Discretized Tikhonov–Phillips regularization for a naturally linearized parameter identification problem

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Abstract

The problem of identification of the diffusion coefficient in the partial differential equation is considered. We discuss a natural linearization of this problem and application of discretized Tikhonov–Phillips regularization to its linear version. Using recent results of regularization theory, we propose a strategy for the choice of regularization and discretization parameters which automatically adapts to unknown smoothness of the coefficient. The estimation of the accuracy will be given and various numerical test supporting theoretical results will be presented.

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1. Introduction

In this paper, we are interested in recovering the (unknown) diffusion coefficient $a = a(x)$ from noisy measurements $u^\delta$ of the solution $u$ of boundary value problem

$$
-\nabla(a\nabla u) = f \quad \text{in } \Omega,
$$

$$
u = g \quad \text{on } \partial\Omega.
$$

(1.1)

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Here $\Omega$ is a convex domain with Lipschitz boundary, $f \in L^2(\Omega)$, $g \in H^2(\partial\Omega)$, and for some fixed noise level $\delta$ we have
\[ \|u^\delta - u\|_{L^2(\Omega)} \leq \delta. \] (1.2)

This inverse problem is extensively discussed in the literature as a model problem for parameter identification (e.g. [1,3–7,15]).

It is usually treated as a nonlinear operator equation
\[ F(a) = u, \] (1.3)
where $F : L^\infty(\Omega) \to H^1(\Omega)$ is a nonlinear coefficient-to-solution map. For example, in [6], projection-regularized Newton (iteration) method has been applied to (1.3) under the assumption that in one-dimensional case $a \in H^1(0,1) : a \geq \underline{a} > 0$, and the boundary values $a(0), a(1)$ are known. Moreover, suboptimal convergence rate with respect to noise level $\delta$ has been proven in case when the real smoothness of $a$ is unknown.

As an alternative, in [4] the so-called equation error method has been suggested for parameter identification. But the results obtained there were under the assumption that both the exact solution and the noisy data satisfied rather strong smoothness condition.

New approach to above parameter identification problem has been proposed in [7]. Using an initial guess $a_0$, the authors of [7] have represented (1.1) as follows:
\[ -\nabla(a_0 \nabla(u - u_0)) = \nabla((a - a_0) \nabla u) \quad \text{in } \Omega, \]
\[ u - u_0 = 0 \quad \text{on } \partial\Omega, \] (1.4)
where $u_0$ solves
\[ -\nabla(a_0 \nabla u_0) = f \quad \text{in } \Omega, \]
\[ u_0 = g \quad \text{on } \partial\Omega. \] (1.5)

Then the following operator equation is linear:
\[ \bar{A}s = \bar{r}, \] (1.6)
where $s = a - a_0$ is the difference between unknown parameter $a$ and the initial guess $a_0$, $\bar{r} = u - u_0$, and the operator $\bar{A}$ maps $s$ to the solution $z$ of
\[ -\nabla(a_0 \nabla z) = \nabla(s \nabla u) \quad \text{in } \Omega, \]
\[ z = 0 \quad \text{on } \partial\Omega. \] (1.7)

Replacing $u$ by a smoothed version $u^\delta_{\text{sm}}$ of $u^\delta$ such that $\nabla u^\delta_{\text{sm}} \in L^\infty$ and the noise level is maintained as $\|u^\delta_{\text{sm}} - u\|_{L^2(\Omega)} \leq C_{\text{sm}} \delta$, we switch to the equation
\[ As = r^\delta, \] (1.8)
with perturbed operator $A = A(u^\delta_{\text{sm}})$ and noisy right-hand side $r^\delta = u^\delta_{\text{sm}} - u_0$, where $A$ maps $s$ to the solution $z$ of the problem
\[ -\nabla(a_0 \nabla z) = \nabla(s \nabla u^\delta_{\text{sm}}) \quad \text{in } \Omega, \]
\[ z = 0 \quad \text{on } \partial\Omega. \] (1.9)
As long as \( \nabla u \) and \( \nabla u^{\delta}_{sm} \) belong to \( L^\infty(\Omega) \), and \( s \) belongs to \( L^2(\Omega) \), we can always seek for the solution \( z \) of \((1.7)\) and \((1.9)\) in \( H_0^1(\Omega) \), which leads to the compactness of the operators \( \bar{A} \) and \( A \), therefore makes \((1.8)\) ill-posed.

Considering \( \bar{A} \) and \( A \) as the operators from \( L^2(\Omega) \) into \( L^2(\Omega) \), we will rely on the estimate

\[
\| \bar{A} - A \| \leq \varepsilon. \tag{1.10}
\]

If \( \{\lambda_k\} \) and \( \{u_k\} \) are, respectively, eigenvalues and orthogonal eigenfunctions of the differential operator \( \nabla(a_0\nabla(\cdot)) \) with zero boundary condition on \( \partial \Omega \), i.e. \( \nabla(a_0\nabla u_k) = \lambda_k u_k \), \( u_k = 0 \) on \( \partial \Omega \), then

\[
(\bar{A} - A)s = \sum_k \lambda_k^{-1} u_k \langle u_k, \nabla(s\nabla(u - u^{\delta}_{sm})) \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is a standard inner product in \( L^2(\Omega) \). Now it is clear that \( \varepsilon \) depends on the approximation of \( \nabla u \) by \( \nabla u^{\delta}_{sm} \). If, for example, \( a_0 \) is such that

\[
c^2(a_0, \Omega) := \sum_k \lambda_k^{-2} \| \nabla u_k \|_{L^\infty(\Omega)}^2 < \infty, \tag{1.11}
\]

then

\[
\| (\bar{A} - A)s \|_{L^2(\Omega)}^2 = \sum_k \lambda_k^{-2} \langle u_k, \nabla(s\nabla(u - u^{\delta}_{sm})) \rangle^2
\]

\[
= \sum_k \lambda_k^{-2} \| \nabla u_k \|^{2} \| s \|_{L^2(\Omega)}^2 \| \nabla(u - u^{\delta}_{sm}) \|_{L^2(\Omega)}^2
\]

and

\[
\varepsilon \leq c(a_0, \Omega) \| \nabla(u - u^{\delta}_{sm}) \|_{L^2(\Omega)}.
\]

Note that \((1.11)\) holds, in particular, for \( \Omega = [0, 2\pi] \), \( a_0 \equiv 1 \), \( \nabla(a_0\nabla u) = u'' \), because in this case \( u_k(x) = \pi^{-1/2} \sin kx \), \( \lambda_k = -k^2 \), \( \| u_k' \|_{L^\infty[0,2\pi]} = k\pi^{-1/2} \).

In [7] a smoothed approximation \( u^{\delta}_{sm} \) has been constructed in such a way that \( \| \nabla(u - u^{\delta}_{sm}) \|_{L^2(\Omega)} \leq c\sqrt{\delta} \) under the additional assumption that \( u \) is smooth enough and \( a \) is bounded away from zero. It means that in this case one can take \( \varepsilon = c\sqrt{\delta} \). At the same time, if above-mentioned assumptions are not satisfied, or some other approximation \( u^{\delta}_{sm} \) is used, then the relation between \( \varepsilon \) and \( \delta \) changes. Therefore, in the sequel we will assume only that \( \varepsilon \) is known and it is much larger than \( \delta \), i.e., \( \varepsilon \gg \delta \). As a result, \( \| \bar{r} - r^\delta \|_{L^2(\Omega)} \leq C_{sm}\delta < \varepsilon \) also holds true.

The authors of [7] have mentioned that the linear equation \((1.8)\) can be regularized by projection. In this case approximate solution would have a form

\[
s = \sum \alpha_j A^*\Phi_j,
\]

where \( \{\Phi_j\} \) is the basis used in projection scheme. In general, the operator \( A^* \) has a quite complicated form, and to avoid this difficulty the authors of [7] study a weaker formulation...
of (1.8) that can be considered as a modified form of equation error method [4]. As a result, the convergence rate and the choice of the regularization and discretization parameter are justified in [7], only under a priori assumption concerning the smoothness of unknown diffusion coefficient.

In this paper, we linearize the problem in the same way as it has been suggested in [7], and then apply recent results [10,11] on the regularization of projection methods. The smoothness assumption is given in the form of general source condition, and applying adaptive regularization/discretization strategy [10,11] we do not assume that this source condition is a priori known. Moreover, since the operator $A$ depends on noisy data, it is natural to give out the source condition in terms of the operator $\tilde{A}$ instead of $A$, which brings some new arguments. In addition, our approximate solution has a form of a linear combination of the basis functions $\Phi_i$, instead of $A^*\Phi_i$, which can simplify a numerical scheme compared to [5,6].

After this introduction, the paper proceeds as follows: in Section 2, we show the process of the regularization and estimate its error; in Section 3, we briefly describe an adaptive strategy for the choice of regularization/discretization parameter; in Section 4 two numerical examples supporting the theoretical results are presented. It is interesting to note that in one of them the standard assumption that $a$ is bounded away from zero is violated. Nevertheless, our algorithm recovers this coefficient quite accurately.

2. Tikhonov–Phillips regularization and the estimation of the accuracy

2.1. Tikhonov–Phillips regularization

We have linearized the initial identification problem into

$$r^\delta = As + \xi_\delta,$$

where $A : L^2(\Omega) \rightarrow L^2(\Omega)$ is a compact operator defined as (1.8) and (1.9), $\xi_\delta$ is a uniformly bounded noise, and we know $\|\xi_\delta\|_{L^2(\Omega)} \leq \varepsilon$.

Assume that the solution $s = a - a_0$ meets a so-called source condition, i.e., it is taken from the set

$$\tilde{A}_\phi := \{ s \in X, s = \phi(\tilde{A}^*\tilde{A})v, \|v\| \leq R \},$$

where the function $\phi$ is some index function on the spectrum of $\tilde{A}^*\tilde{A}$, which will be described in details later. Then we define a regularized approximation for $s$ as $g_\varepsilon(B^*B)B^*r^\delta$, where $g_\varepsilon$ is a regularization method given by the operator function $g_\varepsilon(B^*B)$, $B = AP$, $P = P_n$ is the orthogonal projector from $L^2(\Omega)$ onto $n$-dimensional space $V_n$ of piece-wise linear continuous functions corresponding to triangulation of the domain $\Omega$ with mesh size $h_n$. Let $\{\Phi_i\}_{i=1}^n$ be some basis of $V_n$. Here, we consider Tikhonov–Phillips regularization determined by the function $g_\varepsilon(\lambda) = 1/(\lambda + \varepsilon)$. Applying it to the operator $B = B_n = AP_n$, we approximate the solution $s$ of Eq. (1.8) by the solution $s_{x,\delta,n}$ of regularized equation

$$\varepsilon s + P_nA^*A P_n s = P_nA^*r^\delta.$$
In other words, $s_{x,\delta,n} = g_2(B^*B)B^*r^\delta = \sum_{i=1}^n \gamma_i \Phi_i$, where $\gamma = \{\gamma_i\}_{i=1}^n$ is the solution of linear system
\[(M + \alpha G\Phi)\gamma = Y_\delta,\tag{2.1}\]
with the following matrix and vector:
\[
G\Phi := ((\Phi_i, \Phi_j))_{i,j=1,...,n},
\]
\[
M := ((A\Phi_i, A\Phi_j))_{i,j=1,...,n},
\]
\[
Y_\delta := ((A\Phi_i, r^\delta))_{i=1,...,n}.\tag{2.2}
\]

We would like to note that the adjoint operator $A^*$ of (1.9) is not involved in the construction of $s_{x,\delta,n}$. As to the function $A\Phi_i$, they are the solutions of (1.9), where $s$ is replaced by $\Phi_i$, $i = 1, \ldots, n$. Given a basis $\{\Phi_i\}$, the functions $A\Phi_i$ can theoretically be computed exactly or precomputed numerically in advance. Observe also that we do not need each function $A\Phi_i$ in explicit form, but only its inner products as in $M$ and $Y_\delta$, which can be computed much more accurately than $A\Phi_i$ itself. In any way, the computation error in $M$ and $Y_\delta$ can be made much smaller than observation error $\delta$.

2.2. Source condition for index functions

Recall the properties of the function $g_2(\lambda) = 1/(\alpha + \lambda)$ associated with Tikhonov–Phillips regularization. It is well known that
\[
\sup_{\lambda > 0} \sqrt{\lambda}|g_2(\lambda)| \leq \frac{1}{2\sqrt{2}},\tag{2.3}
\]
and
\[
\sup_{\lambda > 0} \lambda^p|1 - \lambda g_2(\lambda)| \leq 2^p,\tag{2.4}
\]
holds only for $0 \leq p \leq 1$.

To proceed further we should specify the assumptions concerning index function $\phi$. From [10] it follows that when dealing with the discretized Tikhonov–Phillips scheme, it is convenient to assume that the smoothness index function $\phi$ is operator monotone (increasing), because this assumption covers all types of smoothness studied so far in the theory of Tikhonov–Phillips method. Recall that the function $\phi$ is operator monotone on $[0, b]$, if for any pair of self-adjoint operators $U, V$ with spectra in $[0, b]$, such that $U \leq V$, we have $\phi(U) \leq \phi(V)$ (i.e. $\forall f \in X$, $\langle \phi(U)f, f \rangle_X \leq \langle \phi(V)f, f \rangle_X$).

Proposition 2.1. If $\phi$ is operator monotone on $[0, b]$ and $\phi(0) = 0$, then
\[
\sup_{0 < \lambda \leq b} |1 - \lambda g_2(\lambda)| \phi(\lambda) \leq c\phi(x),\tag{2.5}
\]
where the constant $c$ does not depend on $x$. Moreover, for any pair of self-adjoint operators $U, V$ with spectra on $[0, b]$

$$
\|\phi(U) - \phi(V)\| \leq d\phi(\|U - V\|) \quad (2.6)
$$

and $d$ depends only on $\phi$.

**Proof.** Since $\phi$ is operator monotone on $[0, b]$ and $\phi(0) = 0$, then as in [9], such $\phi$ can be represented as a sum of two non-negative functions $\phi = \phi_0 + \phi_1$, where $\phi_0$ is a concave function, $\phi_1$ meets Lipschitz condition with Lipschitz constant $c_1$, and $\phi_0(0) = \phi_1(0) = 0$. Then $\phi_0(\lambda)/\lambda \leq \phi_0(x)/x$ whenever $0 < x \leq \lambda < b$. Thus, for $\lambda \leq x$ we have

$$
\phi(\lambda)/\lambda = (\phi_0(\lambda) + \phi_1(\lambda))/\lambda \leq \phi_0(\lambda)/\lambda + c_1.
$$

Now, put $c := (c_1 b/\phi_0(b) + 1)$, we conclude

$$
\phi(\lambda)/\lambda \leq (c_1 b/\phi_0(b) + 1)\phi_0(x)/x \leq c\phi(x)/x.
$$

That is

$$
\lambda/\phi(\lambda) \geq c^{-1}x/\phi(x), \text{ whenever } 0 < x \leq \lambda < b.
$$

Then (2.5) immediately follows from (2.4) and [11, Definition 2, Proposition 3].

From [9, Theorem 2] we know that for $\phi$ meeting the condition of our proposition, and for any pair of self-adjoint operators $U, V$ with spectra on $[0, b]$

$$
\|\phi(U) - \phi(V)\| \leq \phi(\|U - V\|) + C\|U - V\|, \quad (2.7)
$$

where the constant $C$ depends only on $\phi$. As above from monotonicity of $\phi_0$ it follows that for any given constant $C$, there exists another constant $C' = bC/\phi_0(b)$ such that for any $t \in [0, b]$, $Ct \leq C'\phi_0(t)$. Thus,

$$
C\|U - V\| \leq C'\phi_0(\|U - V\|) \leq C'\phi(\|U - V\|). \quad (2.8)
$$

Now (2.7) and (2.8) lead to (2.6), where we can take $d = 1 + C'$. □

In the sequel, we will assume that index function $\phi$ is operator monotone on $[0, b]$, $b > \|\tilde{A}\|^2$, because such interval contains the spectrum of operator $\tilde{A}^*\tilde{A}$. Therefore, we define the following function class

$$
\mathcal{F} := \{\phi, \phi : (0, b) \to \mathbb{R}_+, \phi(0) = 0, \phi \text{ is operator monotone}\}.
$$

Then as [10] we assume more specifically, either $\phi^2(\lambda)$ to be concave, or $\phi(\lambda) \leq c\sqrt{\lambda}$. The classes of such operator monotone functions will be denoted by $\mathcal{F}_0$ and $\mathcal{F}_{1/2}$, respectively. Observe that up to a certain extent these classes complement each other, because for any $\phi \in \mathcal{F}_0$, $\phi(0) = 0$, $\phi^2(\lambda) \geq \phi^2(b)\lambda/b = c\lambda$, and thus $\phi(\lambda) \geq c\sqrt{\lambda}$. Note that the well-known function classes relative to a Tikhonov–Phillips regularization of ill-posed
operator equations

\[ \phi(\lambda) = \lambda^\mu, \lambda > 0 \quad \text{for} \ 0 < \mu < 1, \]
\[ \phi(\lambda) = \log^{-p}(1/\lambda), 0 < \lambda < 1 \quad \text{for} \ p > 0, \]

are contained in \( \mathcal{F}_0 \cup \mathcal{F}_{1/2} \).

For the sake of simplicity we normalize index functions \( \phi \) in such a way that \( \phi(b) = \sqrt{b} \).

Namely,

\[ \mathcal{F}_0 := \{ \phi \in \mathcal{F}, \phi(b) = \sqrt{b}, \phi^2 \ \text{is concave} \}, \]
\[ \mathcal{F}_{1/2} := \{ \phi \in \mathcal{F}, \phi(b) = \sqrt{b}, \phi(\lambda) \leq \sqrt{\lambda} \}. \]

### 2.3. Estimation of accuracy

The following proposition was proven in [10].

**Proposition 2.2.** Let \( \phi(\lambda) \) be any increasing index function from \( \mathcal{F}_0 \cup \mathcal{F}_{1/2} \). Then for the orthogonal projector \( P \)

\[ \| P \phi(\bar{A}^* A) P - \phi(P \bar{A}^* \bar{A} P) \| \leq d_1 \phi(\| \bar{A}(I - P) \|)^2, \]  \hspace{1cm} (2.9)

where the constant \( d_1 \) depends only on \( \phi \). Moreover, for \( s = \phi(\bar{A}^* \bar{A})v, \| v \| \leq R, \)

\[ \| (I - P)s \| \leq \begin{cases} R \phi(\| \bar{A}(I - P) \|)^2, \quad \phi \in \mathcal{F}_0, \\ R \| \bar{A}(I - P) \|, \quad \phi \in \mathcal{F}_{1/2}. \end{cases} \]  \hspace{1cm} (2.10)

**Proposition 2.3.** Let \( \bar{A} \) be an operator defined by (1.6) and (1.7), where \( \Omega \) is a convex domain with Lipschitz boundary, \( a_0 \) is bounded away from zero, and \( \forall a_0 \in L^\infty(\Omega), \forall u \in L^\infty(\Omega). \) If \( P = P_n \) is the orthogonal projector from \( L^2(\Omega) \) onto \( n \)-dimensional space of piece-wise linear continuous functions corresponding to triangulation of \( \Omega \) with mesh size \( h_n \), then for any \( \gamma \in (\frac{1}{2}, 1) \)

\[ \| \bar{A}(I - P_n) \| \leq c_\gamma h_n^\gamma, \]  \hspace{1cm} (2.11)

where \( c_\gamma \) does not depend on \( h_n \).

**Proof.** Using the same argument as in the proof of [7, Corollary 1], we can prove that \( \bar{A}^* \) acts from \( L^2(\Omega) \) to Sobolev space \( H^\gamma(\Omega), \gamma \in (\frac{1}{2}, 1), \) as a linear bounded operator. Moreover, approximation theory provides us with the following Jackson type inequality:

\[ \| I - P_n \|_{H^\gamma_0(\Omega) \to L^2(\Omega)} \leq d_\gamma h_n^\gamma. \]  \hspace{1cm} (2.12)

Then

\[ \| \bar{A}(I - P_n) \| = \| (I - P_n) \bar{A}^* \| \leq \| I - P_n \|_{H^\gamma_0(\Omega) \to L^2(\Omega)} \| \bar{A}^* \|_{L^2(\Omega) \to H^\gamma(\Omega)} \leq c_\gamma h_n^\gamma. \]  \hspace{1cm} \square
Theorem 2.1. Assume that the solution $s$ of Eq. (1.6) belongs to the set $\mathcal{W}_0$ with $\phi \in F_0 \cup F_{1/2}$. Then for $s_{x,n,\delta} = g_2(B^*B)B^*r^\delta, g_2(\lambda) = 1/(\lambda+\lambda), B = AP_n$, $\parallel A - A \parallel \leq \varepsilon$, $\parallel r^\delta - \tilde{r} \parallel_{L^2(\Omega)} \leq \delta < \varepsilon$ and

$$\parallel \tilde{A}(I - P_n) \parallel \leq \min \left\{ \sqrt{2}, \frac{\varepsilon}{\sqrt{2}} \right\},$$

(2.13)

we have

$$\parallel s - s_{x,n,\delta} \parallel \leq C_1 \phi(x) + C_2 \phi(\varepsilon) + C_R \frac{\varepsilon}{\sqrt{2}},$$

(2.14)

where $C_R \leq (R\sqrt{b} + 3)/2$, and the constants $C_1, C_2$ do not depend on $x$ and $\varepsilon$.

Proof. Note that

$$\parallel s - s_{x,n,\delta} \parallel = \parallel s - g_2(B^*B)B^*r^\delta \parallel$$

$$\leq \parallel s - g_2(B^*B)B^*Bs + \parallel g_2(B^*B)B^*(Bs - r^\delta) \parallel.$$

Moreover,

$$\parallel Bs - r^\delta \parallel \leq \parallel (B - \tilde{A})s \parallel + \parallel \tilde{A}s - r^\delta \parallel$$

$$= \parallel (B - \tilde{A})s \parallel + \parallel \tilde{f} - r^\delta \parallel \leq \parallel (B - \tilde{A})s \parallel + \varepsilon.$$

Meanwhile,

$$\parallel (\tilde{A} - B)s \parallel \leq \parallel (\tilde{A} - A)P_n s \parallel + \parallel \tilde{A}(I - P_n)s \parallel$$

$$\leq \sqrt{b}R \varepsilon + \parallel \tilde{A}(I - P_n) \parallel \parallel (I - P_n)s \parallel.$$

Then (2.3) and Proposition 2.2 give us

$$\parallel g_2(B^*B)B^*(Bs - r^\delta) \parallel$$

$$\leq \begin{cases} \frac{1}{2\sqrt{2}} \left((\sqrt{b}R + 1)\varepsilon + R \parallel \tilde{A}(I - P_n)\parallel \phi(\parallel \tilde{A}(I - P_n)\parallel^2) \right), & \phi \in F_0. \\ \frac{1}{2\sqrt{2}} \left((\sqrt{b}R + 1)\varepsilon + R \parallel \tilde{A}(I - P_n)\parallel^2 \right), & \phi \in F_{1/2}. \end{cases}$$

Keeping in mind (2.5), we can continue

$$\parallel s - g_2(B^*B)B^*Bs \parallel \leq \parallel (I - P_n)s \parallel + \parallel (I - g_2(B^*B)B^*B) \phi(B^*B) \parallel$$

$$+ \parallel (I - g_2(B^*B)B^*B)(P_n \phi(\tilde{A} - \phi(B^*B)) \parallel$$

$$\leq Rc\phi(x) + \parallel (P_n \phi(\tilde{A} - \phi(B^*B)) \parallel + \parallel (I - P_n)s \parallel.$$

The last term has been estimated in Proposition 2.2, and we proceed with the remainder as follows:

$$\parallel (P_n \phi(\tilde{A} - \phi(B^*B)) \parallel \leq \parallel (I - P_n)\phi(\tilde{A} - \phi(B^*B)) \parallel + R \parallel P_n \phi(\tilde{A} - \phi(B^*B)) \parallel$$

$$- \phi(P_n \tilde{A} - \phi(P_n) \parallel \parallel P_n \tilde{A} - \phi(P_n) \parallel$$

$$- \phi(P_n A^* A P_n) \parallel.$$
The first two terms here have been also estimated in Proposition 2.2, and to estimate the last one we use property (2.6).

\[
\| \phi(P_n \tilde{A}^* \tilde{A} P_n) - \phi(P_n A^* A P_n) \| \leq \phi(\| P_n \tilde{A}^* \tilde{A} P_n - P_n A^* A P_n \|) \\
\leq \phi(d_2 \| \tilde{A} - A \|) \leq (|d_2| + 1) \phi(\epsilon),
\]

where \(d_2\) is a positive constant, and \([\cdot]\) denotes the integer part of a positive number.

Summing up we obtain the following inequalities:

\[
\| s - s_{\gamma, n, \delta} \| \leq R c_0(\alpha) + R \phi(\| \tilde{A}(I - P_n) \|^2) + d_1 R \phi(\| \tilde{A}(I - P_n) \|^2) \\
+ R(\lfloor d_2 \rfloor + 1) \phi(\epsilon) + \frac{1}{2 \sqrt{2}} \left( (\sqrt{b} R + 1) \epsilon \right) \quad \text{if} \; \phi \in F_0,
\]

\[
\| s - s_{\gamma, n, \delta} \| \leq R c_0(\alpha) + R \phi(\| \tilde{A}(I - P_n) \|^2) + d_1 R \phi(\| \tilde{A}(I - P_n) \|^2) \\
+ R(\lfloor d_2 \rfloor + 1) \phi(\epsilon) + \frac{1}{2 \sqrt{2}} \left( (\sqrt{b} R + 1) \epsilon \right) \\
+ \| \tilde{A}(I - P_n) \|^2 \quad \text{if} \; \phi \in F_{1/2}.
\]

These inequalities together with (2.13) give us the statement (2.14). \(\square\)

**Corollary 2.1.** Let \(\alpha \leq \epsilon^2\), \(h_n \sim \min \{ \alpha^{1/2}, \epsilon^{1/\gamma} \} \) or \(h_n \sim \epsilon^{1/\gamma}, \gamma \in (\frac{1}{2}, 1)\). Then under the conditions of Proposition 2.3 and Theorem 2.1 the estimation of accuracy (2.14) holds true.

**Proof.** From our assumption it follows that \(\min \{ \alpha^{1/2}, \epsilon^{1/\gamma} \} \geq \epsilon\). On the other hand, under the condition of Proposition 2.3, \(\| \tilde{A}(I - P_n) \| \sim h_n^2\), and for \(h_n\) choosing as in the statement of the corollary assumption (2.13) is satisfied that gives us (2.14). \(\square\)

Note that the assumption \(\alpha \leq \epsilon^2\) is not restrictive. It simply means that the term \(\frac{\epsilon}{\sqrt{\alpha}}\) from the error estimation (2.14) is smaller than 1, which is rather natural.

### 3. Adaptive strategy

Assume that \(h_n\) is chosen as in Corollary 2.1 with \(n = n(\alpha, \epsilon)\). Let \(s_{\alpha, \epsilon} = s_{\gamma, n(\alpha, \epsilon), \delta}\). In view of Theorem 2.1, the optimal choice of the regularization parameter would be \(\alpha = \alpha_{opt}\) for which

\[
\Xi_{\epsilon}(\alpha_{opt}) = C_R \frac{\epsilon}{\sqrt{\alpha_{opt}}},
\]

where \(\Xi_{\epsilon}(\lambda) := C_1 \phi(\lambda) + C_2 \phi(\epsilon)\). Let \(\Theta_{\epsilon}(\alpha) := \Xi_{\epsilon}(\alpha) \sqrt{\alpha}\), then

\[
\alpha_{opt} = \Theta_{\epsilon}^{-1}(C_R \epsilon) \sim \Theta_{\epsilon}^{-1}(\epsilon)
\]
and
\[ \|s - s^{opt,\varepsilon}\| \leq C_3 \Xi_\varepsilon(\Theta_\varepsilon^{-1}(\varepsilon)), \]  
(3.3)
where the constant \( C_3 \) does not depend on \( \varepsilon \).

Of course, for unknown \( \phi \) this optimal choice cannot be realized in practice. At the same
time, estimation (2.14) allows to apply general adaptive strategy from [11] based on the
idea known in statistics as Lepskii's bias-variance balancing. To describe this strategy we
introduce
\[ \Delta_N := \{ x_k = x_0 q^k, k = 0, 1, \ldots, N \} \]
with \( x_0 = \varepsilon^2, q > 1 \); \( N \) is an integer number such that \( x_{N-1} \leq b \leq x_N \). Then the corresponding regularized solutions \( s_{x_k,\varepsilon} \) will be studied successively as long as
\[ \|s_{x_i,\varepsilon} - s_{x_{i-1},\varepsilon}\| \leq 4C_R \varepsilon \sqrt{\varepsilon_{i-1}}. \]

The procedure terminates with
\[ \tilde{x} = \max \left\{ x_i \in \Delta_N : \|s_{x_i,\varepsilon} - s_{x_{i-1},\varepsilon}\| \leq 4C_R \varepsilon \sqrt{\varepsilon_{i-1}} \right\}. \]  
(3.4)
Then, with the same arguments as in the proof of Proposition 4 in [11], we can prove
\[ \|s - s_{\tilde{x},\varepsilon}\| \leq C_4 \varepsilon \sqrt{\varepsilon_{opt}}, \]
where the constant \( C_4 \) does not depend on \( \varepsilon \). Then (3.1) and (3.2) lead to
\[ \|s - s_{\tilde{x},\varepsilon}\| \leq C_5 \Xi_\varepsilon(\Theta_\varepsilon^{-1}(\varepsilon)). \]  
(3.5)

Let \( \Theta(\lambda) := \phi(\lambda) \sqrt{\lambda} \), then \( \Theta_\varepsilon^{-1}(\varepsilon) \leq C_6 \Theta^{-1}(\varepsilon) \), and we can rewrite (3.5) as
\[ \|s - s_{\tilde{x},\varepsilon}\| \leq C_5 \Xi_\varepsilon(\Theta_\varepsilon^{-1}(\varepsilon)) \leq C_5(C_1 \phi(\Theta_\varepsilon^{-1}(\varepsilon)) + C_2 \phi(\varepsilon)) \]
\[ \leq C_7(\phi(\Theta^{-1}(\varepsilon)) + \phi(\varepsilon)). \]

**Theorem 3.1.** *Under the condition of Theorem 2.1, for \( h_n \) chosen as in Corollary 2.1 and \( \tilde{x} \) chosen as in (3.4) we have
\[ \|s - s_{\tilde{x},\varepsilon}\| \leq C(\phi(\Theta^{-1}(\varepsilon)) + \phi(\varepsilon)), \]  
(3.6)
where the constant \( C \) do not depend on \( \varepsilon \).*

**Corollary 3.1.** *If index function \( \phi \in \mathcal{F}_{1/2} \), then
\[ \|s - s_{\tilde{x},\varepsilon}\| \leq C \phi(\Theta^{-1}(\varepsilon)). \]  
(3.7)
**Proof.** For \( \phi \in \mathcal{F}_{1/2} \), \( \phi(\varepsilon) \leq \sqrt{\varepsilon} \). Thus, \( \phi(\varepsilon)\sqrt{\varepsilon} \leq \varepsilon \), which means \( \varepsilon \leq \Theta^{-1}(\varepsilon) \), then (3.6) is reduced to (3.7). \( \square \)
Corollary 3.2. If index function \( \phi \in \mathcal{F}_0 \), then
\[
\|s - s_{\bar{z}, \epsilon}\| \leq C\phi(\epsilon).
\] (3.8)

At the same time, for \( \phi(\lambda) = c\lambda^\mu \), \( 0 \leq \mu < 1/2 \), (3.7) holds true as well. In this case,
\[
\|s - s_{\bar{z}, \epsilon}\| \leq C\epsilon^{2^\mu/\mu}.
\]

Proof. We prove only the last statement. It is well-known (see, e.g. [14, p. 93]) that for \( \phi(\lambda) = c\lambda^\mu \), \( 0 \leq \mu < 1/2 \),
\[
\|\phi(P_n\bar{A}^*A_Pn) - \phi(P_nA^*AP_n)\| = c\|\bar{A}P_n\|^2\mu - |AP_n|^2\mu \leq C_8\|\bar{A} - A\|^{2\mu},
\]
where \( |F| = (F^*F)^{1/2} \). Then \( \phi(\epsilon) \) appearing in (2.14) and (3.6) will be replaced by \( \phi(\epsilon^2) = c\epsilon^{2\mu} \). Therefore, \( \phi(\epsilon^2) \leq C_9\phi(\Theta^{-1}(\epsilon)) \), and (3.7) holds true. \( \square \)

Direct calculations show that the following statement is also true.

Corollary 3.3. If \( \phi(\lambda) = c\log^{-p}(1/\lambda) \), \( c, p > 0 \), then (3.7) holds true.

These corollaries specify the estimation of the accuracy in concrete cases.

4. Numerical examples

In this section, we present two numerical tests to support and verify the theoretical results of this paper. We use MATLAB-code in one-dimensional case, where \( \Omega = [0, 1] \). As in [4], for such \( \Omega \), the situation described in Proposition 2.3 is simplified, and the estimation for \( \bar{A}(I - P_n) \) is still valid. At first we take the same example as in [6].

Example 4.1. Consider
\[
a(x) = \begin{cases} 
1 + \frac{1}{3} \sin^2(\pi \frac{x - 0.5}{0.2}), & x \in [0.3, 0.7], \\
1 & \text{else.}
\end{cases}
\]
\[
u(x) = \begin{cases} 
\frac{x}{1 - 0.2(2 - \sqrt{3})}, & x \in [0, 0.3], \\
\left(0.3 + \frac{0.2\sqrt{3}}{2\pi}(\arctan(\sqrt{3}(\frac{\pi}{2} x - 0.5))) + \arctan(\frac{1}{\sqrt{3}}(\frac{\pi}{2} x - 0.5))) + \pi\right), & x \in [0.3, 0.7], \\
\frac{x - 0.2(2 - \sqrt{3})}{1 - 0.2(2 - \sqrt{3})}, & x \in [0.7, 1],
\end{cases}
\]
satisfying the following one-dimensional problem of the form (1.1):
\[
-(au)_x = 0 \quad \text{in } (0, 1), \\
u(0) = 0, \quad u(1) = 1.
\] (4.1)

We fix initial guess \( a_0 \equiv 1 \), which implies \( u_0(x) = x \).

Figs. 1 and 2 show numerical results. Regularized approximation is produced by the algorithm from Section 2.1. Here, we take the data noise level \( \delta = 0.001 \), \( u_0 = u + \delta \xi \).
where $\zeta$ is random variable with uniform distribution on the interval $[-1, 1]$. The data mollification is done by piece-wise linear interpolation. As it has been discussed in the Introduction, in one-dimensional case \((1.11)\) is satisfied. Thus, we have the noise level $
abla \sim \sqrt{\delta}$. The number of piece-wise linear basis elements for projection is $n = 50$, and components \((2.2)\) were computed using MATLAB-code for numerical integration. The final regularization parameters $\varepsilon = 0.00013$ and 0.00025 are produced by adaptive procedure described in Section 3, where we take $\varepsilon_0 = 0.00008$, $q = 1.1$ and $N = 26$. In Fig. 2, we enlarge the solution function $u(x)$ by fact 10. In this case $\varepsilon$ becomes smaller, and $u_0(x) = 10 \times x$.  

Fig. 1. Regularized approximation with $\varepsilon = 0.00013$ (dashed line) and exact parameter (solid line).

Fig. 2. Regularized approximation with $\varepsilon = 0.00025$ (dashed line) and exact parameter (solid line).
Example 4.2. Consider problem (4.1) with
\[ a(x) = (2x - 1)^{\frac{2}{5}}, \]
\[ u(x) = \frac{1}{2} + \frac{1}{2}(2x - 1)^{\frac{3}{5}}. \]

Figs. 3 and 4 show the results of application of the adaptive procedure described in Section 3 with the same parameters as in Example 4.1. Fig. 4 is again obtained by enlarging the exact solution \( u(x) \) by factor 10.
It is worth to note that in this example the exact coefficient $a$ has a zero point $x = \frac{1}{2}$. It shows that our approach can work without the additional assumption that $a(x)$ is bounded away from zero.

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**References**