Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 260 (2011) 3283–3298
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

On the essential spectrum of complete non-compact manifolds ☆

Zhiqin Lu [a,*], Detang Zhou [b]

[a] *Department of Mathematics, University of California, Irvine, CA 92697, USA*
[b] *Instituto de Matematica, Universidade Federal Fluminense, Niterói, RJ 24020, Brazil*

Received 28 September 2010; accepted 15 October 2010

Available online 15 December 2010

Communicated by Daniel W. Stroock

Abstract

In this paper, we prove that the L^p essential spectra of the Laplacian on functions are $[0, +\infty)$ on a non-compact complete Riemannian manifold with non-negative Ricci curvature at infinity. The similar method applies to gradient shrinking Ricci soliton, which is similar to non-compact manifold with non-negative Ricci curvature in many ways.

Keywords: Essential spectrum; Shrinking soliton

1. Introduction

The spectra of Laplacians on a complete non-compact manifold provide important geometric and topological information of the manifold. In the past two decades, the essential spectra of Laplacians on functions were computed for a large class of manifolds. When the manifold has a soul and the exponential map is a diffeomorphism, Escobar [11], Escobar and Freire [12] proved that the L^2 spectrum of the Laplacian is $[0, +\infty)$, provided that the sectional curvature is non-negative and the manifold satisfies some additional conditions. In [18], the second author proved that those "additional conditions" are superfluous. When the manifold has a pole, J. Li [14]

☆ The first author is partially supported by the NSF award DMS-0904653. The second author is partially supported by CNPq and Faperj of Brazil.
* Corresponding author.
E-mail addresses: zlu@uci.edu (Z. Lu), zhou@impa.br (D. Zhou).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.10.010

proved that the L^2 essential spectrum is $[0, +\infty)$, if the Ricci curvature of the manifold is non-negative. Z. Chen and the first author [7] proved the same result when the radical sectional curvature is non-negative. Among the other results in his paper [10], Donnelly proved that the essential spectrum is $[0, +\infty)$ for manifold with non-negative Ricci curvature and Euclidean volume growth.

In 1997, J.-P. Wang [17] proved that, if the Ricci curvature of a manifold M satisfies $\mathrm{Ric}(M) \geqslant -\delta/r^2$, where r is the distance to a fixed point, and δ is a positive number depending only on the dimension, then the L^p essential spectrum of M is $[0, +\infty)$ for any $p \in [1, +\infty]$. In particular, for a complete non-compact manifold with non-negative Ricci curvature, all L^p essential spectra are $[0, +\infty)$.

Complete gradient shrinking Ricci soliton, which was introduced as singularity model of type I singularities of the Ricci flow, has many similar properties to complete non-compact manifold with non-negative Ricci curvature. From this point of view, we expect the conclusion of Wang's result is true for a larger class of manifolds, including gradient shrinking Ricci solitons.

The first result of this paper is a generalization of Wang's theorem [17].

Theorem 1. *Let M be a complete non-compact Riemannian manifold. Assume that*

$$\varliminf_{x\to\infty} \mathrm{Ric}_M(x) = 0. \tag{1}$$

Then the L^p essential spectrum of M is $[0, +\infty)$ *for any* $p \in [1, +\infty]$.

It should be pointed out that, contrary to the L^2 spectrum, the L^p spectrum of Laplacian may contain non-real numbers. Our proof made essential use of the following result due to Sturm [16]:

Theorem 2 *(Sturm). Let M be a complete non-compact manifold whose Ricci curvature has a lower bound. If the volume of M grows uniformly sub-exponentially, then the L^p spectra are the same for all* $p \in [1, \infty]$.

We say that the volume of M grows *uniformly sub-exponentially*, if for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that, for all $r > 0$ and all $p \in M$,

$$vol\big(B_p(r)\big) \leqslant C(\varepsilon)e^{\varepsilon r} vol\big(B_p(1)\big), \tag{2}$$

where we denote $B_p(r)$ the ball of radius r centered at p.

Remark 1. Note that by the above definition, a manifold with finite volume may not automatically be a manifold of volume growing uniformly sub-exponentially. For example, consider a manifold whose only end is a cusp and the metric $dr^2 + e^{-r} d\theta^2$ on the end $S^1 \times [1, +\infty)$. The volume of such a manifold is finite. However, since the volume of the unit ball centered at any point p decays exponentially, it doesn't satisfy (2).

Remark 2. The assumption that the Ricci curvature has a lower bound is not explicitly stated in Sturm's paper, but is needed in the proof of Theorem 2.

Remark 3. Under the assumptions of Theorem 2, all L^p spectra are contained in $[0, \infty)$. Thus for a fixed p, the L^p spectrum being equal to $[0, \infty)$ is equivalent to the L^p *essential* spectrum being equal to $[0, \infty)$. For the sake of simplicity, in this paper, we don't distinguish the two concepts: the spectrum and the essential spectrum.

In [16, Proposition 1], it is proved that if (1) is true, then the volume of the manifold grows uniformly sub-exponentially. Thus in order to prove Theorem 1, we only need to compute the L^1 spectrum of the manifold.

Using the recent volume estimates obtained by H. Cao and the second author [3], we proved that the essential L^1 spectrum of any complete gradient shrinking soliton contains the half line $[0, +\infty)$ (see Theorem 6). Combining with Sturm's Theorem we have

Theorem 3. *Let M be a complete non-compact gradient shrinking Ricci soliton. If the conclusion of Theorem 2 holds for M, then the L^p essential spectrum of M is $[0, +\infty)$ for any $p \in [1, +\infty]$.*

Finally, under additional curvature conditions, we proved

Theorem 4. *Let (M, g_{ij}, f) be a complete shrinking Ricci soliton. If*

$$\lim_{x\to+\infty} \frac{R}{r^2(x)} = 0,$$

then the L^2 essential spectrum is $[0, +\infty)$, where R is the scalar curvature and $r(x)$ is the distance function.

We believe that the scalar curvature assumption in the above theorem is technical and could be removed. From [3] the average of scalar curvature is bounded and we know no examples of shrinking solitons with unbounded scalar curvature.

2. Preliminaries

Let p_0 be a fixed point of M. Let ρ be the distance function to p_0. Let $\delta(r)$ be a continuous function on $\mathbb{R}^+$ such that

(a) $\lim_{r\to\infty} \delta(r) = 0$;
(b) $\delta(r) > 0$;
(c) $\mathrm{Ric}(x) \geqslant -(n-1)\delta(r)$, if $\rho(x) \geqslant r$.

Note that $\delta(r)$ is a decreasing continuous function. The following lemma is standard:

Lemma 1. *With the assumption (1), we have*

$$\varlimsup_{x\to\infty} \Delta\rho \leqslant 0$$

in the sense of distribution.

Proof. Let g be a smooth function on $\mathbb{R}^+$ such that

$$\begin{cases} g''(r) - \delta(r)g(r) = 0, \\ g(0) = 0, \\ g'(0) = 1. \end{cases}$$

Then by the Laplacian comparison theorem, we have

$$\Delta\rho(x) \leqslant (n-1)g'\big(\rho(x)\big)/g\big(\rho(x)\big)$$

in the sense of distribution. The proof of the lemma will be completed if we can show that

$$\lim_{r\to\infty}\frac{g'(r)}{g(r)}=0.$$

By the definition of $g(r)$, we have $g(r)\geqslant 0$ and $g(r)$ is convex. Thus $g(r)\to+\infty$, as $r\to+\infty$. By the L'Hospital Principal, we have

$$\lim_{r\to+\infty}\frac{(g'(r))^2}{(g(r))^2}=\lim_{r\to+\infty}\frac{2g'(r)g''(r)}{2g(r)g'(r)}=\lim_{r\to+\infty}\delta(r)=0,$$

and this completes the proof of the lemma. □

Without loss of generality, for the rest of this paper, we assume that

$$\frac{g'(r)}{g(r)}\leqslant\delta(r)$$

for all $r>0$.

The following result is well-known:

Proposition 1. *There exists a C^∞ function $\tilde\rho$ on M such that*

(a) $|\tilde\rho-\rho|+|\nabla\tilde\rho-\nabla\rho|\leqslant\delta(\rho(x))$, *and*
(b) $\Delta\tilde\rho\leqslant 2\delta(\rho(x)-1)$

for any $x\in M$ with $\rho(x)>2$.

Proof. Let $\{U_i\}$ be a locally finite cover of M and let $\{\psi_i\}$ be the partition of unity subordinating to the cover. Let $\mathbf{x_i}=(x_i^1,\dots,x_i^n)$ be the local coordinates of U_i. Define $\rho_i=\rho|_{U_i}$.

Let $\xi(\mathbf{x})$ be a non-negative smooth function whose support is within the unit ball of $\mathbb{R}^n$. Assume that

$$\int_{\mathbb{R}^n}\xi(\mathbf{x})\,d\mathbf{x}.$$

Without loss of generality, we assume that all U_i are open subsets of the unit ball of $\mathbb{R}^n$ with coordinates $\mathbf{x_i}$. Then for any $\varepsilon>0$ small enough,

$$\rho_{i,\varepsilon}=\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\xi\left(\frac{\mathbf{x_i}-\mathbf{y_i}}{\varepsilon}\right)\rho_i(\mathbf{y_i})\,d\mathbf{y_i}$$

is a smooth function whose support is within U_i. Let

$$K(x) = \sum_i \big(|\Delta\psi_i| + 2|\nabla\psi_i|\big) + 1.$$

Then $K(x)$ is a smooth positive function on M. On each U_i, we choose ε_i small enough such that

$$\begin{aligned}
&\operatorname{supp}\{\rho_{i,\varepsilon_i}\} \subset U_i,\\
&|\rho_{i,\varepsilon_i} - \rho_i| \leqslant \delta\big(\rho(x)\big)/K(x),\\
&|\nabla\rho_{i,\varepsilon_i} - \nabla\rho_i| \leqslant \delta\big(\rho(x)\big)/K(x),\\
&\Delta\rho_{i,\varepsilon_i} \leqslant \delta\big(\rho(x) - 1\big), \quad \text{for } \rho(x) > 1.
\end{aligned} \tag{3}$$

Here Lemma 1 is used in the last inequality above. We define

$$\tilde\rho = \sum_i \psi_i \rho_{i,\varepsilon_i}.$$

The proof follows from the standard method: let's only prove (b) of the proposition. Since

$$\Delta\tilde\rho = \sum_i \Delta\psi_i \rho_{i,\varepsilon_i} + 2\nabla\psi_i \nabla\rho_{i,\varepsilon_i} + \psi_i \Delta\rho_{i,\varepsilon_i},$$

we have

$$\Delta\tilde\rho = \sum_i \Delta\psi_i (\rho_{i,\varepsilon_i} - \rho_i) + 2\nabla\psi_i (\nabla\rho_{i,\varepsilon_i} - \nabla\rho_i) + \psi_i \Delta\rho_{i,\varepsilon_i}.$$

By (3), we have

$$\Delta\tilde\rho \leqslant \delta\big(\rho(x)\big) + \delta\big(\rho(x) - 1\big),$$

and the proposition is proved. □

Let $p_0 \in M$, and let

$$V(r) = vol\big(B_{p_0}(r)\big)$$

for any $r > 0$.

The main result of this section is (cf. [5,8]).

Lemma 2. *Assume that* (1) *is valid. Then for any* $\varepsilon > 0$*, there is an* $R_1 > 0$ *such that for* $r > R_1$*, we have*

(a) *if* $vol(M) = +\infty$, *then*

$$\int_{B_{p_0}(r)\backslash B_{p_0}(R_1)} |\Delta\tilde{\rho}| \leqslant 2\varepsilon V(r) + 2vol\big(\partial B_{p_0}(R_1)\big);$$

(b) *if* $vol(M) < +\infty$, *then*

$$\int_{M\backslash B_{p_0}(r)} |\Delta\tilde{\rho}| \leqslant 2\varepsilon\big(vol(M) - V(r)\big) + 2vol\big(\partial B_{p_0}(r)\big).$$

Proof. By Proposition 1, for any $\varepsilon > 0$ small enough, we can find R_1 large enough such that

$$\Delta\tilde{\rho} < \varepsilon$$

for $x \in M\backslash B_{p_0}(R_1)$. Thus $|\Delta\tilde{\rho}| \leqslant 2\varepsilon - \Delta\tilde{\rho}$, and we have

$$\int_{B_{p_0}(R_2)\backslash B_{p_0}(r)} |\Delta\tilde{\rho}| \leqslant 2\varepsilon\big(V(R_2) - V(r)\big) - \int_{\partial B_{p_0}(R_2)} \frac{\partial\tilde{\rho}}{\partial n} + \int_{\partial B_{p_0}(r)} \frac{\partial\tilde{\rho}}{\partial n}$$

for any $R_2 > r > R_1$ by the Stokes' Theorem, where $\frac{\partial}{\partial n}$ is the derivative of the outward normal direction of the boundary $\partial B_{p_0}(r)$. By (3), we get

$$\int_{B_{p_0}(R_2)\backslash B_{p_0}(r)} |\Delta\tilde{\rho}| \leqslant 2\varepsilon\big(V(R_2) - V(r)\big) - \frac{1}{2}vol\big(\partial B_{p_0}(R_2)\big) + 2vol\big(\partial B_{p_0}(r)\big). \tag{4}$$

If $vol(M) = +\infty$, then we take $R_2 = r$, $r = R_1$ in the above inequality and we get (a).
If $vol(M) < +\infty$, taking $R_2 \to +\infty$ in (4), we get (b). □

3. Proof of Theorem 1

In this section we prove the following result which implies Theorem 1.

Theorem 5. *Let M be a complete non-compact manifold satisfying*

(1) *the volume of M grows uniformly sub-exponentially*;
(2) *the Ricci curvature of M has a lower bound*;
(3) *M satisfies the assertions in Lemma* 2.

Then the L^1 essential spectrum of the Laplacian is $[0, \infty)$.

Proof. We essentially follow Wang's proof [17]. First, using the characterization of the essential spectrum (cf. Donnelly [9, Proposition 2.2]), we only need to prove the following: for any $\lambda \in \mathbb{R}$ positive and any positive real numbers ε, μ, there exists a smooth function $\xi \neq 0$ such that

(1) $\operatorname{supp}(\xi) \subset M \backslash B_{p_0}(\mu)$ and is compact;
(2) $\|\Delta\xi + \lambda\xi\|_{L^1} < \varepsilon\|\xi\|_{L^1}$.

Let R, x, y be big positive real numbers. Assume that $y > x + 2R$ and $x > 2R > 2\mu + 4$. Define a cut-off function $\psi : \mathbb{R} \to \mathbb{R}$ such that

(1) $\operatorname{supp}\psi \subset [x/R - 1, y/R + 1]$;
(2) $\psi \equiv 1$ on $[x/R, y/R]$, $0 \leqslant \psi \leqslant 1$;
(3) $|\psi'| + |\psi''| < 10$.

For any given ε, μ and λ, let

$$\phi = \psi\left(\frac{\tilde{\rho}}{R}\right)e^{i\sqrt{\lambda}\tilde{\rho}}.$$

A straightforward computation shows that

$$\Delta\phi + \lambda\phi = \left(\frac{1}{R^2}\psi''|\nabla\tilde{\rho}|^2 + i\sqrt{\lambda}\frac{2}{R}\psi'|\nabla\tilde{\rho}|^2 + \left(i\sqrt{\lambda}\psi + \frac{\psi'}{R}\right)\Delta\tilde{\rho}\right)e^{i\sqrt{\lambda}\tilde{\rho}} + \lambda\phi\bigl(-|\nabla\tilde{\rho}|^2 + 1\bigr).$$

By Proposition 1,

$$|\Delta\phi + \lambda\phi| \leqslant \frac{C}{R} + C|\Delta\tilde{\rho}| + C\delta\bigl(\rho(x)\bigr),$$

where C is a constant depending only on λ. Thus we have

$$\|\Delta\phi + \lambda\phi\|_{L^1} \leqslant \left(\frac{C}{R} + C\delta(x - R)\right)\bigl(V(y + R) - V(x - R)\bigr) + C\int_{B_{p_0}(y+R) - B_{p_0}(x-R)} |\Delta\tilde{\rho}|. \tag{5}$$

Case 1: $vol(M) = +\infty$. By Lemma 2, if we choose ε/C small enough and R, x big enough and then assume y is large if necessary, we get

$$\|\Delta\phi + \lambda\phi\|_{L^1} \leqslant 4\varepsilon V(y + R). \tag{6}$$

Note that $\|\phi\|_{L^1} \geqslant V(y) - V(x)$. If we choose y big enough, then we have

$$\|\phi\|_{L^1} \geqslant \frac{1}{2}V(y). \tag{7}$$

We claim that there exists a sequence $y_k \to \infty$ such that $V(y_k + R) \leqslant 2V(y_k)$. If not, then for a fixed number y, we have

$$V(y + kR) > 2^k V(y)$$

for any $k \in \mathbb{Z}$ positive. On the other hand, by the uniform sub-exponentially growth of the volume, we have

$$2^k V(y) \leqslant V(y+kR) \leqslant C(\varepsilon)V(1)e^{\varepsilon(y+kR)}$$

for any k large and for any $\varepsilon > 0$. This is a contradiction if $\varepsilon R < \log 2$. Thus there is a y such that $V(y+R) \leqslant 2V(y)$, and thus by (6), (7), we have

$$\|\Delta\phi + \lambda\phi\|_{L^1} \leqslant 16\varepsilon\|\phi\|_{L^1}.$$

The case when M is of infinite volume is proved.

Case 2: $vol(M) < +\infty$. By Lemma 2,

$$\|\Delta\psi + \lambda\psi\|_{L^1} \leqslant C\left(\frac{1}{R} + 2\varepsilon + \delta(x-R)\right)\bigl(vol(M) - V(x-R)\bigr) + 2C vol\bigl(\partial B_{p_0}(x-R)\bigr).$$

Let $f(r) = vol(M) - V(r)$. Like above, we choose ε small and R, x big. Then

$$\|\Delta\phi + \lambda\phi\|_{L^1} \leqslant 4\varepsilon f(x-R) - 2Cf'(x-R)$$

for any x, y large enough. On the other hand, we always have

$$\|\phi\|_{L^1} \geqslant f(x) - f(y).$$

Since the volume is finite, we choose y large enough such that

$$\|\phi\|_{L^1} \geqslant \frac{1}{2}f(x).$$

Similar to the case of $vol(M) = +\infty$, the theorem is proved if the following statement is true: there is a sequence $x_k \to +\infty$ such that

$$2\varepsilon f(x_k - R) - Cf'(x_k - R) \leqslant 4\varepsilon f(x_k)$$

for all k.

If there doesn't exist such a sequence, then for x large enough, we have

$$2\varepsilon f(x-R) - Cf'(x-R) \geqslant 4\varepsilon f(x).$$

Replacing ε by ε/C, we have

$$2\varepsilon f(x-R) - f'(x-R) \geqslant 4\varepsilon f(x),$$

which is equivalent to

$$-\bigl(e^{-2\varepsilon x} f(x-R)\bigr)' \geqslant 4\varepsilon e^{-2\varepsilon x} f(x).$$

Integrating the expression from x to $x+R$, using the monotonicity of $f(x)$, we get

$$-e^{-2\varepsilon(x+R)}f(x)+e^{-2\varepsilon x}f(x-R)\geqslant 2e^{-2\varepsilon x}\left(1-e^{-2\varepsilon R}\right)f(x+R),$$

which implies

$$f(x-R)\geqslant 2\left(1-e^{-2\varepsilon R}\right)f(x+R).$$

Let R be big so that

$$2\left(1-e^{-2\varepsilon R}\right)>\frac{5}{4}.$$

Then we have

$$f(x-R)\geqslant \frac{5}{4}f(x+R)$$

for x large enough. Iterating the inequality, we get

$$f(x-R)\geqslant \left(\frac{5}{4}\right)^k f\left(x+(2k-1)R\right) \tag{8}$$

for all positive integer k.

On the other hand, we pick points p_k so that $dist(p_k,p_0)=x+(2k-1)R+1$. Then by the uniform sub-exponential growth of the volume, for any $\varepsilon>0$, since $B_{p_k}(1)\subset M\backslash B_{p_0}(x+(2k-1)R)$, we have

$$\begin{aligned} f\left(x+(2k-1)R\right)&\geqslant vol\left(B_{p_k}(1)\right)\\ &\geqslant \frac{1}{C(\varepsilon)}e^{-\varepsilon(x+(2k-1)R+2)}vol\left(B_{p_k}\left(x+(2k-1)R+2\right)\right).\end{aligned}$$

But $B_{p_k}(x+(2k-1)R+2)\supset B_{p_0}(1)$ so that there is a constant C, depending on ε and x only such that

$$f\left(x+(2k-1)R\right)\geqslant CV(1)e^{-2\varepsilon kR}.$$

Choosing ε small enough such that $2\varepsilon R<\log\frac{5}{4}$, we get a contradiction to (8) when $k\to\infty$. □

4. Gradient shrinking Ricci soliton

A complete Riemannian metric g_{ij} on a smooth manifold M is called a *gradient shrinking Ricci soliton*, if there exists a smooth function f on M^n such that the Ricci tensor R_{ij} of the metric g_{ij} is given by

$$R_{ij}+\nabla_i\nabla_j f=\rho g_{ij}$$

for some positive constant $\rho > 0$. The function f is called a *potential function*. Note that by scaling g_{ij} we can rewrite the soliton equation as

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij} \tag{9}$$

without loss of generality.

The following basic result on Ricci soliton is due to Hamilton (cf. [13, Theorem 20.1]).

Lemma 3. *Let* (M, g_{ij}, f) *be a complete gradient shrinking Ricci soliton satisfying* (9)*. Let* R *be the scalar curvature of* g_{ij}*. Then we have*

$$\nabla_i R = 2R_{ij} \nabla_j f,$$

and

$$R + |\nabla f|^2 - f = C_0$$

for some constant C_0*.*

By adding the constant C_0 to f, we can assume

$$R + |\nabla f|^2 - f = 0. \tag{2.1}$$

We fix this normalization of f throughout this paper.

Definition 1. We define the following notations:

(i) since $R \geqslant 0$ by Lemma 4 below, $f(x) \geqslant 0$. Let

$$\rho(x) = 2\sqrt{f(x)};$$

(ii) for any $r > 0$, let

$$D(r) = \{x \in M\colon\ \rho(x) < r\} \quad \text{and} \quad V(r) = \int_{D(r)} dV;$$

(iii) for any $r > 0$, let

$$\chi(r) = \int_{D(r)} R\, dV.$$

The function $\rho(x)$ is similar to the distance function in many ways. For example, by [3, Theorem 20.1], we have

$$r(x) - c \leqslant \rho(x) \leqslant r(x) + c,$$

where c is a constant and $r(x)$ is the distance function to a fixed reference point.

We summarize some useful results of gradient shrinking Ricci soliton in the following lemma without proof:

Lemma 4. *Let (M, g_{ij}, f) be a complete non-compact gradient shrinking Ricci soliton of dimension n. Then*

(1) *The scalar curvature $R \geqslant 0$ (B.-L. Chen* [6], *see also Proposition* 5.5 *in* [2]).
(2) *The volume is of Euclidean growth. That is, there is a constant C such that $V(r) \leqslant Cr^n$ (Theorem* 2 *of* [3]).
(3) *We have*

$$nV(r) - 2\chi(r) = rV'(r) - \frac{4}{r}\chi'(r) \geqslant 0.$$

In particular, the average scalar curvature over $D(r)$ is bounded by $\frac{n}{2}$, i.e. $\chi(r) \leqslant \frac{n}{2}V(r)$ (Lemma 3.1 *in* [3]).

(4) *We have*

$$\nabla\rho = \frac{\nabla f}{\sqrt{f}} \quad \text{and} \quad |\nabla\rho|^2 = \frac{|\nabla f|^2}{f} = 1 - \frac{R}{f} \leqslant 1.$$

Using the above lemma, we prove the following result which is similar to Lemma 2.

Lemma 5. *Let (M, g_{ij}, f) be a complete non-compact gradient shrinking Ricci soliton of dimension n. Then for any two positive numbers x, r with $x > r$, we have*

$$\int_{D(x)\setminus D(r)} |\Delta\rho| \leqslant \frac{2n}{r}\big[V(x) - V(r)\big] + V'(r),$$

$$\int_{D(x)\setminus D(r)} |\Delta\rho|^2 \leqslant \left(\frac{n^2}{r^2} + 2n \max_{\rho\in[r,x]} \frac{R}{\rho^2}\right)V(x).$$

Proof. Since $R + \Delta f = \frac{n}{2}$ and $R \geqslant 0$, we have

$$\Delta\rho = \frac{\Delta f}{\sqrt{f}} - \frac{1}{2}\frac{|\nabla f|^2}{(\sqrt{f})^3} \leqslant \frac{\Delta f}{\sqrt{f}} \leqslant \frac{n}{\rho}. \tag{10}$$

By the Co–Area formula (cf. [15]), we have,

$$V(r) = \int_0^r ds \int_{\partial D(s)} \frac{1}{|\nabla\rho|}\, dA.$$

Therefore,

$$V'(r)=\int_{\partial D(r)}\frac{1}{|\nabla\rho|}\,dA=\frac{r}{2}\int_{\partial D(r)}\frac{1}{|\nabla f|}\,dA.$$

Thus we have

$$\begin{aligned}\int_{D(x)\setminus D(r)}|\Delta\rho| &\leqslant 2\int_{D(x)\setminus D(r)}\frac{n}{\rho}-\int_{D(x)\setminus D(r)}\Delta\rho\\ &=2\int_{D(x)\setminus D(r)}\frac{n}{\rho}-\int_{\partial D(x)}\frac{\partial\rho}{\partial\nu}+\int_{\partial D(r)}\frac{\partial\rho}{\partial\nu}\\ &\leqslant 2\int_{D(x)\setminus D(r)}\frac{n}{\rho}+\int_{\partial D(r)}\frac{1}{|\nabla\rho|}\\ &\leqslant\frac{2n}{r}\big[V(x)-V(r)\big]+V'(r), \qquad (11)\end{aligned}$$

where $\nu=\frac{\nabla\rho}{|\nabla\rho|}$ is the outward normal vector to ∂D. This completes the proof of the first part of the lemma.

Now we prove the second part of the lemma. From (10), we have

$$\begin{aligned}\Delta\rho &=\frac{2\Delta f}{\rho}-\frac{|\nabla\rho|^2}{\rho}\\ &=\frac{2}{\rho}\left(\frac{n}{2}-R\right)-\frac{1}{\rho}\left(1-\frac{R}{f}\right)\\ &=\frac{n-1}{\rho}-\frac{2R}{\rho}+\frac{4R}{\rho^2}\\ &\geqslant-\frac{2R}{\rho}. \qquad (12)\end{aligned}$$

Then

$$\begin{aligned}\int_{D(x)\setminus D(r)}|\Delta\rho|^2 &\leqslant\int_{D(x)\setminus D(r)}\frac{n^2}{\rho^2}+\int_{D(x)\setminus D(r)}\frac{4R^2}{\rho^2}\\ &\leqslant\frac{n^2}{r^2}\big[V(x)-V(r)\big]+\left(\max_{\rho\in[r,x]}\frac{4R}{\rho^2}\right)\chi(x)\\ &\leqslant\left(\frac{n^2}{r^2}+2n\max_{\rho\in[r,x]}\frac{R}{\rho^2}\right)V(x), \qquad (13)\end{aligned}$$

where in the last inequality above we used (3) of Lemma 4. □

Now we are ready to prove

Theorem 6. *Let (M, g_{ij}, f) be a complete gradient shrinking Ricci soliton. Then the L^1 essential spectrum contains* $[0, +\infty)$.

Proof. Similar to that of Theorem 1, we only need to prove the following: for any $\lambda \in \mathbb{R}$ positive and any positive real numbers ε, μ, there exists a smooth function $\xi \neq 0$ such that

(1) $\operatorname{supp}(\xi) \subset M \backslash B_{p_0}(\mu)$ and is compact;
(2) $\|\Delta\xi + \lambda\xi\|_{L^1} < \varepsilon\|\xi\|_{L^1}$.

Let $a \geqslant 2$ be a positive number. Define a cut-off function $\psi : \mathbb{R} \to \mathbb{R}$ such that

(1) $\operatorname{supp}\psi \subset [0, a+2]$;
(2) $\psi \equiv 1$ on $[1, a+1]$, $0 \leqslant \psi \leqslant 1$;
(3) $|\psi'| + |\psi''| < 10$.

For any given $b \geqslant 2 + \mu$, $l \geqslant 2$ and $\lambda > 0$, let

$$\phi = \psi\left(\frac{\rho - b}{l}\right)e^{i\sqrt{\lambda}\rho}. \tag{14}$$

A straightforward computation shows that

$$\Delta\phi + \lambda\phi = \left(\frac{\psi''}{l^2}|\nabla\rho|^2 + i\sqrt{\lambda}\frac{2\psi'}{l}|\nabla\rho|^2\right)e^{i\sqrt{\lambda}\rho} + \left(i\sqrt{\lambda}\psi + \frac{\psi'}{l}\right)\Delta\rho e^{i\sqrt{\lambda}\rho} + \lambda\phi\left(-|\nabla\rho|^2 + 1\right).$$

By Lemma 4, we have

$$|\Delta\phi + \lambda\phi| \leqslant \frac{C}{l} + C|\Delta\rho| + \lambda\frac{R}{f}, \tag{15}$$

where C is a constant depending only on λ. By Lemma 5, we have

$$\begin{aligned}\|\Delta\phi + \lambda\phi\|_{L^1} &\leqslant \frac{C}{l}\left[V\left(b + (a+2)l\right) - V(b)\right] + C\int\limits_{D(b+(a+2)l)\backslash D(b)} |\Delta\rho| \\ &\quad + \lambda\int\limits_{D(b+(a+2)l)\backslash D(b)} \frac{4R}{\rho^2} \\ &\leqslant \left(\frac{C}{l} + \frac{2nC}{b}\right)\left[V\left(b + (a+2)l\right) - V(b)\right] + CV'(b) \\ &\quad + \frac{4\lambda}{b^2}\int\limits_{D(b+(a+2)l)\backslash D(b)} R\end{aligned}$$

$$\leqslant \left(\frac{C}{l} + \frac{2nC}{b}\right)\big[V\big(b+(a+2)l\big) - V(b)\big] + CV'(b) + \frac{4\lambda}{b^2}\chi\big(b+(a+2)l\big). \tag{16}$$

From Lemma 4, we can choose l and b large enough so that

$$\|\Delta\phi + \lambda\phi\|_{L^1} \leqslant \varepsilon V\big(b+(a+2)l\big) + CV'(b).$$

By a result of Cao–Zhu (cf. [1, Theorem 3.1]), the volume of M is infinite. Therefore we can fix b and let l be large enough so that

$$\|\Delta\phi + \lambda\phi\|_{L^1} \leqslant 2\varepsilon V\big(b+(a+2)l\big). \tag{17}$$

On the other hand, note that $\|\phi\|_{L^1} \geqslant V(b+(a+1)l) - V(b+l)$. If we choose a large enough, then we have

$$\|\phi\|_{L^1} \geqslant \frac{1}{2}V\big(b+(a+1)l\big). \tag{18}$$

We claim that there exists a sequence $a_k \to \infty$ such that $V(b+(a_{k+1}+2)l) \leqslant 2V(b+(a_k+1)l)$. Otherwise for some fixed number a, we have

$$V\big(b+(a+k)l\big) > 2^{k-1}V\big(b+(a+1)l\big)$$

for any $k \geqslant 2$, which contradicts to the fact that the volume is of Euclidean growth (Lemma 4). Let a be a constant large enough such that $V(b+(a+2)l) \leqslant 2V(b+(a+1)l)$. By (17), (18), we have

$$\|\Delta\phi + \lambda\phi\|_{L^1} \leqslant 8\varepsilon\|\phi\|_{L^1},$$

and the proof is complete. □

Proof of Theorem 4. The proof is similar to that of Theorem 6: it suffices to prove the following: for any $\lambda \in \mathbb{R}$ positive and any positive real numbers ε, μ, there exists a smooth function $\xi \neq 0$ such that

(1) $\operatorname{supp}(\xi) \subset M \backslash B_{p_0}(\mu)$ and is compact;
(2) $\|\Delta\xi + \lambda\xi\|_{L^2} < \varepsilon\|\xi\|_{L^2}$.

Let $a \geqslant 2$ be a positive number. For any given $b \geqslant 2+\mu$, $l \geqslant 2$ and $\lambda > 0$, let ϕ be defined as in (14). By (15), we have

$$|\Delta\phi + \lambda\phi|^2 \leqslant \frac{C}{l^2} + C|\Delta\rho|^2 + C\frac{R^2}{f^2},$$

where C is a constant depending only on λ. Thus we have

$$
\begin{aligned}
\|\Delta\phi+\lambda\phi\|_{L^2}^2 &\leqslant \frac{C}{l^2}\big[V\big(b+(a+2)l\big)-V(b)\big] \\
&\quad + C\int_{D(b+(a+2)l)\setminus D(b)} |\Delta\rho|^2 + C\int_{D(b+(a+2)l)\setminus D(b)} \frac{16R^2}{\rho^4} \\
&\leqslant C\left(\frac{1}{l^2}+\frac{n^2}{b^2}+2n\max_{\rho\in[b,b+(a+2)l]}\frac{R}{\rho^2}\right)V\big(b+(a+2)l\big) \\
&\quad + \frac{4C}{b^2}\int_{D(b+(a+2)l)\setminus D(b)} R \\
&\leqslant C\left(\frac{1}{l^2}+\frac{n^2}{b^2}+2n\max_{\rho\in[b,b+(a+2)l]}\frac{R}{\rho^2}\right)V\big(b+(a+2)l\big) \\
&\quad + \frac{4C}{b^2}\chi\big(b+(a+2)l\big),
\end{aligned}
\tag{19}
$$

where we used Lemma 5 and the fact $R \leqslant f = \frac{1}{4}\rho^2$. From Lemma 4, we can choose l and b large enough so that

$$\|\Delta\phi+\lambda\phi\|_{L^2}^2 \leqslant \varepsilon V\big(b+(a+2)l\big).$$

Note that $\|\phi\|_{L^2}^2 \geqslant V(b+(a+1)l) - V(b+l)$. If we choose a big enough, then we have

$$\|\phi\|_{L^2}^2 \geqslant \frac{1}{2}V\big(b+(a+1)l\big). \tag{20}$$

Since the volume of M is of Euclidean growth, there is a positive number $a > 0$ such that

$$V\big(b+(a+1)l\big) \geqslant \frac{1}{2}V\big(b+(a+2)l\big),$$

and therefore we have

$$\|\Delta\phi+\lambda\phi\|_{L^2}^2 \leqslant 4\varepsilon\|\phi\|_{L^2}^2.$$

The theorem is proved. □

5. Further discussions

As can be seen clearly in the above context, the key of the proof is the L^1 boundedness of $\Delta\rho$. The Laplacian comparison theorem implies the volume comparison theorem. The converse is, in general, not true. On the other hand, the formula[1]

[1] In the sense of distribution.

$$\int_{B(R)\setminus B(r)} \Delta\rho = vol\big(\partial B(R)\big) - vol\big(\partial B(r)\big)$$

clearly shows that volume growth restriction gives the bound of the integral of $\Delta\rho$. Based on this observation, we make the following conjecture

Conjecture 1. *Let M be a complete non-compact Riemannian manifold whose Ricci curvature has a lower bound. Assume that the volume of M grows uniformly sub-exponentially. Then the L^p essential spectrum of M is $[0, +\infty)$ for any $p \in [1, +\infty]$.*

Such a conjecture, if true, would give a complete answer to the computation of the essential spectrum of non-compact manifold with uniform sub-exponential volume growth.

The parallel Sturm's theorem on p-forms was proved by Charalambous [4]. Using that, a similar result of Theorem 1 also holds for p-forms under certain conditions.

References

[1] H.D. Cao, Geometry of complete gradient shrinking solitons, arXiv:0903.3927v2.

[2] H.D. Cao, Recent Progress on Ricci Solitons, Adv. Lectures Math., vol. 11, 2009, pp. 1–38.

[3] H.D. Cao, D. Zhou, On complete gradient shrinking Ricci solitons, J. Differential Geom. 85 (2010) 175–185.

[4] Nelia Charalambous, On the L^p independence of the spectrum of the Hodge Laplacian on non-compact manifolds, J. Funct. Anal. 224 (1) (2005) 22–48, MR2139103 (2006e:58044).

[5] Jeff Cheeger, Tobias H. Colding, Lower bounds on Ricci curvature and the almost rigidity of warped products, Ann. of Math. (2) 144 (1) (1996) 189–237, doi:10.2307/2118589, MR1405949 (97h:53038).

[6] Bing-Long Chen, Strong uniqueness of the Ricci flow, J. Differential Geom. 82 (2) (2009) 363–382, MR2520796.

[7] Zhi Hua Chen, Zhi Qin Lu, Essential spectrum of complete Riemannian manifolds, Sci. China Ser. A 35 (3) (1992) 276–282, MR1183713 (93k:58221).

[8] Tobias H. Colding, Ricci curvature and volume convergence, Ann. of Math. (2) 145 (3) (1997) 477–501, doi:10.2307/2951841, MR1454700 (98d:53050).

[9] Harold Donnelly, On the essential spectrum of a complete Riemannian manifold, Topology 20 (1) (1981) 1–14, doi:10.1016/0040-9383(81)90012-4, MR592568 (81j:58081).

[10] Harold Donnelly, Exhaustion functions and the spectrum of Riemannian manifolds, Indiana Univ. Math. J. 46 (2) (1997) 505–527, doi:10.1512/iumj.1997.46.1338, MR1481601 (99b:58230).

[11] José F. Escobar, On the spectrum of the Laplacian on complete Riemannian manifolds, Comm. Partial Differential Equations 11 (1) (1986) 63–85, doi:10.1080/03605308608820418, MR814547 (87a:58155).

[12] José F. Escobar, Alexandre Freire, The spectrum of the Laplacian of manifolds of positive curvature, Duke Math. J. 65 (1) (1992) 1–21, doi:10.1215/S0012-7094-92-06501-X, MR1148983 (93d:58174).

[13] Richard S. Hamilton, The formation of singularities in the Ricci flow, in: Surveys in Differential Geometry, vol. II, Cambridge, MA, 1993, Int. Press, Cambridge, MA, 1995, pp. 7–136, MR1375255 (97e:53075).

[14] Jia Yu Li, Spectrum of the Laplacian on a complete Riemannian manifold with nonnegative Ricci curvature which possess a pole, J. Math. Soc. Japan 46 (2) (1994) 213–216, doi:10.2969/jmsj/04620213, MR1264938 (95g:58248).

[15] R. Schoen, S.-T. Yau, Lectures on differential geometry, in: Conference Proceedings and Lecture Notes in Geometry and Topology, I, Int. Press, Cambridge, MA, 1994, Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu; translated from the Chinese by Ding and S.Y. Cheng; Preface translated from the Chinese by Kaising Tso, MR1333601 (97d:53001).

[16] Karl-Theodor Sturm, On the L^p-spectrum of uniformly elliptic operators on Riemannian manifolds, J. Funct. Anal. 118 (2) (1993) 442–453, doi:10.1006/jfan.1993.1150, MR1250269 (94m:58227).

[17] Jiaping Wang, The spectrum of the Laplacian on a manifold of nonnegative Ricci curvature, Math. Res. Lett. 4 (4) (1997) 473–479, MR1470419 (98h:58194).

[18] De Tang Zhou, Essential spectrum of the Laplacian on manifolds of nonnegative curvature, Int. Math. Res. Not. 5 (1994), doi:10.1155/S1073792894000231, 209 ff., approx. 6 pp. (electronic) MR1270134 (95g:58250).