Discrete Mathematics 312 (2012) 1144–1147

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

The eigenvalues of q-Kneser graphs

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ARTICLE INFO

Article history:
Received 11 November 2010
Received in revised form 29 November 2011
Accepted 30 November 2011
Available online 24 December 2011

Keywords:
q-Kneser graph
Eigenvalue

ABSTRACT

In this note, by proving some combinatorial identities, we obtain a simple form for the eigenvalues of q-Kneser graphs.

1. Introduction

It is well known that the eigenvalue technique plays an important role in studying structures of a graph. In particular, the second-largest eigenvalue of a graph gives information about expansion and randomness properties (see [2]), and the least eigenvalue gives independence number and chromatic number bounds (see [2,4]).

The *Kneser graph*, denoted by $K(v, k)$, is the graph whose vertices are k-subsets of a fixed v-set, with two vertices adjacent if they are disjoint. The eigenvalues of Kneser graphs are computed in [4, Theorem 9.4.3], and have appeared in various applications, for example in relation to the Erdös–Ko–Rado theorem [4], the hyperenergetic property of graphs [1], chromatic polynomials [6], algebraically independent quantities of simplices [5] and so on.

In this note, we are concerned with eigenvalues of the q-analogues of Kneser graphs. Let $\mathbb{F}_q^v$ be a v-dimensional vector space over a finite field $\mathbb{F}_q$. The *q-Kneser graph* $qK(v, k)$ has as its vertex set the collection of k-dimensional subspaces of $\mathbb{F}_q^v$, and two vertices are adjacent if they intersect trivially. If $k \le v < 2k$, then $qK(v, k)$ is the null graph, so we only consider the case $v \ge 2k$.

Let q be a prime power. For any integer n and positive integer i, the Gaussian coefficient is defined by

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \prod_{j=0}^{i-1} \frac{q^{n-j} - 1}{q^{i-j} - 1}.$$

By convention, $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$. From now on, we will omit the subscript q. Note that

$$\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}. \tag{1}$$

Delsarte [3] calculated the eigenvalues of Grassmann schemes. In particular, the eigenvalues of $qK(v, k)$ were given.

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doi:10.1016/j.disc.2011.11.042

Theorem 1 (*[3, Theorem 10]*). *All the eigenvalues of* $qK(v,k)$ *are*

$$\lambda_j = (-1)^j q^{(k-j)j+\binom{j}{2}} \sum_{s=0}^{k-j} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} k-j \\ s \end{bmatrix} \begin{bmatrix} v-2j-s \\ v-k-j \end{bmatrix}, \tag{2}$$

where $j = 0, 1, \ldots, k$.

In this note, we obtain a simple form for the eigenvalues of $qK(v,k)$ as follows.

Theorem 2. *The distinct eigenvalues of* $qK(v,k)$ *are*

$$\lambda_j = (-1)^j q^{\binom{k}{2}+\binom{k-j+1}{2}} \begin{bmatrix} v-k-j \\ v-2k \end{bmatrix}, \quad j = 0, 1, \ldots, k.$$

Moreover, the multiplicity of λ_j *is 1 if* $j = 0$, *and* $\begin{bmatrix} v \\ j \end{bmatrix} - \begin{bmatrix} v \\ j-1 \end{bmatrix}$ *if* $j \geq 1$.

2. Proof of Theorem 2

We start with some useful combinatorial identities.

Lemma 3. *For any integer* n *and nonnegative integer* i, *we have*

$$\begin{bmatrix} n \\ i \end{bmatrix} = (-1)^i q^{ni-\binom{i}{2}} \begin{bmatrix} -n+i-1 \\ i \end{bmatrix}.$$

Proof. If $i = 0$, the identity is obvious. If $i > 0$, then

$$\begin{aligned} \begin{bmatrix} n \\ i \end{bmatrix} &= (-1)^i \prod_{j=0}^{i-1} \frac{1-q^{n-j}}{q^{i-j}-1} \\ &= (-1)^i q^{\frac{(2n-i+1)i}{2}} \prod_{j=0}^{i-1} \frac{q^{-n+j}-1}{q^{i-j}-1} \\ &= (-1)^i q^{ni-\binom{i}{2}} \prod_{j=0}^{i-1} \frac{q^{-n+i-1-j}-1}{q^{i-j}-1} \\ &= (-1)^i q^{ni-\binom{i}{2}} \begin{bmatrix} -n+i-1 \\ i \end{bmatrix}, \end{aligned}$$

as desired. □

The following identity is a generalization of [7, Theorem 2.14].

Lemma 4. *For any integer* n *and nonnegative integer* a, *we have*

$$\sum_{s=0}^{a} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} n \\ s \end{bmatrix} = q^{na} \begin{bmatrix} a-n \\ a \end{bmatrix}.$$

Proof. We prove the result by induction on a. If $a = 0$, then the result is trivial. Suppose $a \geq 1$. By induction and Lemma 3,

$$\begin{aligned} \sum_{s=0}^{a} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} n \\ s \end{bmatrix} &= \sum_{s=0}^{a-1} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} n \\ s \end{bmatrix} + (-1)^a q^{\binom{a}{2}} \begin{bmatrix} n \\ a \end{bmatrix} \\ &= q^{n(a-1)} \begin{bmatrix} a-1-n \\ a-1 \end{bmatrix} + q^{na} \begin{bmatrix} a-1-n \\ a \end{bmatrix} \\ &= q^{na} \begin{bmatrix} a-n \\ a \end{bmatrix}. \end{aligned}$$

Hence, the desired result follows. □

Lemma 5. *Let m, a, t be nonnegative integers with $t \le a \le m$. Then*

$$\sum_{s=0}^{a}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} a-s \\ t \end{bmatrix} = q^{m(a-t)} \begin{bmatrix} a-m \\ a-t \end{bmatrix}. \tag{3}$$

Proof. We prove the result by induction on a and t. If $t = 0$, (3) is immediate by Lemma 4. If $a = t$, (3) is straightforward. Now suppose $1 \le t < a$. By (1) and induction,

$$\begin{aligned}
\sum_{s=0}^{a}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} a-s \\ t \end{bmatrix} &= \sum_{s=0}^{a-1}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} a-s \\ t \end{bmatrix} \\
&= \sum_{s=0}^{a-1}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} a-1-s \\ t-1 \end{bmatrix} + q^t \sum_{s=0}^{a-1}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} a-1-s \\ t \end{bmatrix} \\
&= q^{m(a-t)} \begin{bmatrix} a-1-m \\ a-t \end{bmatrix} + q^t q^{m(a-1-t)} \begin{bmatrix} a-1-m \\ a-1-t \end{bmatrix} \\
&= q^{m(a-t)} \begin{bmatrix} a-m \\ a-t \end{bmatrix},
\end{aligned}$$

as desired. □

Theorem 6. *Let m, a, t be nonnegative integers with $a \ge m$ and $a \ge t$. Then*

$$\sum_{s=0}^{m}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} a-s \\ t \end{bmatrix} = q^{m(a-t)} \begin{bmatrix} a-m \\ a-t \end{bmatrix}. \tag{4}$$

Proof. We prove the result by induction on a and t. If $t = 0$, (4) is immediate from Lemma 4. If $a = m$, (4) holds by Lemma 5. If $a = t$, (4) is straightforward. Suppose $a \ge m + 1$ and $1 \le t < a$. By (1) and induction,

$$\begin{aligned}
\sum_{s=0}^{m}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} a-s \\ t \end{bmatrix} &= \sum_{s=0}^{m}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} a-1-s \\ t-1 \end{bmatrix} + q^t \sum_{s=0}^{m}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} a-1-s \\ t \end{bmatrix} \\
&= q^{m(a-t)} \begin{bmatrix} a-1-m \\ a-t \end{bmatrix} + q^t q^{m(a-1-t)} \begin{bmatrix} a-1-m \\ a-t-1 \end{bmatrix} \\
&= q^{m(a-t)} \begin{bmatrix} a-m \\ a-t \end{bmatrix}.
\end{aligned}$$

Therefore, (4) holds. □

Substituting $t = a - m$ in (4), we obtain

Corollary 7. *For nonnegative integers $a \ge m$, we have*

$$\sum_{s=0}^{m}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} a-s \\ a-m \end{bmatrix} = q^{m^2} \begin{bmatrix} a-m \\ m \end{bmatrix}.$$

Proof of Theorem 2. By (2) and Corollary 7, we have

$$\begin{aligned}
\lambda_j &= (-1)^j q^{(k-j)j+\binom{j}{2}} \sum_{s=0}^{k-j}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} k-j \\ s \end{bmatrix} \begin{bmatrix} v-2j-s \\ v-2j-(k-j) \end{bmatrix} \\
&= (-1)^j q^{(k-j)j+\binom{j}{2}} q^{(k-j)^2} \begin{bmatrix} v-2j-(k-j) \\ k-j \end{bmatrix} \\
&= (-1)^j q^{\binom{k}{2}+\binom{k-j+1}{2}} \begin{bmatrix} v-k-j \\ v-2k \end{bmatrix}.
\end{aligned}$$

By arguments similar to those in [3,8], the multiplicity of each λ_j may be computed. □

Acknowledgment

This research was partially supported by NCET-08-0052, NSF of China (10871027) and the Fundamental Research Funds for the Central Universities of China.

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