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Nonlinear Monotone Operators with Values in $\mathscr{L}(X, Y)$

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1. INTRODUCTION

Let X be a topological vector space, Y an ordered topological vector space, and $\mathscr{L}(X, Y)$ the space of all linear and continuous mappings from X into Y. If T is an operator from X into $\mathscr{L}(X, Y)$ (generally multivalued) with domain $D(T)$, T is said to be *monotone* if

$$(A_1 - A_2)(x_1 - x_2) \geqslant 0$$

for all $x_i \in D(T)$ and $A_i \in T(x_i)$, $i = 1, 2$.

In the scalar case $Y = \mathbb{R}$, this definition coincides with the well known definition of monotone operator (cf. [12, 4]). An important subclass of monotone operators $T: X \to \mathscr{L}(X, Y)$ consists of the subdifferentials of convex operators from X into Y, which have been studied by Valadier [17], Kusraev and Kutateladze [11], Papageorgiou [13], and others. Some properties of monotone operators have been investigated by Kirov [7, 8], in connection with the study of the differentiability of convex operators.

In this paper we begin our investigation of the properties of monotone operators from X into $\mathscr{L}(X, Y)$ by introducing, in Section 3, the notion of hereditary order convexity of subsets of $\mathscr{L}(X, Y)$. This notion plays a key role in our discussion and expresses a kind of separability between sets and points. Its interest relies on the fact that the images of maximal monotone operators are hereditarily order-convex.

The main result of Section 4 is the equivalence, under suitable hypotheses, of upper demicontinuity to upper hemicontinuity for monotone operators.

In Section 5, we prove the local boundedness of monotone operators $T: X \to \mathscr{L}(X, Y)$ in case X is a Fréchet space and Y a normed space with normal cone. For corresponding continuity and boundedness results in the scalar case we refer to [6, 4, 10, 5, 15].

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As an application of the above results, we prove in Section 6 that the subdifferential ∂F of a convex operator $F: X \to Y$ with closed epigraph is maximal monotone. For $Y = \mathbb{R}$, this is a well known theorem of Minty [12].

2. Notations and Definitions

Throughout this paper we shall denote by X a real locally convex Hausdorff space and by Y a real locally convex Hausdorff space which is also an ordered linear space with closed positive cone Y_+ (cf. [14]). Let $\mathscr{L} = \mathscr{L}(X, Y)$ be the space of all linear and continuous mappings from X into Y. We denote by $\mathscr{L}_s(X, Y)$ the space $\mathscr{L}(X, Y)$ endowed with the topology of simple convergence (cf. [9]).

Let T be a nonlinear multivalued operator from X into $\mathscr{L}$. The effective domain of T is the set $D(T) = \{x \in X: T(x) \neq \emptyset\}$ and the graph of T is the subset of $X \times \mathscr{L}$ given by

$$G(T) = \{(x, A): x \in D(T), A \in T(x)\}.$$

T is said to be *monotone*, if

$$(A_1 - A_2)(x_1 - x_2) \geqslant 0 \qquad \text{for all} \quad (x_i, A_i) \in G(T), i = 1, 2.$$

A monotone operator T is called *D-maximal* (resp. *maximal*) if the following condition is satisfied: if $x_0 \in D(T)$ (resp. $x_0 \in X$) and $A_0 \in \mathscr{L}(X, Y)$ are such that $(A_0 - A)(x_0 - x) \geqslant 0$ for all $(x, A) \in G(T)$, then $A_0 \in T(x_0)$. An operator $T: X \to \mathscr{L}$ is said to be *locally bounded* at $x_0 \in D(T)$ if there exists a neighborhood U of x_0 such that the set

$$T(U) = \bigcup \{T(x): x \in U\}$$

is an equicontinuous subset of $\mathscr{L}(X, Y)$.

If $K \subset \mathscr{L}$ and $x \in X$, we denote by Kx the set $\{Ax: A \in K\}$. We write $\langle x^*, x\rangle$ in place of $x^*(x)$ for $x \in X$ and $x^* \in X^*$.

3. Hereditary Order-Convex Subsets of $\mathscr{L}(X, Y)$

Let S be a subset of Y. We denote by $[S]$ the order-convex cover of S, that is,

$$[S] = \{y \in Y: a \leqslant y \leqslant b \text{ for some } a, b \in S\}.$$

S is called order-convex iff $S = [S]$.

DEFINITION 1. Let K be a subset of $\mathscr{L}(X, Y)$. The *hereditarily order-convex cover* of K is defined by

$$[K]^h = \{A \in \mathscr{L}: Ax \in [Kx] \text{ for all } x \in X\}.$$

Let $K \subset \mathscr{L}$ and A an element of $\mathscr{L}$ with the property: for each $x \in X$ there exists $A' \in K$ such that $A'x \geqslant Ax$. Then there exists $A'' \in K$ such that $A''(-x) \geqslant A(-x)$. Hence $Ax \in [Kx]$. Consequently, $[K]^h$ can be defined equivalently by

$$[K]^h = \{A \in \mathscr{L}: \text{for all } x \in X \text{ there exists } A' \in K: A'x \geqslant Ax\}.$$

DEFINITION 2. A subset K of $\mathscr{L}(X, Y)$ is said to be *hereditarily order-convex* (briefly, HOC) if $K = [K]^h$.

Thus K is HOC if and only if for every $A_0 \notin K$ there exists $x \in X$ that "separates" A_0 and K in the sense $Ax \ngeqslant A_0 x$, for all $A \in K$. The interest of HOC subsets of $\mathscr{L}$ to the study of maximal monotone operators relies on the following proposition.

PROPOSITION 1. *Let $T: X \to \mathscr{L}(X, Y)$ be D-maximal monotone. Then for each $x \in D(T)$, $T(x)$ is s-closed, convex, and HOC.*

Proof. As in the special case $Y = \mathbb{R}$ one can prove that $T(x)$ is s-closed and convex [4]. Now, let $A \in [T(x_0)]^h$, $x_0 \in D(T)$.

Then for each $x \in X$ there exists $A' \in T(x_0)$ such that

$$A'(x - x_0) \geqslant A(x - x_0).$$

For any $(x, B) \in G(T)$ we have

$$\begin{aligned}(B - A)(x - x_0) &= B(x - x_0) - A(x - x_0) \geqslant B(x - x_0) - A'(x - x_0)\\ &= (B - A')(x - x_0) \geqslant 0.\end{aligned}$$

Since T is D-maximal, we conclude that $A \in T(x_0)$. Thus $T(x_0)$ is HOC.

One can easily verify that $[K]^h$ is the smallest HOC subset of $\mathscr{L}$ containing K.

In the special case $Y = \mathbb{R}$, the HOC-cover of a set $K \subset \mathscr{L}(X, \mathbb{R}) = X^*$ is given by the following proposition which is an easy consequence of the Hahn–Banach Theorem.

PROPOSITION 2. *For each $K \subset X^*$, $[K]^h$ is the intersection of the family of all w^*-open half-spaces containing K, whenever this family is not void. Otherwise, $[K]^h = X^*$.*

Since each convex and w^*-closed (resp. w^*-open) subset K of X^* ($K \neq X^*$) is the intersection of all w^*-open half-spaces containing it, we deduce the following properties of HOC subsets of X^*.

COROLLARY 1. (i) *Every convex and w^*-closed (resp. w^*-open) subset of X^* is HOC.*

(ii) *Every HOC subset of X^* is convex.*

We note that a convex subset of X^* is not necessarily HOC, as the following example shows.

EXAMPLE 1. Let $X = c_0$, $X^* = l_1$ and let K be the convex subset of l_1 defined by $K = \{f = (f_i) \in l_1 : \sum_{i=1}^{\infty} f_i = 1\}$. K is a hyperplane defined by the element $(1, 1, ..., 1, ...) \in l_\infty$. Thus K is not w^*-closed and so K is w^*-dense in l_1. Therefore K cannot be contained in any w^*-open half-space, so $[K]^h = X^*$. In particular, K is not HOC.

We note that when $Y \neq \mathbb{R}$, convexity is not related to hereditary order convexity.

EXAMPLE 2. Let $X = \mathbb{R}$ and $Y = \mathbb{R}^2$ with the usual order. Then the space $\mathscr{L}(\mathbb{R}, \mathbb{R}^2)$ is isomorphic to $\mathbb{R}^2$, since to every $A \in \mathscr{L}$ corresponds an element $\hat{A} = (a, b) \in \mathbb{R}^2$ such that $Ax = x\hat{A}$, $x \in \mathbb{R}$. It can be easily seen that a subset K of $\mathscr{L}$ is HOC iff the corresponding subset $\hat{K}$ of $\mathbb{R}^2$ is order-convex. Now, the set N corresponding to $\hat{N} = \{(1, -1), (-1, 1)\}$ is HOC but not convex. On the other hand, the set Q corresponding to $\hat{Q} = \{(a, a): 0 \leqslant a \leqslant 1\}$ is convex and closed but not HOC.

4. CONTINTUITY PROPERTIES OF MONOTONE OPERATORS

Let F and G be topological spaces. An operator T from F into 2^G is said to be *upper semicontinuous* if, for each $x_0 \in F$ and each open set V in G with $T(x_0) \subset V$, there exists a neighborhood U of x_0 such that $T(x) \subset V$ whenever $x \in U$. A multivalued operator T: $X \to \mathscr{L}(X, Y)$ which is upper semicontinuous from $D(T)$ into $\mathscr{L}_s(X, Y)$ is said to be *upper demicontinuous.* If T is upper semicontinuous from each segment $Q \subset D(T)$ into $\mathscr{L}_s(X, Y)$ then T is said to be *upper hemicontinuous.* Let B be a subset of X. A point $x_0 \in B$ is an algebraic interior point of B if for each $x \in X$ there exists $\lambda_0 > 0$ such that $x_0 + \lambda x \in B$ for all $0 < \lambda < \lambda_0$. The set of all algebraic interior points of B is denoted by *corB*. If $B = corB$ then B is called linearly open. In case $Y = \mathbb{R}$, it is known that if the monotone operator T: $X \to 2^{X^*}$ is upper hemicontinuous, $D(T)$ is linearly open and for all $x \in D(T)$, $T(x)$ is an equicontinuous, w^*-closed and convex subset of X^*, then T is D-maximal monotone [10]. This no longer valid in case $Y \neq \mathbb{R}$ as the following example shows.

EXAMPLE 3. Let $X=\mathbb{R}$ and $Y=\mathbb{R}^2$. We define the operator T: $\mathbb{R} \to \mathscr{L}(\mathbb{R}, \mathbb{R}^2)$ by $T(x)=0$ if $x<0$, $T(0)=K$ where $\hat{K}=\{(a, a)\colon 0 \leqslant a \leqslant 1\}$ and $T(x)=A$, where $\hat{A}=(1, 1)$, if $x>0$ (see Example 2). This operator satisfies all the above requirements but it is not D-maximal since $T(0)$ is not HOC.

If T: $X \to \mathscr{L}$ is monotone, then the operator $x \to [T(x)]^h$ is also monotone. We define the operator $\tilde{T}$: $X \to \mathscr{L}$ by

$$\tilde{T}(x)=[\overline{T(x)}]^h,$$

where $\overline{T(x)}$ is the closure of $T(x)$ in $\mathscr{L}_s(X, Y)$. It is clear that $\tilde{T}$ is also monotone.

LEMMA 1. *Let F be a topological space, G be a linear topological space, and T: $F \to 2^G$ an upper semicontinuous operator such that for any $x \in F$, $T(x)$ is relatively compact. If K is a relatively compact subset of F, then $T(K)$ is relatively compact in G.*

The proof of the lemma is an easy consequence of the fact that the operator $\bar{T}$ defined by $\bar{T}(x)=\overline{T(x)}$ is upper semicontinuous, and of Theorem 3 in [1, p. 110].

THEOREM 1. *Let T: $X \to \mathscr{L}_s(X, Y)$ be monotone. If T is upper hemicontinuous, $D(T)$ is linearly open and for each $x \in D(T)$, $T(x)$ is relatively compact, then $\tilde{T}$ is D-maximal monotone.*

Proof. Let $x_0 \in D(T)$ and $A_0 \in \mathscr{L}$ such that

$$(A-A_0)(x-x_0) \geqslant 0 \qquad \text{for all} \quad (x, A) \in G(\tilde{T}).$$

We shall show that $A_0 \in \tilde{T}(x_0)$. Suppose that $A_0 \notin \tilde{T}(x_0)=[\overline{T(x_0)}]^h$. Then there exists $x \in X$ such that

$$(A'-A_0)\, x \notin Y_+ \qquad \text{for all} \quad A' \in \overline{T(x_0)}, \tag{1}$$

Let $\{t_n\}$ be a sequence of positive numbers such that $t_n \to 0$ as $n \to \infty$. We set $x_n=x_0+t_n x$. Since $D(T)$ is linearly open there exists $n_0 \in \mathbb{N}$ such that $x_n \in D(T)$ for all $n \geqslant n_0$. For $A_n \in T(x_n)$ we have

$$(A_n-A_0)(x_n-x_0) \geqslant 0$$

which implies that

$$(A_n-A_0)\, x \geqslant 0. \tag{2}$$

By Lemma 1, $\bigcup_n T(x_n)$ is relatively compact, so there exists a subnet $\{A_\alpha\}$ of $\{A_n\}$ such that $A_\alpha \to A$ in $\mathscr{L}_s(X, Y)$. The upper semicontinuity of $\bar{T}$ implies that $A \in \bar{T}(x_0)$. From (2) and the closedness of the positive cone Y_+ we get

$$(A - A_0)\, x \geqslant 0,$$

which contradicts (1). Thus T is D-maximal monotone.

As we shall show now, the relative compactness of $T(x)$ involved in the above theorem is ensured under suitable assumptions for X and Y.

A subset K of $\mathscr{L}(X, Y)$ is said to be *weakly order bounded* if for each $x \in X$, Kx is an order bounded subset of Y.

PROPOSITION 3. *Let T: $X \to \mathscr{L}_s(X, Y)$ be a monotone operator and x_0 an algebraic interior point of $D(T)$. Then*

(i) *$T(x_0)$ is weakly order bounded.*

(ii) *If X is barrelled and Y_+ is normal, then $T(x_0)$ is equicontinuous.*

(iii) *If, in addition, Y has compact order intervals, then $T(x_0)$ is relatively compact.*

Proof. (i) For each $x \in X$ there exists $\lambda > 0$ such that $x_0 + \lambda x \in D(T)$. Let $B \in T(x_0 + \lambda x)$ be fixed. Then for each $A \in T(x_0)$ we have

$$(B - A)(x_0 + \lambda x - x_0) \geqslant 0,$$

which implies that

$$Ax \leqslant Bx.$$

Likewise, there exist $\lambda_1 > 0$ and $B_1 \in T(x_0 + \lambda_1(-x))$ such that

$$Ax \geqslant B_1 x \qquad \text{for each} \quad A \in T(x_0).$$

Hence $T(x_0)$ is weakly order bounded.

(ii) It is sufficient to show that $T(x_0)$ is bounded in $\mathscr{L}_s(X, Y)$ [9, p. 137]. Any neighborhood of 0 in $\mathscr{L}_s(X, Y)$ contains a neighborhood of the form

$$V = \bigcap_{1 \leqslant i \leqslant m} \{A \in L\colon Ax_i \in W\},$$

where $x_i \in X$ and W is an order-convex neighborhood of 0 in Y. Since $T(x_0)$ is weakly order bounded there exist y_i, y_i', $i = 1, 2, ..., m$ such that $y_i' \leqslant Ax_i \leqslant y_i$ for any $A \in T(x_0)$. The set W is absorbing, so there exists $\lambda > 0$ such that $\lambda y_i, \lambda y_i' \in W$ for all i. Hence for all $A \in T(x_0)$ one has

$\lambda y_i' \leqslant \lambda A x_i \leqslant \lambda y_i$ and from the order convexity of W we deduce that $\lambda A x_i \in W$. Thus $\lambda A \in V$ and $T(x_0)$ is bounded in $\mathscr{L}_s(X, Y)$.

(iii) By the weak order boundedness of $T(x_0)$, for each $x \in X$, the set $\{Ax\colon A \in T(x_0)\}$ is relatively compact. The assertion follows from [3, p. 23].

THEOREM 2. *Let $T\colon X \to \mathscr{L}_s(X, Y)$ be a D-maximal monotone operator. Suppose that for each $x \in D(T)$ there exists a neighborhood V of x such that $T(V)$ is equicontinuous and relatively compact. Then T is upper demicontinuous.*

Proof. Suppose that T is not upper demicontinuous. Then there exists a point $x_0 \in D(T)$ and an open set W in $\mathscr{L}_s(X, Y)$ with $T(x_0) \subset W$, such that for each neighborhood U of x_0 there exist $x \in U$ and $A \in T(x)$ with $A \notin W$. By our assumptions, one can find a neighborhood V of x_0 such that $T(V)$ is an equicontinuous and relatively compact subset of $\mathscr{L}_s(X, Y)$. Hence there exist a net $\{x_\alpha\}$ in V and $A_\alpha \in T(x_\alpha)$ such that $x_\alpha \to x_0$ and $A_\alpha \notin W$ for all α. It then follows that there exists a subnet $\{A_\beta\}$ of $\{A_\alpha\}$ with $A_\beta \to A_0 \notin T(x_0)$. Now, for each $x \in D(T)$ and $A \in T(x)$ we have

$$(A_\beta - A)(x_\beta - x) = A_\beta x_\beta - A_\beta x - A x_\beta + Ax \geqslant 0. \tag{3}$$

It is clear that $A_\beta x \to A_0 x$ and $A x_\beta \to A x_0$. On the other hand, $A_\beta x_\beta \to A_0 x_0$. Indeed, we have $A_\beta x_\beta - A_0 x_0 = A_\beta(x_\beta - x_0) + (A_\beta - A_0)\, x_0$ and $(A_\beta - A_0)\, x_0 \to 0$, while by the equicontinuity of $\{A_\beta\}$, $A_\beta(x_\beta - x_0) \to 0$. It follows from (3) that

$$(A_0 - A)(x_0 - x) \geqslant 0.$$

Since T is D-maximal monotone, we conclude that $A_0 \in T(x_0)$ which is a contradiction. Therefore, T is upper demicontinuous.

The following theorem is a simple consequence of Theorems 1 and 2.

THEOREM 3. *Let $T\colon X \to \mathscr{L}_s(X, Y)$ be a monotone operator with $D(T)$ a linearly open subset of X. Suppose that*

(i) *For each $x \in D(T)$ there exists a neighborhood V of x, such that $T(V)$ is equicontinuous and relatively compact.*

(ii) *For each $x \in D(T)$, $T(x)$ is closed and HOC.*

Then T is upper demicontinuous iff it is upper hemicontinuous..

Remark. When $Y = \mathbb{R}$, the equicontinuity of a subset of $\mathscr{L}(X, \mathbb{R}) = X^*$ implies its w^*-relative compactness. In addition, a w^*-closed subset of X^* is HOC iff it is convex. Thus Theorem 3 generalizes an analogous result proved in [10].

5. Local Boundedness of Monotone Operators

Let T: $X \to \mathscr{L}(X, Y)$ be a monotone operator. It is known [5] that if X is a Fréchet space and $Y = \mathbb{R}$, then T is locally bounded at any interior point of $D(T)$. The same conclusion has been obtained by Kirov [7] in case X is a Banach space, Y a normed lattice, and $D(T) = X$. In what follows we generalize these results when X is a Fréchet space and Y a normed space with normal cone.

The proof of the following lemma is similar to the proof of the lemma in [5].

Lemma 2. *Let X be a Fréchet space, Y a normed space, $\{x_n\}$ a sequence in X converging to 0, and $\{A_n\}$ a sequence in $\mathscr{L}(X, Y)$ such that the set $\{A_n, n \in \mathbb{N}\}$ is not equicontinuous. If $a_n = \max\{1, \|A_n x_n\|\}$ and $B_n = A_n/a_n$, then there exists $x_0 \in X$ and a subsequence $\{B_{n_k}\}$ of $\{B_n\}$ such that $\|B_{n_k} x_0\| \to \infty$.*

Proof. We shall first show that the set $\{B_n$: $n \in \mathbb{N}\}$ is not equicontinuous. If $a_n = 1$ for all sufficiently large n, the assertion is obvious. Otherwise, for some suitable subsequence we shall have $\|B_n x_n\| = 1$. Since $x_n \to 0$, $\{B_n\}$ is not equicontinuous. Therefore, $\{B_n\}$ is not bounded in $\mathscr{L}_s(X, Y)$ [9, p. 137]. Thus there exists a neighborhood $W = \{A \in L$: $\|Ax_0\| < 1\}$ for some fixed $x_0 \in X$, such that for all $\lambda > 0$, $\{B_n$: $n \in \mathbb{N}\}$ will not be a subset of λW. So for any $k \in \mathbb{N}$ there exists B_{n_k} such that $B_{n_k} \notin kW$. Hence $\|B_{n_k} x_0\| \geqslant k$, which implies $\|B_{n_k} x_0\| \to \infty$.

Theorem 4. *Let X be a Fréchet space, Y a normed space with a normal cone, and T: $X \to \mathscr{L}(X, Y)$ be monotone. Then T is locally bounded at any algebraic interior point of $D(T)$.*

Proof. Suppose that x' is an algebraic interior point of $D(T)$ and T is not locally bounded at x'. Without loss of generality, we may assume that $x' = 0$. Let d be a metric defining the topology of X.

If $U_n = \{x \in X$: $d(0, x) < 1/n\}$, then $T(U_n)$ is not equicontinuous. Therefore, $T(U_n)$ is not bounded in $\mathscr{L}(X, Y)$ with respect to the topology of bounded convergence [9, p. 137]. This topology is generated by a metric ρ [9, p. 140]. Hence there exists $A_n \in T(U_n)$ such that $\rho(0, A_n) > n$. If we choose $x_n \in U_n$ such that $A_n \in T(x_n)$, then $x_n \to 0$ and the set $\{A_n : n \in \mathbb{N}\}$ is not equicontinuous. We now define a sequence $\{B_n\}$ as in Lemma 2. Then there exists $x_0 \in X$ and a suitable subsequence of $\{B_n\}$ (which we denote again by $\{B_n\}$) such that $\|B_n x_0\| \to \infty$. Since 0 is an algebraic interior point of $D(T)$, there exists $\lambda > 0$ such that $\pm z_0 \in D(T)$, where $z_0 = \lambda x_0$. Let $A_0 \in T(z_0)$, $A_0' \in T(-z_0)$. One has

$$(A_n - A_0)(x_n - z_0) \geqslant 0$$

and

$$(A_n - A'_0)(x_n + z_0) \geqslant 0$$

from which follows

$$B_n z_0 \leqslant \frac{A_0}{a_n}(z_0 - x_n) + B_n x_n := u_n$$

and

$$-B_n z_0 \leqslant \frac{-A'_0}{a_n}(z_0 + x_n) + B_n x_n := v_n.$$

Since Y_+ is normal, there exists $\alpha \in \mathbb{R}_+$ such that for each $x, y \in Y$, $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant \alpha \|y\|$ [14]. Thus, from $0 \leqslant B_n z_0 + v_n \leqslant v_n + u_n$, we get

$$\|B_n z_0 + v_n\| \leqslant \alpha \|v_n + u_n\| \Rightarrow \|B_n z_0\| \leqslant (\alpha + 1)\|v_n\| + \|u_n\|.$$

As one can easily verify, $\{\|v_n\|\}$ and $\{\|u_n\|\}$ are bounded, which contradicts $\|B_n z_0\| \to \infty$.

In Theorems 1, 2, 3 no reference has been made as to the topology of Y, which usually is taken to be the weak topology (see [17, 18]). From the next corollary it follows that in this case and under suitable assumptions the assertion of Theorem 4 can be strengthened and the hypothesis (i) of Theorem 3 is redundant.

COROLLARY 2. *Let X be a Fréchet space, Y a normed space with a normal cone, and $T: X \to \mathscr{L}(X, Y)$ a monotone operator. Let further σ be a topology on Y weaker than the norm topology with the property that every σ-bounded subset of Y is σ-relatively compact, and Y_σ be the space Y endowed with the topology σ. Then for each algebraic interior point x of $D(T)$ there exists a neighborhood U of x such that $T(U)$ is an equicontinuous and relatively compact subset of $\mathscr{L}_s(X, Y_\sigma)$.*

Proof. By Theorem 4, there exists a neighborhood U of x such that $T(U)$ is equicontinuous in $\mathscr{L}(X, Y)$. Hence, $T(U)$ is equicontinuous in $\mathscr{L}(X, Y_\sigma)$. By a theorem of Grothendieck [9, p. 140] $T(U)$ is relatively compact in $\mathscr{L}_s(X, Y_\sigma)$.

6. THE MAXIMALITY OF THE SUBDIFFERENTIAL OPERATOR

Let $F: X \to Y$ be a convex operator, that is, an operator such that

$$F(\lambda x + (1-\lambda) y) \leqslant \lambda F(x) + (1-\lambda) F(y)$$

for all $x, y \in X$ and $0 \leqslant \lambda \leqslant 1$. The epigraph of F is defined by epi $F = \{(x, y): y \geqslant F(x), x \in X\}$. The subdifferential of F at x_0 is the set

$$\partial F(x_0) = \{A \in \mathscr{L}(X, Y): A(x - x_0) \leqslant F(x) - F(x_0) \text{ for all } x \in X\}.$$

The subdifferential operator ∂F is obviously monotone. When Y is order complete, then $A \in \partial F(x)$ if and only if $Ay \leqslant F'(x, y)$ for all $y \in X$, where $F'(x, y)$ is defined by $F'(x, y) = \inf\{(F(x + \lambda y) - F(x))/\lambda: \lambda > 0\}$ (cf. [17]). It is easy to see that the subdifferential $\partial F(x)$ is HOC for every $x \in X$. Indeed, let $A \in [\partial F(x)]^h$. Then for each $y \in X$ there exists $A' \in \partial F(x)$ such that $A(y - x) \leqslant A'(y - x)$. It then follows that

$$A(y - x) \leqslant A'(y - x) \leqslant F(y) - F(x),$$

which means that $A \in \partial F(x)$. Thus $\partial F(x)$ is HOC.

As an application of our preceding results, we shall now prove a generalization of Minty's theorem on the maximality of the subdifferential operator [12].

THEOREM 5. *Let X be a Fréchet space and suppose that Y satisfies one of the following conditions*:

(i) *Y is a dual Banach lattice.*

(ii) *Y is a Banach lattice with weakly compact intervals.*

If F: $X \to Y$ is a convex operator with closed epigraph, then ∂F is maximal monotone.

Proof. (i) Let $Y = Z^*$, where Z is a Banach lattice, and let Y_σ be the space Y endowed with the $\sigma(Y, Z)$ topology. Since F has closed epigraph it is continuous [2], so $\partial F(x) \neq \varnothing$ for all $x \in X$ [17]. By Corollary 2, for each $x \in X$ there exists a neighborhood U of x such that $\partial F(U)$ is relatively compact in $\mathscr{L}_s(X, Y_\sigma)$. We show now that the graph $G(\partial F)$ is closed in $X \times \mathscr{L}_s(X, Y_\sigma)$. This by [1, p. 112] implies that ∂F is upper semicontinuous and $\partial F(x)$ is compact, hence by Theorem 1 ∂F is maximal monotone.

Let $\{(x_i, A_i)\}$ be a net in $G(\partial F)$ such that $(x_i, A_i) \to (x_0, A_0)$, so $x_i \to x_0$ and $A_i x \to A_0 x$ in Y_σ for all $x \in X$. Since $A_i \in \partial F(x_i)$ one has

$$A_i x \leqslant F(x + x_i) - F(x_i) \qquad \text{for all} \quad x \in X.$$

Now the cone Y_+ is σ-closed, so taking limits in the above inequality we deduce

$$A_0 x \leqslant F(x + x_0) - F(x_0) \qquad \text{for all} \quad x \in X.$$

Hence $(x_0, A_0) \in G(\partial F)$. So $G(\partial F)$ is closed and ∂F is maximal.

(ii) Let J be the canonical injection from Y into Y^{**}. As before the mapping F is continuous, so the mapping $J \circ F$ is convex and continuous. It follows from part (i) of the theorem that the operator $\partial(J \circ F)$ is maximal monotone.

We shall now need the following lemma:

LEMMA 3. *For any $x \in X$ one has $\partial(J \circ F)(x) = [J \circ \partial F(x)]^h$.*

Proof. It is obvious that $J \circ \partial F(x) \subset \partial(J \circ F)(x)$. Since $\partial(J \circ F)(x)$ is HOC, we deduce that $[J \circ \partial F(x)]^h \subset \partial(J \circ F)(x)$. Now let $B \in \partial(J \circ F)(x)$ and $y \in X$. Then $By \leqslant (J \circ F)'(x, y)$. But

$$\begin{aligned}(J \circ F)'(x, y) &= \inf\{J(F(x + \lambda y) - F(x))/\lambda : \lambda > 0\} \\ &= J(\inf\{(F(x + \lambda y) - F(x))/\lambda : \lambda > 0\}) = J(F'(x, y)),\end{aligned}$$

since by the assumption on Y, the injection J preserves the infimum of any decreasing net [16, Theorem 5.10]. On the other hand, by [17, Theorem 6], there exists $A \in \partial F(x)$ such that $Ay = F'(x, y)$. Hence, $By \leqslant J \circ A(y)$ and $B \in [J \circ \partial F(x)]^h$, which proves the lemma.

Proof of Theorem 5 *completed.* Let $x_0 \in X$ and $A_0 \in \mathscr{L}(X, Y)$ such that

$$(A - A_0)(x - x_0) \geqslant 0 \qquad \text{for all } x \in X \text{ and } A \in \partial F(x).$$

Then for any $B \in \partial(J \circ F)(x)$, there exists by the lemma $A \in \partial F(x)$ such that $B(x - x_0) \geqslant J \circ A(x - x_0)$. Hence

$$(B - J \circ A_0)(x - x_0) \geqslant (J \circ A - J \circ A_0)(x - x_0) \geqslant 0.$$

Since $\partial(J \circ F)$ is maximal, we deduce that $J \circ A_0 \in \partial(J \circ F)(x_0)$, so $A_0 \in \partial F(x_0)$ and ∂F is maximal.

Remark. As examples of spaces Y satisfying the assumptions of the above theorem, we mention c_0, l^p, and L^p $(1 \leqslant p \leqslant \infty)$.

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