g.\lambda x.fx(gx)$, and the constants as themselves, and so M_t is Barendregt's M_{λ}, see [3, 7.3.1], extended to handle the constants. We rely on context to tell which of the translations is meant.

In the other direction we translate terms M of combinatory logic (respectively, the $\lambda\beta$-calculus, the $\lambda\beta\eta$-calculus) to terms t_M of $\mathscr{T}_{\mathrm{CL}}$ (respectively, $\mathscr{T}_{\beta}$, $\mathscr{T}_{\beta\eta}$). In the case of combinatory logic t_M is simply M and in the other two cases t_M is Barendregt's t_{CL}, see [3, 7.3.1], extended to handle the constants. We again rely on context to tell which of the translations is meant. We write the Curry abstraction operator applied to a first-order term t and variable x as $\lambda^* x.t$ [3, 5.1].

One has that $t = t_{M_t}$ is provable in the relevant first-order theory and that M and M_{t_M} are convertible in the relevant calculus. One further has that M and N are convertible in a given calculus if and only if $t_M = t_N$ is provable in the corresponding theory and that $t = u$ is provable in a theory if and only if M_t and M_u are convertible in the relevant calculus. This is trivial for combinatory logic; for the $\lambda\beta$- and the $\lambda\beta\eta$-calculi, one uses the results in [3, 7.3], extended to handle the constants.

We consider three kinds of equational theory Th. Combinatory logic theories are equational theories extending the above two equations for K and S; one also allows constants in the terms and equations, taken from a given set of constants. For the $\lambda\beta$-calculus one considers λ-theories in the sense of [3] and for the $\lambda\beta\eta$-calculus one considers his $\lambda\eta$-theories, allowing additional constants in both cases. We write $M =_{\mathrm{Th}} N$ to assert that the equation $M = N$ is in the theory Th (and so $=_{\mathrm{Th}}$ is the relation $\{(M,N) | M =_{\mathrm{Th}} N\}$) and we write CL, $\lambda\beta$ and $\lambda\beta\eta$ for, respectively, the minimal combinatory logic theory, λ-theory and $\lambda\eta$-theory with no constants.

Given an equationally consistent theory Th, i.e., one that does not contain the equation $x = y$, one can construct the *open term model* $\mathscr{M}_{\mathrm{Th}}$ of the relevant theory $\mathscr{T}$. The domain of the model is the collection of equivalence classes $[M]$ of open terms, where the equivalence relation is $=_{\mathrm{Th}}$, that of the relevant equational theory. Application is defined by $[M][N] = [MN]$; K and S are interpreted by [K] and [S] or by $[\lambda x.\lambda y.x]$ and $[\lambda f.\lambda g.\lambda x.fx(gx)]$ as appropriate. The denotation of a term t, assigning $[M_1], \ldots, [M_n]$ to its free variables $x_1, \ldots, x_n$, is $[M_t[M_1/x_1, \ldots, M_n/x_n]]$ (we employ a different notation for substitution than that used in [3]). Note that $\mathscr{M}_{\mathrm{CL}}$, $\mathscr{M}_{\lambda\beta}$ and $\mathscr{M}_{\lambda\beta\eta}$ are the usual open term models.

One can also obtain theories from models. Let $\mathscr{M}$ be a model of one of our theories which also interprets the given set of constants. Then $\{M = N \mid \mathscr{M} \models t_M = t_N\}$ is a consistent equational theory of the corresponding kind.

Our results concern partially ordered or po models. An interpretation is *partially ordered* (a *po* interpretation) if its carrier is a partial order and application is monotone; it is further *pointed* (a *ppo* interpretation) if it has a least element, written $\perp$. Strictly speaking, po interpretations are not interpretations as they have additional structure: we should rather speak of interpretations admitting a compatible partial order. However, the looser way of speaking will not create any difficulties and we continue with it.

A po interpretation of one of our three theories $\mathscr{T}$ is a model if it is in the usual first-order sense. If it is a model of CL one has $\perp \cdot x = \perp$. It is natural to consider partially ordered versions of one of the axioms for **1** and of extensionality. We say that a po interpretation of $\mathscr{T}_\beta$ is **1**-*order-extensional* if we have:

$$\forall z.\,(xz \leq yz) \Rightarrow \mathbf{1}x \leq \mathbf{1}y$$

and that a po interpretation of $\mathscr{T}_{\beta\eta}$ is *order-extensional* if we have:

$$\forall z.\,(xz \leq yz) \Rightarrow x \leq y$$

A ppo interpretation is a *pointed cpo* (or *cppo*) interpretation if its carrier is a pointed cpo and application is continuous in both arguments. More precisely, we consider two cases: in the first by cpo we mean ω-cpo where lubs of increasing ω-chains exist and continuity means preserving those lubs; and in the second we mean dcpo where lubs of all directed sets exist and continuity means preserving those lubs. For the most part, our results do not depend on this distinction and so we only make it when needed.

In the cases of $\mathscr{T}_\beta$ or $\mathscr{T}_{\beta\eta}$ we define a *retract* model to be a cppo that has as a retract the space of its continuous self-maps with application, K and S then given in the usual way; in the case of $\mathscr{T}_{\beta\eta}$ the retract is then necessarily an isomorphism. Every cppo model of CL has a least-fixed point operator, i.e., an element Y such that $\mathrm{Y} \cdot x = \bigvee x^n \cdot \perp$: one takes Y to be $\bigvee_{n \geq 0} \mathrm{Y}^{(n)}$, where $\mathrm{Y}^{(0)} = \perp$ and $\mathrm{Y}^{(n+1)} = \mathrm{SKY}^{(n)}$.

We classify sentences according to quantifier and matrix complexity. By Π_n we mean the Π_n-sentences, and similarly for Σ_n. By Π_n(POS) (respectively, Π_n(EQ)) we mean those sentences equivalent to a Π_n-sentence with positive matrix (respectively, equational matrix), and similarly for Σ_n(POS) and Σ_n(EQ). By EQ we (evidently) mean Π_0(EQ), and similarly for POS. As combinatory logic permits pairing, repeated universal or existential quantifiers can be reduced to single ones and conjunctions of equations can be reduced to single equations. Further, when working with $\mathscr{T}_\beta$ or $\mathscr{T}_{\beta\eta}$, $\forall x.t = u$ is equivalent to $\lambda x.t = \lambda x.u$, which serves to reduce quantifier complexity for these two theories.

Finally, note that for any set $\mathscr{F}$ of sentences, $\mathscr{F}$-consistency is equivalent to $\mathscr{F}'$-completeness, where $\mathscr{F}' = \{\neg\varphi \mid \varphi \in \mathscr{F}\}$. We therefore look for counterexamples with positive matrix or, better, equational matrix; we also prefer natural counterexamples. For positive results we can generally do no better than all positive matrices of some prefix class.

3. Completeness

We begin with two counterexamples to completeness. The first is natural in that it relates to a standard property of fixed-points; the second is, rather, of a technical nature but it applies to a wider class of models of the λ-calculus.

Lemma 1. *For any combinatory logic or λ-calculus term M and variables $f, g \notin \mathrm{FV}(M)$, the terms $f(M(gf))$ and $M(fg)$ are not convertible.*

Proof. Any reduct of $f(M(ff))$ has an odd number of f's, but any reduct of $M(ff)$ has an even number. □

Theorem 1. *The cppo models of $\mathscr{T}_{\mathrm{CL}}$ are not $\Sigma_2(\mathrm{EQ})$-complete and the cppo models of $\mathscr{T}_\beta$ or $\mathscr{T}_{\beta\eta}$ are not $\Sigma_1(\mathrm{EQ})$-complete.*

Proof. Consider the following sentence:

$$\varphi_1 \equiv_{\mathrm{def}} \exists y.\forall f, g. f(y(\mathrm{B}gf)) = y(\mathrm{B}fg)$$

where $\mathrm{B} =_{\mathrm{def}} \lambda^* f.\lambda^* g.\lambda^* x.f(gx)$. This holds in every cppo interpretation of CL by the above remark on least-fixed points. However, by Lemma 1, it does not hold for open term models. □

The following lemma is immediate from [11]:

Lemma 2. *There is a closed term A of the λ-calculus such that the terms:*

$$A(xy)(xy)(xy)(xz) \quad \text{and} \quad A(xy)(xz)(xz)(xz)$$

are λβ-convertible, but the terms:

$$A(xM)(xM)(xM)(xN) \quad \text{and} \quad A(xM)(xM)(xN)(xN)$$

are not λβη-convertible for any non-λβ-convertible terms M, N not containing x as a free variable.

We then have:

Theorem 2. *The ppo models of $\mathscr{T}_\beta$ or $\mathscr{T}_{\beta\eta}$ are not $\Sigma_1(\mathrm{EQ})$-complete.*

Proof. Consider the following sentence:

$$\varphi_2 \equiv_{\mathrm{def}} \exists y.\forall x, z.A(xy)(xy)(xy)(xz) = A(xy)(xy)(xz)(xz)$$

By the second part of Lemma 2 this is false in the λβη open term model. However, it is true in any ppo model of $\mathscr{T}_\beta$ or $\mathscr{T}_{\beta\eta}$ as we then have for all elements *x*,*z* that:

$$\begin{aligned} A(x\perp)(x\perp)(x\perp)(xz) &\leq A(x\perp)(x\perp)(xz)(xz) \\ &\leq A(x\perp)(xz)(xz)(xz) \\ &= A(x\perp)(x\perp)(x\perp)(xz) \end{aligned}$$

with the last equality holding by the first part of the lemma. □

We do not know whether the ppo models of $\mathscr{T}_{\mathrm{CL}}$ are $\Sigma_2(\mathrm{EQ})$-incomplete, or even, for that matter, whether they are $\Sigma_2(\mathrm{POS})$-incomplete.

We next present some positive results on completeness, now working our way in the opposite direction: from partially ordered models to cpo ones. We consider only the case of $\mathscr{T}_{\beta\eta}$ in detail, contenting ourselves with remarks on the other two cases as they are very similar. Our method is proof-theoretic: we define a $\lambda\beta\eta\perp$-calculus, an ordered version of the λβη-calculus with a least element. Its syntax is that of the λ-calculus with an additional constant ⊥. Its axiom system has inequational judgments $M \leq N$ and the following axioms and rules:

$$M \leq M$$

$$\frac{L \leq M \quad M \leq N}{L \leq N}$$

$$\frac{M \leq M' \quad N \leq N'}{MN \leq M'N'}$$

$$\frac{M \leq N}{\lambda x.M \leq \lambda x.N}$$

$$\perp \leq M$$

$$(\lambda x.M)N = M[N/x]$$

$$\lambda x.Mx = M \ (\text{if } x \notin \mathrm{FV}(M))$$

where $M = N$ stands for the conjunction of the judgments $M \leq N$ and $N \leq M$. Note that the *order-exensionality rule*:

$$\frac{Mx \leq Nx}{M \leq N} \quad (\text{if } x \notin \mathrm{FV}(M) \cup \mathrm{FV}(N))$$

is derivable.

A $\lambda\eta\ \bot$-*inequational theory* Th is a set of inequations $M \le N$ between λ-terms, possibly containing ⊥ and other constants from a given set, and closed under the rules and axioms of the $\lambda\beta\eta\ \bot$-calculus; any such theory is closed under substitution. We write $M \le_{\mathrm{Th}} N$ to assert that $M \le N$ is in Th and $M =_{\mathrm{Th}} N$ to assert that $M = N$ is in it, meaning that both $M \le N$ and $N \le M$ are. Such a theory Th is *(inequationally) consistent* if $x \le y$ is not in it; the set of equations it contains forms a $\lambda\eta$-theory, consistent if, and only if, Th is inequationally consistent; and the resulting open term model $\mathscr{M}_{\mathrm{Th}}$ of $\mathscr{T}_{\beta\eta}$ (assuming consistency) is an order-extensional ppo model, setting $[M] \le [N]$ if and only if $M \le_{\mathrm{Th}} N$.

Let $\lambda\eta\ \bot$ be the least $\lambda\eta\ \bot$-inequational theory with ⊥ the only given constant. We seek a characterisation of it in terms of reduction relations. Let $\rightarrow_{\beta\eta}$ be one step of β- or η-reduction; let $\rightarrow_{\bot}$ be the contextual (called compatible in [3]) closure of the δ-*rule* $\bot \rightarrow M$ (where M is any term); and let $\rightarrow_{\beta\eta\bot}$ be their union.

Lemma 3

(i) Suppose that $M \rightarrow^*_{\beta\eta} M'$ and $M \rightarrow^*_{\beta\eta\bot} M''$. Then there is an N such that $M' \rightarrow^*_{\beta\eta\bot} N$ and $M'' \rightarrow^*_{\beta\eta} N$.

(ii) $M \le_{\lambda\eta\bot} N$ if and only if there is a term L such that $M \rightarrow^*_{\beta\eta\bot} L$ and $N \rightarrow^*_{\beta\eta} L$.

(iii) Let M and N be terms not containing ⊥ . Then $M \le_{\lambda\eta\bot} N$ if and only if M and N are βη-convertible (and so $\lambda\eta\ \bot$ is inequationally consistent).

Proof

(i) First of all if $M \rightarrow_{\beta\eta} M'$ and $M \rightarrow_{\bot} M''$ then there is an N such that $M' \rightarrow^*_{\bot} N$ and $M'' \rightarrow_{\beta\eta} N$. This is obvious for the case of one step of η-reduction. For the case of one step of β-reduction, the case of nonoverlap is trivial; if the δ-rule has been applied to a redex $(\lambda x.L)N$, then one can complete the diagram with one δ-reduction step in case it was applied to L and with as many as there are occurrences of x in L in case it was applied to N.
The rest of the proof of part 1 is a sequence of diagram chases. It follows first that if $M \rightarrow_{\beta\eta} M'$ and $M \rightarrow^*_{\bot} M''$ then there is an N such that $M' \rightarrow^*_{\bot} N$ and $M'' \rightarrow_{\beta\eta} N$; one then has that if $M \rightarrow^*_{\beta\eta} M'$ and $M \rightarrow^*_{\bot} M''$ then there is an N such that $M' \rightarrow^*_{\bot} N$ and $M'' \rightarrow *_{\beta\eta} N$; and this yields the conclusion.

(ii) Clearly, if there is a term L such that $M \rightarrow^*_{\beta\eta\bot} L$ and $N \rightarrow^*_{\beta\eta} L$ then $M \le_{\lambda\eta\bot} N$. Conversely, we need to show that the relation between terms M and N of the existence of such an L is closed under the rules and axioms of the $\lambda\beta\eta\ \bot$-calculus. The only non-obvious matter is transitivity and that is immediate from part 1.

(iii) Immediate from part 2. □

Theorem 3. *The order-extensional ppo models of $\mathscr{T}_{\beta\eta}$ are Π_1(POS)-complete.*

Proof. Any sentence in Π_1(POS) is provably equivalent to a sentence φ of the form $\forall x. (t_1 = u_1 \vee \cdots \vee t_n = u_n)$. If true in all order extensional ppo models of $\mathscr{T}_{\beta\eta}$, φ is, in particular, true in $\mathscr{M}_{\lambda\eta\bot}$, and so, assigning $[x]$ to x, we find that $M_{t_i} =_{\lambda\eta\bot} M_{u_i}$ for some i. So, by the last part of Lemma 3, M_{t_i} and M_{u_i} are βη-convertible. We then have that $t_i = u_i$, and so φ, is provable in $\mathscr{T}_{\beta\eta}$. We conclude that φ is valid. □

The Π_1(POS)-completeness of (**1**-order-extensional) ppo models of $\mathscr{T}_{\mathrm{CL}}$ or $\mathscr{T}_{\beta}$ is established in much the same way. One introduces $\mathrm{CL}_{\bot}$- and $\lambda\ \bot$-inequational theories, proves an analogue of Lemma 3 for the minimal such theories $CL_{\bot}$ and $\lambda\ \bot$ and takes the open term model of the corresponding theory.

In order to get similar positive results for the smaller class of ω-cppo models we need a few notions concerning ideals in partial orders.

Definition 1. An *ideal* in a partial order P is a downwards-closed subset of P; for any subset X of P we write $X \downarrow$ for $\{x \in P \mid \exists y \in X. x \le y\}$, the least ideal including X; and for any $x \in P$ we write $x \downarrow$ for $\{x\} \downarrow$. An ideal $\mathscr{I}$ is *directed* if it is nonempty and any two elements of the ideal have an upper bound in the ideal; it is *denumerably generated* if $\mathscr{I} = X \downarrow$ for some denumerable subset X of $\mathscr{I}$. We write $\mathrm{I}_{\omega}(P)$ (respectively, $\mathrm{I}_d(P)$) for the collection of all denumerably generated directed ideals (respectively, all directed ideals) of P, and partially order them by subset; $\mathrm{I}_{\omega}(P)$ is an ω-cpo and $\mathrm{I}_d(P)$ is a dcpo: both are pointed if, and only if, P is.

Now, given a ppo model $\mathscr{M} = (X, \mathrm{K}, \mathrm{S}, \cdot)$ of $\mathscr{T}_{\mathrm{CL}}$, we can define an ω-cppo model $\mathrm{I}_{\omega}(\mathscr{M})$ of $\mathscr{T}_{\mathrm{CL}}$, viz $(\mathrm{I}_{\omega}(X), \mathrm{K}\downarrow, \mathrm{S}\downarrow, \cdot)$ where:

$$\mathscr{I} \cdot \mathscr{J} =_{\mathrm{def}} \{x \cdot y \mid x \in \mathscr{I}, y \in \mathscr{J}\} \downarrow$$

and there is a similar dcppo model $\mathrm{I}_d(\mathscr{M})$ defined using all directed ideals. Taking a model of the form $\mathrm{I}_{\omega}(\mathscr{M}_{\mathrm{Th}})$, where Th is a $\lambda\eta\ \bot$-inequational theory, it is not hard to see that the denotation of a term t is $[M_t[M_1/x_1, \ldots, M_n/x_n]] \downarrow$, if we assign $[M_1] \downarrow, \ldots, [M_n] \downarrow$ to its free variables $x_1, \ldots, x_n$.

There is no general reason why either $\mathrm{I}_{\omega}(\mathscr{M})$ or $\mathrm{I}_d(\mathscr{M})$ should be order-extensional, even if $\mathscr{M}$ is. Fortunately, however, the former is for suitably chosen term models. The *extension of a $\lambda\eta\ \bot$-inequational theory* Th *by a set of constants C* is the least $\lambda\eta\ \bot$-inequational theory Th(C) which includes Th and whose terms may contain elements of C as constants; it is assumed here that C is a set not containing any constant occurring in a term of Th. It is not hard to show that Th(C) is the set of inequations:

$$\{M[c_1/x_1,\ldots,c_n/x_n] \le N[c_1/x_1,\ldots,c_n/x_n] \mid M \le_{\mathrm{Th}} N, c_1,\ldots,c_n \in C\}$$

(and so, in particular, Th(C) is inequationally consistent of Th is).

Lemma 4. *If* Th *is inequationally consistent then* $\mathrm{I}_\omega(\mathscr{M}_{\mathrm{Th}(C)})$ *is an order-extensional model of* $\mathscr{T}_{\beta\eta}$.

Proof. Suppose that $\mathscr{I}\cdot\mathscr{K} \subseteq \mathscr{J}\cdot\mathscr{K}$ for all directed, countably generated ideals $\mathscr{K}$. Let $[M]$ be an element of $\mathscr{I}$; we show it is in $\mathscr{J}$. As $\mathscr{J}$ is denumerably generated, it has the form $\{[N_i]\}\downarrow$, and so, as there are uncountably many constants, we can choose a constant c which does not occur in M or in any of the N_i. Taking $\mathscr{K}$ to be $[c]\downarrow$ we find that $[Mc] \in \mathscr{J}\cdot[c]\downarrow$ and so, for some i, we have $Mc \le_{\mathrm{Th}(C)} N_i c$. Using the above characterisation of Th(C), it follows that $Mx \le_{\mathrm{Th}} N_i x$, where x does not appear in M or N; we therefore have that $M \le_{\mathrm{Th}} N_i$ and so that $[M] \in \mathscr{J}$, as required. □

Theorem 4. *The class of order-extensional ω-cppo models of $\mathscr{T}_{\beta\eta}$ is Π_1(POS)-complete.*

Proof. If a sentence $\forall x.\,(t_1 = u_1 \vee \cdots \vee t_n = u_n)$ is true in all order-extensional ω-cppo models then, by Lemma 4, it is true in $\mathrm{I}_\omega(\mathscr{M}_{\lambda\eta\perp(C)})$, choosing C to be uncountably infinite. But then, assigning $[x]\downarrow$ to x, we find that for some i, $[M_{t_i}]\downarrow = [M_{u_i}]\downarrow$ holds, i.e., that $M_{t_i} =_{\lambda\eta\perp(C)} M_{u_i}$. Using the above characterisation of $\lambda\eta\perp(C)$ we then have $M_{t_i} =_{\lambda\eta\perp} M_{u_i}$ and so φ is valid. □

Similar ideal-theoretic methods establish the Π_1(POS)-completeness of the **1**-order-extensional ω-cppo models of $\mathscr{T}_\beta$ and of ω-cppo models of $\mathscr{T}_{\mathrm{CL}}$. In the latter case one does not have to establish any implication analogous to extensionality and so it is not necessary to take an uncountable supply of constants. For the same reason it is straightforward to obtain the Π_1(POS)-completeness of dcppo models of $\mathscr{T}_{\mathrm{CL}}$ using the version of the ideal construction with all directed ideals.

It may be that $\mathrm{I}_\omega(\mathscr{M}_{\lambda\eta\perp})$ is order-extensional: we leave this as an open problem. We also do not know if $\mathrm{I}_d(\mathscr{M}_{\lambda\eta\perp(C)})$, is order-extensional for some set of constants C; if it were then Π_1(POS)-completeness would also hold for order-extensional dcppo models of $\mathscr{T}_{\beta\eta}$. One could, presumably, then establish Π_1(POS)-completeness for **1**-order-extensional dcppo models of $\mathscr{T}_\beta$ similarly.

4. Consistency

We begin by showing the Π_2(EQ)-inconsistency of proper partially ordered models for all three theories, where a partial order is *proper* if it contains distinct elements x and y such that $x \le y$. An *applicative structure* $(X,\cdot)$ is a set X equipped with a binary operation $\cdot$ termed *application*; every interpretation of $\mathscr{T}_{\mathrm{CL}}$ cuts down to such a structure.

Definition 2

(i) A subset A of an applicative structure $(X,\cdot)$ is *separable* if for each function $f: A \to X$ there exists $\bar{f} \in X$ such that $f(a) = \bar{f}\cdot a$ for all $a \in A$.
(ii) An applicative structure $(X,\cdot)$ is *ω-separable* if each finite subset of X is separable.

It was shown in [8] that there exists an ω-separable model of $\mathscr{T}_{\beta\eta}$.

Theorem 5. *The proper partially ordered models of any of $\mathscr{T}_{\mathrm{CL}}, \mathscr{T}_\beta$ or $\mathscr{T}_{\beta\eta}$ are not Π_2(EQ)-consistent.*

Proof. Consider the following sentence:

$$\varphi_3 \equiv_{\mathrm{def}} \forall x, y.\exists f.fx = y \wedge fy = x$$

It is evidently satisfied by any ω-separable applicative structure. On the other hand no applicative structure which admits a proper partial order can satisfy it. For otherwise choose $x \le y$, with x, y distinct. Then there is an f such that $fx = y$ and $fy = x$ and we have $y = fx \le fy = x$, yielding a contradiction. □

We now consider Σ_2(EQ)-consistency for $\mathscr{T}_{\beta\eta}$, which is the same as Σ_1(EQ)-consistency for that theory; we shall return to the other two theories later. As we shall see, whether or not Σ_1(EQ)-consistency holds depends on the consistency with $\mathscr{T}_{\beta\eta}$ of the following *critical sentences*:

$$\varphi_{n,m} \equiv_{\mathrm{def}} \exists\,\mathbf{a},\mathbf{c}.\forall u.Q_{n,m}$$

where $n \ge 1$, $m \ge 2$, and $\mathbf{a}$ and $\mathbf{c}$, respectively, abbreviate the lists of variables $a_1,\ldots,a_n$ and $c_1,\ldots,c_m$, and $Q_{n,m}$ is the conjunction of the following equations:

$$\mathrm{I} = a_1(\lambda x.\mathbf{u})$$

$a_1(\lambda x.\mathbf{c}u) = a_2(\lambda x.\mathbf{u})$

...

$a_{n-1}(\lambda x.\mathbf{c}u) = a_n(\lambda x.\mathbf{u})$

$a_n(\lambda x.\mathbf{c}u) = u$

where, for $i = 1, n$, we write $a_i(\lambda x.\mathbf{c}u)$ and $a_i(\lambda x.\mathbf{u})$ to abbreviate the respective terms $a_i(\lambda x.c_1 u)\ldots(\lambda x.c_m u)$ and $a_i(\lambda x.u)\ldots(\lambda x.u)$, the latter with m occurrences of $\lambda x.u$.

Note that no critical sentence $\varphi_{n,m}$ can be satisfied by an order-extensional ppo model. For, if it were, we would have $\mathrm{I} \leq \perp$, as we could calculate, using corresponding abbreviations for $\perp$:

$$\begin{array}{llcl} \mathrm{I} & = a_1(\lambda x.\perp) & \leq & a_1(\lambda x.\mathbf{c}\perp) \\ & = a_2(\lambda x.\perp) & \leq\cdots\leq & a_{n-1}(\lambda x.\mathbf{c}\perp) \\ & = a_n(\lambda x.\perp) & \leq & a_n(\lambda x.\mathbf{c}\perp) \\ & = \perp & & \end{array}$$

and it would then follow, for any x, that $x = \mathrm{I}x \leq \perp x = \perp$, contradicting non-triviality.

The problem of determining the consistency of a critical sentence $\varphi_{n,m}$ with $\mathscr{T}_{\beta\eta}$ seems to be very difficult. Substituting fresh constants for the c_j, one obtains an equivalent formulation in terms of the consistency with the $\lambda\beta\eta$-calculus of $n+1$ equations, and one could then try a Church–Rosser argument. However, this method runs into difficulties when, as here, there are non-linear equations: a well-known example is the failure of confluence for the $\lambda\beta$-calculus with surjective pairing [13, 10.4]. On the other hand, neither is there any obvious proof of inconsistency.

A similar situation arises with generalized Mal'cev operators which were first discussed in the context of the λ-calculus by Selinger [11] and whose consideration helped us find the critical sentences; these operators correspond to proper partially ordered models rather than pointed ones. The corresponding critical sentences assert their existence:

$$\mu_n \equiv_{\mathrm{def}} \exists m_1,\ldots,m_n.\forall x,y.R'_n$$

where R'_n is the conjunction of the following set of equations:

$x = m_1 xyy$

$m_1 xxy = m_2 xyy$

...

$m_{n-1} xxy = m_n xyy$

$m_n xxy = y$

These equations are also nonlinear and it is not known if any of the μ_n are consistent with $\mathscr{T}_{CL}$ (it is known that the first two are inconsistent [11]). There is a connection with our critical sentences that is worth noticing, that $\mathscr{T}_{\mathrm{CL}} \vdash \mu_n \Rightarrow \psi_{n,2}$.

The discussion now forks into two cases. In the first, let us assume that some critical sentence $\varphi_{n,m}$ is consistent with $\mathscr{T}_{\beta\eta}$. In this case, by the above remarks, the order-extensional ppo models are $\Sigma_1(\mathrm{EQ})$-inconsistent for $\mathscr{T}_{\beta\eta}$. One further then conjectures, under the same assumption, that substituting $\Omega\underline{1}\underline{i}$ for a_i $(1 \leq i \leq m)$ and $\Omega\underline{2}\underline{j}$ for c_j $(1 \leq j \leq n)$ in the matrix of the critical formula, the resulting formula $\varphi'_{n,m}$ remains consistent (where Ω is (SII)(SII) and $\underline{k}$ is the kth Church numeral, viz $\lambda f.\lambda x.f^k x$). One sees, as before, that $\varphi'_{n,m}$ is not true in any order-extensional ppo model of $\mathscr{T}_{\beta}$, and so, under this additional assumption one has that the order-extensional ppo models are EQ-inconsistent for $\mathscr{T}_{\beta\eta}$. Let us remark, finally, that the discussion here a fortiori also applies to the narrower class of order-extensional cppo models.

We now turn to the second case where all the critical sentences $\varphi_{n,m}$ are inconsistent with $\mathscr{T}_{\beta\eta}$ aiming to establish $\Sigma_1(\mathrm{EQ})$-consistency. For any $\lambda\eta$-theory Th whose terms do not contain the constant $\perp$, let Th^* be the least $\lambda\eta$ $\perp$-inequational theory containing all the equations of Th, and let $\mathrm{Th}_\perp$ be the least $\lambda\eta$-theory including Th whose terms may contain $\perp$; it is easily verified that it consists of all equations $M[\perp/x] = N[\perp/x]$ where $M =_{\mathrm{Th}} N$ and x is any variable.

We next characterise Th^* in terms of $\mathrm{Th}_\perp$ and a relation $\trianglelefteq$, where $M \trianglelefteq N$ holds if and only if for some $m \geq 0$, there are terms $A, C_1,\ldots,C_m$ such that:

$$M =_{\mathrm{Th}_\perp} A(\lambda x.\perp)\ldots(\lambda x.\perp)$$

with m $(\lambda x.\perp)$'s and:

$$A(\lambda x.C_1\perp)\ldots(\lambda x.C_m\perp) =_{\mathrm{Th}_\perp} N$$

As will be clear, without loss of generality we can insist that the terms $A, C_1,\ldots,C_m$ do not contain $\perp$. Also, by adding dummy arguments to A, we can increase m.

Lemma 5

(i) *The relation $\trianglelefteq$ includes $\leq_{\mathrm{Th}_\perp}$; it is closed under application and λ-abstraction; and $\perp \trianglelefteq M$ holds for all terms M.*
(ii) *The relations $\trianglelefteq^*$ and $\leq_{\mathrm{Th}^*}$ coincide.*

Proof

(i) First $\trianglelefteq$ includes $Th_\perp$ as if $M =_{Th_\perp} N$ then, taking $m = 0$ and $A =_{\text{def}} M$, we see that $M \trianglelefteq N$. Next $\perp \trianglelefteq M$ holds for any term M, as we see if we take $m = 1$, $A =_{\text{def}} \lambda x.x \perp$ and $C_1 =_{\text{def}} \lambda y.M$, with $y \notin \text{FV}(M)$.
To show closure under application, suppose that $M \trianglelefteq N$ and $M' \trianglelefteq N'$ so that:

$$M =_{\text{Th}_\perp} A(\lambda x.\, \perp) \ldots (\lambda x.\, \perp)$$

$$N =_{\text{Th}_\perp} A(\lambda x.\, C_1 \perp) \ldots (\lambda x.\, C_m \perp)$$

$$M' =_{\text{Th}_\perp} A'(\lambda x.\, \perp) \ldots (\lambda x.\, \perp)$$

$$N' =_{\text{Th}_\perp} A'(\lambda x.\, C'_1 \perp) \ldots (\lambda x.\, C'_{m'} \perp)$$

hold, for some terms $A, C_1, \ldots, C_m$ and $A', C'_1, \ldots, C'_{m'}$. We then have:

$$MM' =_{\text{Th}_\perp} A''(\lambda x.\, \perp) \ldots (\lambda x.\, \perp)(\lambda x.\, \perp) \ldots (\lambda x.\, \perp)$$

$$NN' =_{\text{Th}_\perp} A''(\lambda x.\, C_1 \perp) \ldots (\lambda x.\, C_m \perp)(\lambda x.\, C'_1 \perp) \ldots (\lambda x.\, C'_{m'} \perp)$$

where A'' is $\lambda u_1, \ldots, u_m.\lambda u'_1, \ldots, u'_{m'} A u_1 \ldots u_m (A' u'_1 \ldots u'_{m'})$, and so, as desired, we have $MM' \trianglelefteq NN'$.
To show closure under λ-abstraction, suppose that $M \trianglelefteq N$ so that:

$$M =_{\text{Th}_\perp} A(\lambda x.\, \perp) \ldots (\lambda x.\, \perp)$$

$$N =_{\text{Th}_\perp} A(\lambda x.\, C_1 \perp) \ldots (\lambda x.\, C_m \perp)$$

hold, for some terms $A, C_1, \ldots, C_m$. We then have that:

$$\lambda y.M =_{\text{Th}_\perp} A'(\lambda y.\lambda x.\, \perp) \ldots (\lambda y.\lambda x.\, \perp)$$

$$\lambda y.N =_{\text{Th}_\perp} A'(\lambda y.\lambda x.\, C_1 \perp) \ldots (\lambda y.\lambda x.\, C_m \perp)$$

where A' is $\lambda u_1, \ldots, u_m.\lambda y.A(u_1 y) \ldots (u_m y)$. But then we have that:

$$\lambda y.M =_{\text{Th}_\perp} A''(\lambda z.\, \perp) \ldots (\lambda z.\, \perp)$$

$$\lambda y.N =_{\text{Th}_\perp} A''(\lambda z.\, C_1[(z)_0/y, (z)_1/x] \perp) \ldots (\lambda z.\, C_m[(z)_0/y, (z)_1/x] \perp)$$

where A'' is $\lambda u_1, \ldots, u_m.A'((\lambda y.\lambda x.u_1[y,x]) \ldots (\lambda y \lambda x.u_m[y,x]))$, and we are using the pairing notation of [3, 6.2]. This shows that, as desired, $\lambda y.M \trianglelefteq \lambda y.N$.

(ii) It follows at once from part 1 that $\trianglelefteq^*$ is a $\lambda\eta$ $\perp$-theory including Th. To show it is the least such, let Th$'$ be any other. Then if $M \trianglelefteq N$ we have:

$$M =_{\text{Th}_\perp} A(\lambda x.\, \perp) \ldots (\lambda x.\, \perp) \leq_{\text{Th}'} A(\lambda x.\, C_1 \perp) \ldots (\lambda x.\, C_m \perp) =_{\text{Th}_\perp} N$$

for some terms $A, C_1, \ldots, C_m$ and so $M \leq_{\text{Th}'} N$. This shows that Th$'$ includes $\trianglelefteq$ and so, as required, $\trianglelefteq^*$. □

It follows from this lemma that Th* is inequationally consistent if and only if it is not the case that $\text{I} \trianglelefteq^* \perp$; for the the next lemma assume that Th is consistent.

Lemma 6. *If* $\text{I} \trianglelefteq^* \perp$ *then some critical sentence is satisfied by the open term model of* Th.

Proof. By assumption, and adding an extra reflexive link if needed, we have a chain:

$$\text{I} \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_n \trianglelefteq \perp$$

for some $n \geq 1$. We therefore have an $m \geq 2$ and terms A_i $(i = 1, n)$ and terms $C_{i,j}$ $(i = 1, n, j = 1, m)$ not containing $\perp$ such that:

$$I =_{\text{Th}_\perp} A_1(\lambda x.\, \perp) \ldots (\lambda x.\, \perp)$$

$$A_1(\lambda x.C_{0,1} \perp) \ldots (\lambda x.C_{0,m} \perp) =_{\text{Th}_\perp} M_1 =_{\text{Th}_\perp} A_2(\lambda x.\, \perp) \ldots (\lambda x.\, \perp)$$

$$\ldots$$

$$A_n(\lambda x.C_{n,1} \perp) \ldots (\lambda x.C_{n,m} \perp) =_{\text{Th}_\perp} \perp$$

It follows that:

$$I =_{\mathrm{Th}} A_1(\lambda x.u)\ldots(\lambda x.u)$$

$$A_1(\lambda x.C_{0,1}u)\ldots(\lambda x.C_{0,m}u) =_{\mathrm{Th}} A_2(\lambda x.u)\ldots(\lambda x.u)$$

$$\ldots$$

$$A_n(\lambda x.C_{n,1}u)\ldots(\lambda x.C_{n,m}u) =_{\mathrm{Th}} u$$

We need a version of these equations in which the $C_{i,j}$ do not depend on i. To that end define A'_i (for $i = 1, n$) by:

$$A'_1 =_{\mathrm{def}} \lambda x_{1,1}, \ldots, x_{1,m}, \ldots, x_{n,1}, \ldots, x_{n,m}.A_1 x_{1,1}, \ldots, x_{1,m}$$

$$\ldots$$

$$A'_n =_{\mathrm{def}} \lambda x_{1,1}, \ldots, x_{1,m}, \ldots, x_{n,1}, \ldots, x_{n,m}.A_n x_{n,1}, \ldots, x_{n,m}$$

take $m' =_{\mathrm{def}} nm$ and define C'_k (for $k = 1, m'$) by setting:

$$C'_1, \ldots, C'_{m'} =_{\mathrm{def}} C_{1,1}, \ldots, C_{1,m}, \ldots, C_{n,1}, \ldots, C_{n,m}$$

We can then rewrite our equations in the form:

$$I =_{\mathrm{Th}_u} A'_1(\lambda x.u)\ldots(\lambda x.u)$$

$$A'_1(\lambda x.C'_1 u)\ldots(\lambda x.C'_{m'}u) =_{\mathrm{Th}} A'_2(\lambda x.u)\ldots(\lambda x.u)$$

$$\ldots$$

$$A'_n(\lambda x.C'_1 u)\ldots(\lambda x.C'_{m'}u) =_{\mathrm{Th}} u$$

and so the open term model of Th satisfies the critical sentence $\varphi_{n,m'}$, which concludes the proof. □

Theorem 6. *Suppose that all the critical sentences $\varphi_{n,m}$ are inconsistent with $\mathscr{T}_{\beta\eta}$. Then the order-extensional ω-cppo models of $\mathscr{T}_{\beta\eta}$ are $\Sigma_1(\mathrm{EQ})$-consistent.*

Proof. Let φ be a $\Sigma_1(\mathrm{EQ})$-sentence. Without loss of generality, we can take it to be of the form $\exists x.\, t = u$. If there is a model of $\mathscr{T}_{\beta\eta}$ that satisfies it, then there is one of $\mathscr{T}_{\beta\eta}$ extended by a constant c that satisfies $t[c/x] = u[c/x]$, and there is therefore a consistent $\lambda\eta$-theory Th containing the equation $M_{t[c/x]} = M_{u[c/x]}$.

So by the assumption that all the critical sentences $\varphi_{n,m}$ are inconsistent with $\mathscr{T}_{\beta\eta}$, we get, using Lemmas 5 and 6, that Th^* is inequationally consistent. So $\mathscr{M}_{\mathrm{Th}^*}$ is an order-extensional ppo model of $\mathscr{T}_{\beta\eta}$ satisfying $t[c/x] = u[c/x]$ and so φ. Taking an extension $\mathrm{Th}^*(C)$ of Th^* by uncountably many constants, one further has, by Lemma 4, that $\mathrm{I}_\omega(\mathscr{M}_{\mathrm{Th}^*(C)})$ is an order-extensional ω-cppo model of $\mathscr{T}_{\beta\eta}$ satisfying φ. □

If, as discussed above, one could establish order-extensionality for directed ideal models one would further have $\Sigma_1(\mathrm{EQ})$-consistency for order-extensional dcppo models under the asme assumption.

Let us now turn to the other two theories. First, in the case of $\mathscr{T}_\beta$, $\Sigma_2(\mathrm{EQ})$-consistency is again the same as $\Sigma_1(\mathrm{EQ})$-consistency, the same critical sentences $\varphi_{n,m}$ serve, and the problem of determining their consistency with $\mathscr{T}_\beta$ again seems to be difficult. No critical sentence can be satisfied by an **1**-order-extensional ppo model and so, if some critical sentence is consistent we have $\Sigma_1(\mathrm{EQ})$-inconsistency and, under the same assumption, one further conjectures EQ-inconsistency. On the other hand, if every critical sentence is inconsistent with $\mathscr{T}_\beta$ one deduces $\Sigma_1(\mathrm{EQ})$-consistency for **1**-order-extensional ω-cppo models. The proof is entirely analogous to that of Theorem 6, starting from the $\lambda\beta\ \bot$-calculus, which is just the $\lambda\beta\eta\ \bot$-calculus minus η-conversion. Finally, as before, if one could establish **1**-order-extensionality for directed ideal models one would also have such consistency for **1**-order-extensional dcppo models.

Next, in the case of $\mathscr{T}_{\mathrm{CL}}$, one does not have λ-abstraction and the simpler critical sentences:

$$\psi_{n,m} \equiv_{\mathrm{def}} \exists\, \mathbf{a}, \mathbf{c}.\forall u.R_{n,m}$$

serve, where $n \geq 1$, $m \geq 2$, and **a** and **c**, respectively, abbreviate the lists of variables $a_1, \ldots, a_n$ and $c_1, \ldots, c_m$, and $R_{n,m}$ is the conjunction of the following equations:

$$\mathrm{I} = a_1\mathbf{u}$$

$a_1(\mathbf{c}u) = a_2\mathbf{u}$

...

$a_{n-1}(\mathbf{c}u) = a_n\mathbf{u}$

$a_n(\mathbf{c}u) = u$

where for $i = 1, n$, $a_i(\mathbf{c}u)$ and $a_i\mathbf{u}$ abbreviate $a_i(c_1 u)\ldots(c_m u)$ and $a_i u \ldots u$, respectively, both with m u's. The problem of determining their consistency with $\mathscr{T}_{\mathrm{CL}}$ may be easier than in the case of the λ-calculus, but, of course, non-linearity remains.

No critical sentence $\psi_{n,m}$ can be satisfied by a ppo model and so, if one were consistent, we would have $\Sigma_2(\mathrm{EQ})$-inconsistency and, under the same assumption, one further conjectures $\Pi_1(\mathrm{EQ})$-inconsistency. On the other hand, assuming every such sentence inconsistent, one deduces $\Sigma_2(\mathrm{EQ})$-consistency for cppo models (both forms of ideal model work). The proof is, as before, entirely analogous to that of Theorem 6, but is simpler as there are no abstractions to consider: the only point is to use the Curry abstraction operator instead, wherever necessary.

Finally, let us consider sentences with positive matrix. We have already established $\Pi_2(\mathrm{EQ})$-inconsistency and $\Sigma_2(\mathrm{EQ})$-inconsistency for all three theories, assuming, in the latter case, that some critical sentence is inconsistent. Assuming instead that every critical sentence is consistent it may nonetheless be that $\Sigma_2(\mathrm{POS})$- or even $\Pi_1(\mathrm{POS})$-inconsistency holds; we leave these as open problems. Note that $\Sigma_1(\mathrm{POS})$-consistency is equivalent to $\Sigma_1(\mathrm{EQ})$-consistency as every $\Sigma_1(\mathrm{POS})$ sentence is equivalent to a finite disjunction of $\Sigma_1(\mathrm{EQ})$ sentences.

4.1. A possible natural counterexample

We give natural $\Sigma_2(\mathrm{EQ})$-sentences (φ_4' and φ_4'', below) which may provide counterexamples to the $\Sigma_2(\mathrm{EQ})$-consistency, or even the $\Pi_1(\mathrm{EQ})$-consistency of cppo models. Consider the following sentence:

$$\varphi_4 \equiv_{\mathrm{def}} \exists x, y.\ \forall z.\, x(yz) = y(xz) \wedge \forall z. xz \neq yz$$

As commuting continuous functions on a cppo have a common fixed-point, this sentence is satisfied by no cppo models of $\mathscr{T}_{CL}$. However, as we now show, it is consistent with $\mathscr{T}_{\beta\eta}$. The following result is interesting especially because it answers (negatively) an open question raised more than 25 years ago by Dana Scott,[2] as to whether in a formal λ-theory commuting functions always have a common fixed point. Let Com be the $\lambda\eta$-theory with two constants f and g and the equation:

$$f(g(x)) = g(f(x))$$

Proposition 1. *The theory* Com *is consistent. There is, however, no term M such that* $f(M) =_{\mathrm{Com}} g(M)$.

Proof. Let δ-reduction, $\rightarrow_\delta$, be the contextual closure of the relation generated by following two rules:

$$f(gM) \rightarrow g(fM) \quad g(fM) \rightarrow f(gM)$$

We begin by showing that if $M =_{\mathrm{Com}} N$ then we have that $M \rightarrow^*_{\beta\eta} M' \rightarrow^*_\delta N'$ and $N \rightarrow^*_{\beta\eta} N'$ for some M' and N' (the converse is evident). It is enough to show that the relation between M and N of the existence of such an M' and N' is transitive. To that end, one first shows that an application of a δ-rule "commutes" with one step of β- or η-reduction in the sense that if $M \rightarrow_\delta N \rightarrow_{\beta\eta} P$, then there exists Q such that $M \rightarrow_{\beta\eta} Q \rightarrow^n_\delta P$. This is clear for η-reduction. For β-reduction, if the δ-rule has been applied to a subterm of the form $f(gP)$, then if the β-reduction involves only P the result is trivial, otherwise the whole of the subterm $g(fP)$ is involved. But then the application of the δ-rule can be postponed, possibly using it more than once. It then immediately follows that if $M \rightarrow^*_\delta N \rightarrow^*_{\beta\eta} P$, then there exists Q such that $M \rightarrow^*_{\beta\eta} Q \rightarrow^*_\delta P$. Transitivity follows from this result and the Church–Rosser theorem for the $\lambda\beta\eta$-calculus with extra constants.

We next need to look at δ-reduction in more detail. For any $\pi \in \{f, g\}^*$, define $\pi \cdot M$ recursively for terms M by: $\epsilon \cdot M = M$, $(f\pi) \cdot M = f(\pi \cdot M)$ and $(g\pi) \cdot M = g(\pi \cdot M)$. Every term M can be analysed uniquely in the form $\pi_M \cdot M_0$ where M_0 is not of either of the forms $f(M_1)$ or $g(M_1)$; we call π_M the *prefix* of M. If M and N are δ-convertible then π_M and π_N are permutations of each other and if $M \rightarrow^*_{\beta\eta} N$ then π_M is a prefix of π_N.

Now to establish the rest of proposition suppose, for the sake of contradiction, that there exists M such that $f(M) =_{\mathrm{Com}} g(M)$. From the above it follows that there exist M' and M'', such that $M \rightarrow^*_{\beta\eta} M'$, $f(M') \rightarrow^*_\delta g(M'')$ and $M \rightarrow^*_{\beta\eta} M''$. It follows that $f\pi_{M'}$ and $g\pi_{M''}$ are permutations of each other, and so, in particular, $\pi_{M'}$ and $\pi_{M''}$ have the same length. But by Church–Rosser M' and M'' have a common $\beta\eta$-reduct, and so $\pi_{M'} = \pi_{M''}$. But this contradicts the fact that $f\pi_{M'}$ and $g\pi_{M''}$ are permutations of each other, concluding the proof. □

Now consider the following sentence:

[2] Personal communication, March, 1981.

$$\varphi'_4 \equiv_{\text{def}} \exists x, y, d.\ \forall z.x(yz) = y(xz) \wedge \forall z.dz(xz) = \mathrm{T} \wedge dz(yz) = \mathrm{F}$$

where T is $\lambda^* x.\lambda^* y.x$ and F is $\lambda^* x.\lambda^* y.y$. It implies φ_4, relative to CL. So if we can prove it is consistent with one of our three theories, then we have a counterexample to $\Sigma_2(\mathrm{EQ})$-consistency of cppo models of that theory. As before the difficulty with finding a Church–Rosser argument is the non-linearity of the equations. Note too that if we replace the non-linear equations with the linear $d(xz) = \mathrm{T} \wedge d(yz) = \mathrm{F}$ then the resulting sentence is inconsistent with $\mathscr{T}_{\mathrm{CL}}$, so the non-linearity certainly plays a rôle. Finally if, following a previous thought, we could prove consistency then, replacing x, y and d by $\Omega\underline{1}$, $\Omega\underline{2}$ and $\Omega\underline{3}$, the resulting sentence φ''_4 might even provide a counterexample to $\Pi_1(\mathrm{EQ})$-consistency. Whether these sentences do indeed provide counterexamples we leave as open problems.

4.2. On quantifier-free inconsistency

We consider consistency for sentences with arbitrary matrices, but, at least for the λ-calculus, quantifier-free. Let Th be the $\lambda\eta$-theory generated by the equation:

$$\Omega xx = \Omega$$

Lemma 7. *The theory* Th *is consistent and, for any terms M and N, we have that* $\Omega MN =_{\mathrm{Th}} \Omega$ *holds if, and only if,* $M =_{\mathrm{Th}} N$ *does.*

Proof. Consistency follows from, e.g., a Church–Rosser argument that the stronger theory generated by the equation $\Omega MN = \Omega$ is consistent with the $\lambda\beta\eta$-calculus. The rest is Lemma 3.1 of [9], but adapted to the $\lambda\beta\eta$-calculus; the proof is exactly Salibra's, but with the evident additions to account for η-conversion. □

Theorem 7. *The sentence:*

$$\forall x.\Omega xx = \Omega\ \wedge\ \Omega \neq \Omega\Omega(\Omega \mathrm{KI})$$

is consistent with $\mathscr{T}_{\beta\eta}$ *but no ppo model of* $\mathscr{T}_{\mathrm{CL}}$ *satisfies it.*

Proof. For the first part note that the equation $\Omega = \Omega\Omega(\Omega\mathrm{KI})$ is not in Th. For, if it were, by Lemma 7 we would have, successively that $\Omega = \Omega\mathrm{KI}$ and $\mathrm{K} = \mathrm{I}$ were too, contradicting the consistency of Th. For the second part, one follows the argument in the proof of Theorem 3.5 of [9] to show that in any ppo model of CL the sentence $\Omega = \Omega\Omega(\Omega\mathrm{KI})$ holds. □

We therefore have that consistency fails for quantifier-free sentences and ppo models for our two λ-calculus theories $\mathscr{T}_{\beta}$ and $\mathscr{T}_{\beta\eta}$, and also that Π_1-consistency fails for $\mathscr{T}_{\mathrm{CL}}$. The formula:

$$\forall x.\Omega\underline{1}xx = \Omega\underline{1}\ \wedge\ \Omega\underline{2}(\Omega\underline{1}) = \mathrm{T}\ \wedge\ \Omega\underline{2}(\Omega\underline{1}\Omega\underline{1}(\Omega\underline{1}\mathrm{KI})) = \mathrm{F}$$

might then provide a counterexample to EQ or $\Pi_1(\mathrm{EQ})$-consistency. The difficulty, as always, is non-linearity, here of the first equation, and it is an open question as to whether the sentence is indeed a counterexample.

In [5], Honsell and Ronchi Della Rocca give a consistent λ-theory which is not the theory of any retract model. One can extract another example, if of narrower scope, from this work, namely that the following quantifier-free sentence φ_5 is consistent with $\mathscr{T}_{\beta}$ but has no retract models:

$$\Omega\Omega = \Omega\ \wedge\ \lambda x.\Omega(\Omega x) = \lambda x.\Omega x\ \wedge$$

$$\lambda xy.\Omega(x(\Omega y)\Omega)(x\Omega(\Omega y)) = \lambda xy.\Omega(x(\Omega y(\Omega y))\ \wedge\ \Omega \neq \lambda x.\Omega$$

Note that we still have non-linearity. A similar quantifier-free non-linear sentence φ'_5 can be found using the work of Manzonetto and Salibra [7]; the sentence in question is to the effect that Ω is a central element in their sense that differs from both T and F.

5. Concluding remarks

Considering just the $\lambda\beta\eta$-calculus, an interesting example of a Σ_2-sentence true in all retract models but not in every model is the following:

$$\varphi_6 \equiv_{\text{def}} \exists a, b, c, d, e.\ a \neq b \wedge (cd = a) \wedge (ce = b) \wedge (\forall x.\ cx = a \vee cx = b)$$

In fact, in any retract model $(X, \mathrm{K}, \mathrm{S}, \cdot)$ the function:

$$x \in X \mapsto \text{if } x \leq a \text{ then } \bot \text{ else } a$$

where a is any element of the model different from the bottom element is continuous, hence representable. So each retract model satisfies φ_6. On the other hand, as shown in [3], the open term model of $\lambda\beta\eta$ satisfies the *range* property, whereby the range of any term is either infinite or else the term denotes a constant, so falsifying φ_6. The question whether there is an order-extensional ppo, or even cppo, model which satisfies the range property remains open.

One can relativize the basic notions of this paper. Given two classes of models $\mathscr{C} \subseteq \mathscr{C}'$ of a theory $\mathscr{T}$ and a set of sentences $\mathscr{F}$, one can enquire if $\mathscr{C}$ is *$\mathscr{F}$-complete* or *$\mathscr{F}$-consistent* relative to $\mathscr{C}'$. As an example, the directed ideal constructions show that cppo models of $\mathscr{T}_{\mathrm{CL}}$ are consistent relative to ppo models for $\Sigma_2(\mathrm{POS})$. This may fail for Σ_2; the sentence φ_4 would provide a counterexample if one could show, for example, that Proposition 1 held for Com^*, the least $\lambda\perp$-inequational theory including Com.

Another such question is whether one can separate the cppo models from the retract ones. For example, assuming that all the critical sentences $\varphi_{n,m}$ are inconsistent with $\mathscr{T}_\beta$, are the retract models quantifier-free-consistent relative to the order-extensional ω-cppo models of $\mathscr{T}_\beta$? One of the sentences φ_5 or φ'_5 may provide a counterexample.

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