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A characterization of the Hamming graph by strongly closed subgraphs

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Abstract

The Hamming graph $H(d, q)$ satisfies the following conditions:

(i) For any pair (u, v) of vertices there exists a strongly closed subgraph containing them whose diameter is the distance between u and v. In particular, any strongly closed subgraph is distance-regular.
(ii) For any pair (x, y) of vertices at distance $d - 1$ the subgraph induced by the neighbors of y at distance d from x is a clique of size $a_1 + 1$.

In this paper we prove that a distance-regular graph which satisfies these conditions is a Hamming graph.

1. Introduction

The reader is referred to the next section or [4] for the definitions.

The known distance-regular graphs have many subgraphs of high regularity. For example, the Odd graphs, the doubled Odd graphs, the doubled Grassmann graphs, the Hamming graphs and the dual polar graphs satisfy the following condition:

($*$) For any pair (u, v) of vertices there exists a strongly closed subgraph containing them whose diameter is the distance between u and v.

Our problem is to classify distance-regular graphs which satisfy this condition. Lots of partial answers have been obtained in [9,11,13,14]. In [9] we proved that a distance-regular graph which satisfies the condition ($*$) and contains a strongly closed subgraph which is a non-regular distance-biregular graph is either the Odd graph, the doubled Odd graph or the doubled Grassmann graph.

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(i) For any pair (u, v) of vertices there exists a strongly closed subgraph containing them whose diameter is the distance between u and v. In particular, any strongly closed subgraph is distance-regular.

(ii) For any pair (x, y) of vertices at distance $d - 1$ the subgraph induced by the neighbors of y at distance d from x is a clique of size $a_1 + 1$.

The main purpose of this paper is to show that a distance-regular graph which satisfies these conditions is a Hamming graph.

Let Γ be a distance-regular graph of diameter $d \geq 2$. Suppose that for any pair of vertices at distance $d-1$ there exists a strongly closed subgraph of diameter $d-1$ containing them. Then it can be shown that there exists a positive integer n such that for any given pair (x, y) of vertices at distance $d - 1$ the subgraph induced by the neighbors of y at distance d from x is a disjoint union of n cliques of size $a_1 + 1$. In particular, $b_{d-1} = n(a_1 + 1)$ (see Lemma 17). Suppose the condition (i) holds. Then the condition (ii) holds if and only if $b_{d-1} = a_1 + 1$. This is an extremal case of this direction.

The following are our main results.

Theorem 1. *Let Γ be a distance-regular graph of diameter $d \geq 3$ and valency $k \geq 3$. Suppose that $b_{d-1} = a_1 + 1$ holds and for any pair of vertices at distance $d - 1$ there exists a strongly closed subgraph of diameter $d - 1$ containing them. Then Γ is either the collinearity graph of a generalized $2d$-gon of order $(a_1 + 1, 1)$ with $d \in \{3, 4, 6\}$, the Pappus graph, the Coxeter graph, the doubled Odd graph $2O_k$ or the Hamming graph $H(d, a_1 + 2)$.*

Theorem 2. *Let Γ be a distance-regular graph of diameter $d \geq 3$ and valency $k \geq 4$. Then the following three conditions are equivalent.*

(i) *$b_{d-1} = a_1 + 1$ holds and for any pair of vertices at distance $d - 1$ there exists a strongly closed subgraph of diameter $d - 1$ containing them which is distance-regular.*

(ii) *For any integer m with $1 \leq m \leq d - 1$ and for any pair of vertices at distance m there exists a strongly closed subgraph of diameter m containing them which is distance-regular. Moreover for any pair (x, y) of vertices at distance $d - 1$ the subgraph induced by the neighbors of y at distance d from x is a clique of size $a_1 + 1$.*

(iii) *Γ is the Hamming graph $H(d, a_1 + 2)$.*

Theorem 3. *Let Γ be a distance-regular graph of diameter $d \geq 3$ and valency $k \geq 3$. Suppose that $b_{d-2} = 2b_{d-1}$ and for any pair of vertices at distance $d - 1$ there exists a strongly closed subgraph of diameter $d - 1$ containing them. Then Γ is either the Pappus graph, the Coxeter graph or the Hamming graph $H(d, a_1 + 2)$.*

This paper is organized as follows. In Section 2 we recall some definitions and basic terminology for distance-regular graphs and strongly closed subgraphs. We collect several known results for strongly closed subgraphs and give some consequences. In Section 3 we study a design obtained from the strongly closed subgraphs of a distance-regular graph Γ. It gives us some conditions for the intersection numbers of Γ. By using them, we prove our main results in Section 4.

2. Preliminaries

First we recall our notation and terminology. Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph with usual distance ∂_Γ and diameter $d = d(\Gamma)$. For a vertex u in Γ we denote by $\Gamma_j(u)$ the set of vertices which are at distance j from u, where $\Gamma_{-1}(u) = \Gamma_{d+1}(\Gamma) = \emptyset$. For two vertices x and y in Γ with $\partial_\Gamma(x, y) = j$, let

$$C(x, y) := \Gamma_{j-1}(x) \cap \Gamma_1(y), \qquad A(x, y) := \Gamma_j(x) \cap \Gamma_1(y),$$
$$B(x, y) := \Gamma_{j+1}(x) \cap \Gamma_1(y).$$

Definition 4. Let i be an integer with $0 \le i \le d$.

(i) We say $c_i(\Gamma)$-*exists* if $c_i(\Gamma) = |C(x, y)|$ is a constant whenever $\partial_\Gamma(x, y) = i$.
(ii) We say $a_i(\Gamma)$-*exists* if $a_i(\Gamma) = |A(x, y)|$ is a constant whenever $\partial_\Gamma(x, y) = i$.
(iii) We say $b_i(\Gamma)$-*exists* if $b_i(\Gamma) = |B(x, y)|$ is a constant whenever $\partial_\Gamma(x, y) = i$.

A connected graph Γ of diameter d is said to be *distance-regular* if $c_i(\Gamma)$-exists and $b_i(\Gamma)$-exists for all $i = 0, \ldots, d$. Then Γ is a regular graph of valency $k = k(\Gamma) = b_0(\Gamma)$ and $a_i(\Gamma)$-exists with $a_i(\Gamma) = k(\Gamma) - c_i(\Gamma) - b_i(\Gamma)$ for all $i = 0, \ldots, d$. Remark $c_0(\Gamma) = a_0(\Gamma) = b_d(\Gamma) = 0$ and $c_1(\Gamma) = 1$. The constants $c_i(\Gamma)$, $a_i(\Gamma)$ and $b_i(\Gamma)$ $(i = 0, \ldots, d)$ are called the *intersection numbers* of Γ.

A connected bipartite graph Γ with bipartition $\Gamma^+ \cup \Gamma^-$ is called *distance-biregular* if for any $x \in \Gamma^+$ and for any $y \in \Gamma_i(x)$

$$c_i^+(\Gamma) = |C(x, y)|, \qquad b_i^+(\Gamma) = |B(x, y)|$$

depend only on i, and for any $x' \in \Gamma^-$ and $y' \in \Gamma_i(x')$

$$c_i^-(\Gamma) = |C(x', y')|, \qquad b_i^-(\Gamma) = |B(x', y')|$$

depend only on i.

For more background information about distance-regular graphs and distance-biregular graphs we refer the reader to [1,4].

Next we recall the definition and some facts for the Hamming graph (see [1, Section III,2], [4, Section 9.2]).

Let q and d be integers at last 2 and let X be a set with q elements. The *Hamming graph* $H(d, q)$ is the graph with the vertex set X^d the cartesian product of d copies of X and two vertices are adjacent whenever they differ in precisely one coordinate. Then two vertices are at distance i if and only if they differ in precisely i coordinates. Let Γ be the Hamming graph $H(d, q)$. Then Γ is a distance-regular graph of diameter d with the intersection numbers

$$c_j(\Gamma) = j, \qquad a_j(\Gamma) = j(q-2), \qquad b_j(\Gamma) = (d-j)(q-1), \quad (j = 0, 1, \ldots, d).$$

Moreover for any integer j with $0 \le j \le d$ and any vertices x, y at distance j in Γ the following hold:

- The subgraph induced by $C(x, y)$ is a coclique of size j.
- The subgraph induced by $A(x, y)$ is a disjoint union of j cliques of size $q - 2$.
- The subgraph induced by $B(x, y)$ is a disjoint union of $d - j$ cliques of size $q - 1$.
- There exists an induced subgraph which is the Hamming graph $H(j, q)$ containing x and y.

Here only we recall the last fact. Let $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. Set $T = \{i \mid 1 \le i \le d, x_i \ne y_i\}$. Then $|T| = j$ since $\partial_\Gamma(x, y) = j$. Define

$$\Lambda = \{z = (z_1, \ldots, z_d) \in X^d \mid z_i = x_i \text{ for all } i \in (X \setminus T)\}.$$

Then the induced subgraph on Λ is the Hamming graph $H(j, q)$ containing x and y. Moreover, it can be shown that $C(w, w') \cup A(w, w') \subseteq \Lambda$ for any vertices $w, w' \in \Lambda$.

In the rest of this paper let Γ be a distance-regular graph of diameter $d = d(\Gamma) \geq 2$ and valency $k = k(\Gamma) \geq 3$. We denote c_i, a_i and b_i for the intersection numbers $c_i(\Gamma), a_i(\Gamma)$ and $b_i(\Gamma)$ of Γ. Define

$$r = r(\Gamma) := \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}.$$

It is known that $b_r > b_{r+1}$ holds (see [4, Proposition 5.4.4]).

Next we recall the definition and some results for strongly closed subgraphs.

Definition 5. Let Γ be a distance-regular graph of diameter $d = d(\Gamma) \geq 2$ and valency $k = k(\Gamma) \geq 3$. Let Λ be a subset of vertices in Γ. We identify Λ with the induced subgraph on it.

(i) A subgraph Λ is called *strongly closed* if $C(x, y) \cup A(x, y) \subseteq \Lambda$ for any $x, y \in \Lambda$.
(ii) We say that *the condition $(SC)_m$ holds* if for any given pair of vertices at distance m there exists a strongly closed subgraph of diameter m containing them.
(iii) Let (x, y) be a pair of vertices in Γ. Define $\Delta(x, y)$ to be the intersection of all strongly closed subgraphs in Γ containing x and y.

We remark that Γ itself is a strongly closed subgraph in Γ. Hence $\Delta(x, y)$ can be defined. It follows, by definition, that the intersection of strongly closed subgraphs is also strongly closed unless it is the empty set. Hence $\Delta(x, y)$ is the smallest strongly closed subgraph containing x and y. Suppose that Γ satisfies the condition $(SC)_j$ for some integer j with $1 \leq j \leq d-1$. Then for any pair (x, y) of vertices at distance j there exists a strongly closed subgraph Ψ of diameter j containing x and y. Then $\Delta(x, y)$ has diameter j since $x, y \in \Delta(x, y) \subseteq \Psi$.

Let Λ be a strongly closed subgraph of Γ with diameter $m = d(\Lambda)$. Take vertices x and y in Λ. Then any shortest path between x and y in Γ is contained in the subgraph Λ. So the distance in Λ coincides with the distance in Γ. Hence

$$\Lambda_i(u) = \{z \in \Lambda \mid \partial_\Lambda(u, z) = i\} = \Gamma_i(u) \cap \Lambda$$

for any $u \in \Lambda$ and $i = 0, 1, \ldots, m$. It follows that if $\partial_\Lambda(x, y) = \partial_\Gamma(x, y) = j$, then

$$\Lambda_{j-1}(x) \cap \Lambda_1(y) = C(x, y) \quad \text{and} \quad \Lambda_j(x) \cap \Lambda_1(y) = A(x, y).$$

Thus $c_i(\Lambda)$-exists and $a_i(\Lambda)$-exists with $c_i(\Lambda) = c_i$ and $a_i(\Lambda) = a_i$ for all $i = 1, \ldots, m$. Moreover if Λ is a regular graph of valency $k(\Lambda)$, then $b_i(\Lambda)$-exists with $b_i(\Lambda) = k(\Lambda) - c_i - a_i$ for all $i = 1, \ldots, m$, and thus Λ is distance-regular. However there exist several examples of non-regular strongly closed subgraphs in a distance-regular graph.

Let G be a connected graph. We define the *n-subdivision graph of G*, denoted by $G^{(n)}$, the graph obtained from G by replacing each edge by a path of length n. For any pair of vertices at distance 6 in the Foster graph there exists a 2-subdivision graph of the Peterson graph containing them as a strongly closed subgraph (see [11, Theorem 1.4] and [4, Section 13.2A]), and for any pair of vertices at distance 5 in the Biggs–Smith graph there exists a 3-subdivision graph of the complete graph K_4 containing them as a strongly closed subgraph (see [11, Theorem 1.5] and [4, Section 13.4]).

There are lots of non-regular distance-biregular graphs as strongly closed subgraphs in the doubled Grassmann graphs, the doubled Odd graphs, and the Odd graphs (see [9, Section 2] and [4, Section 9.1D, Section 9.3]).

In the rest of this section we recall several known results for strongly closed subgraphs. They imply that these examples of distance-regular graphs are almost all distance-regular graphs which satisfy the condition $(SC)_m$ for some m and contain a non-regular strongly closed subgraph of diameter m.

The following result is proved by Suzuki in [11, Theorem 1.1].

Proposition 6. *Let Γ be a distance-regular graph of diameter d, valency k and $r = r(\Gamma)$. Let Λ be a strongly closed subgraph of diameter $m := d(\Lambda)$. Then one of the following holds.*

(i) *Λ is distance-regular.*
(ii) $2 \le m \le r$.
(iii) *Λ is (non-regular) distance-biregular with $c_{2i-1} = c_{2i}$ for all i with $2i \le m$. Moreover $r \equiv m \equiv 0 (\mathrm{mod}\ 2)$.*
(iv) *Λ is the 3-subdivision graph $K_{k+1}^{(3)}$ of a complete graph K_{k+1} or the 3-subdivision graph $M_k^{(3)}$ of a Moore graph M_k. Moreover $m = r + 2 \in \{5, 8\}, a_1 = 0$ and $c_{r+1} = c_{r+2} = a_{r+1} = a_{r+2} = 1$.*

In particular, $(c_{m-1}, a_{m-1}, b_{m-1}) = (c_m, a_m, b_m)$ holds except the case (i).

Suppose that there exists a strongly closed subgraph Λ of diameter m which is distance-regular. Then $c_i(\Lambda) = c_i$ and $a_i(\Lambda) = a_i$ for $i = 1, \ldots, m$. Hence $k(\Lambda) = c_m + a_m$ and $b_i(\Lambda) = k(\Lambda) - c_i(\Lambda) - a_i(\Lambda) = (c_m + a_m) - c_i - a_i = b_i - b_m$ for $i = 1, \ldots, m$. In particular, we obtain $b_{m-1} - b_m = b_{m-1}(\Lambda) > 0$. Hence a strongly closed subgraph of diameter m is distance-regular if and only if $b_{m-1} > b_m$ holds.

Let x and y be vertices in Γ with $\partial_\Gamma(x, y) = m$. Suppose that $b_{m-1} > b_m$ and there exists a strongly closed subgraph of diameter m containing x and y. Let $\Delta(x, y)$ be as in Definition 5(iii), and let Ψ be any strongly closed subgraph of diameter m containing x and y. Then $\Delta(x, y)$ has diameter m since $x, y \in \Delta(x, y) \subseteq \Psi$. The above observation shows that both of $\Delta(x, y)$ and Ψ are distance-regular graphs with the same intersection numbers such that $\Delta(x, y) \subseteq \Psi$. This implies that $\Delta(x, y) = \Psi$ and it is a unique strongly closed subgraph of diameter m containing x and y.

A graph Γ is said to be *of order* (s, t) if for any vertex u in Γ the subgraph induced by $\Gamma_1(u)$ is a disjoint union of $t + 1$ cliques of size $s + 1$. Since a strongly closed subgraph of diameter 1 is a clique of size $a_1 + 2$, it is straightforward to see the following lemma (see [4, Proposition 1.2.1]).

Lemma 7. *Let Γ be a distance-regular graph. Then the following conditions are equivalent.*

(i) *The condition $(SC)_1$ holds.*
(ii) *Each edge lies on a clique of size $a_1 + 2$.*
(iii) *Γ is of order (s, t), where $s = a_1 + 1$ and $t = \frac{b_1}{a_1+1}$.* □

A connected graph E is called an *expanded tree* if there are no induced cycles except triangles. Moreover if each edge lies on a clique of size $s + 1$, then E is called an *expanded tree of order s*.

We remark that an expanded tree of order 1 is a tree.

Suppose $r = r(\Gamma) \ge 2$ and consider a connected induced subgraph Λ of Γ such that $2 \le d(\Lambda) \le r$. By the definition of $r(\Gamma)$ there is no induced cycle of length less than $2r + 2$ in Γ except triangles. So Λ is an expanded tree. Let x and y be distinct vertices in Λ and let $i = \partial_\Lambda(x, y)$. Then there exists a path P of length i connecting x and y in Λ, and there is no other path of length less than or equal to i connecting x and y in Γ as Γ has no induced cycle of length less than $2r + 2$ except triangles. Hence P is a unique shortest path connecting x and y in Γ which is contained in Λ. In particular, $\partial_\Lambda(x, y) = \partial_\Gamma(x, y)$ for any $x, y \in \Lambda$. We remark that each edge of Γ lies on a clique of size $a_1 + 2$ since $c_2 = 1$ (see [4, Proposition 1.2.1]).

If Λ is strongly closed, then $\{z, z'\} \cup A(z, z') \subseteq \Lambda$ for any edge (z, z') in Λ, and hence Λ is an expanded tree of order $a_1 + 1$. Conversely we assume that Λ is an expanded tree of order $a_1 + 1$ and prove that it is strongly closed in Γ. Let u and v be distinct vertices in Λ and let $m = \partial_\Gamma(u, v)$. Then $m = \partial_\Gamma(u, v) = \partial_\Lambda(u, v) \leq d(\Lambda) \leq r$. As $c_m = 1$ and $a_1 = \cdots = a_m$, there exists a unique vertex w such that $\{w\} = C(u, v)$ and $A(u, v) = A(w, v)$. Since the unique shortest path connecting u and v is contained in Λ, we have $w \in \Lambda$. Since Λ is an expanded tree of order $a_1 + 1$ and (w, v) is an edge in Λ, we have $\{w, v\} \cup A(w, v) \subseteq \Lambda$. Hence $C(u, v) \cup A(u, v) = \{w\} \cup A(w, v) \subseteq \Lambda$ and Λ is strongly closed. Therefore an induced subgraph Λ with $2 \leq d(\Lambda) \leq r$ is strongly closed if and only if it is an expanded tree of order $a_1 + 1$.

Let x and y be distinct vertices with $m := \partial_\Gamma(x, y) \leq r$ in Γ. Then there exists a unique shortest path $(x = x_0, x_1, \ldots, x_m = y)$ of length m connecting x and y as $c_1 = \cdots = c_m = 1$. Define

$$\Sigma(x, y) := \bigcup_{i=1}^{m} (\{x_{i-1}, x_i\} \cup A(x_{i-1}, x_i)).$$

Then $\Sigma(x, y)$ is an expanded tree of order $a_1 + 1$ with diameter m such that $m \leq r$. Hence $\Sigma(x, y)$ is a strongly closed subgraph of diameter m. Then $\Sigma(x, y) = \Delta(x, y)$ since any strongly closed subgraph Λ containing x and y also contains $(x = x_0, x_1, \ldots, x_m = y)$ and $A(x_{i-1}, x_i) \subseteq \Lambda$ for $i = 1, \ldots, m$.

This observation shows that a distance-regular graph with $r = r(\Gamma) \geq 2$ always satisfies the condition $(SC)_m$ for $m = 1, 2, \ldots, r$. Moreover any strongly closed subgraph of diameter m with $2 \leq m \leq r$ is an expanded tree of order $a_1 + 1$.

We say a strongly closed subgraph Λ is *trivial* if $2 \leq d(\Lambda) \leq r$. We should consider non-trivial strongly closed subgraphs.

The following results are proved in [9, Proposition 4.5] and [8, Theorem 1].

Proposition 8. *Let Γ be a distance-regular graph of diameter d, valency k and $r = r(\Gamma)$. Suppose that the condition $(SC)_m$ holds for some m with $r+1 \leq m \leq d-1$ and a strongly closed subgraph of diameter m is a non-regular distance-biregular graph. Then one of the following holds.*

(i) *Γ is either the doubled Grassmann graph, the doubled Odd graph, or the Odd graph.*
(ii) $r = 4, m = 6, a_1 = \cdots = a_6 = 0, c_5 = c_6 = 2$ *and* $k \in \{3, 57\}$.

Proposition 9. *Let Γ be a distance-regular graph of diameter d and $r = r(\Gamma)$. Let m be an integer with $r + 1 \leq m \leq d - 1$. If the condition $(SC)_m$ holds, then the condition $(SC)_i$ holds for all i with $1 \leq i \leq m$.*

The following corollary is a direct consequence of Propositions 6, 8 and 9.

Corollary 10. *Let Γ be a distance-regular graph of diameter d, valency $k \geq 3$ and $r = r(\Gamma)$. Let m be an integer with $r + 1 \leq m \leq d - 1$. Suppose the condition $(SC)_m$ holds. Then the condition $(SC)_i$ holds for all i with $1 \leq i \leq m$. Moreover one of the following holds.*

(i) *$b_{m-1} > b_m$ and any strongly closed subgraph of diameter m is distance-regular.*
(ii) *Γ is either the doubled Grassmann graph, the doubled Odd graph, or the Odd graph.*
(iii) $r = 4, m = 6, a_1 = \cdots = a_6 = 0, c_5 = c_6 = 2$ *and* $k \in \{3, 57\}$.
(iv) $m = r + 2, r \in \{3, 6\}, a_1 = 0$ *and* $c_{r+1} = c_{r+2} = a_{r+1} = a_{r+2} = 1$.

Proof. The first assertion follows by Proposition 9. Let Λ be a strongly closed subgraph of diameter m. Then Λ is either one of the graphs in Proposition 6(i), (iii), (iv). If Λ is distance-regular, then $b_{m-1} > b_m$ and (i) holds. If Λ is non-regular distance-biregular, then either (ii) or (iii) holds by Proposition 8. If Λ is a graph as in Proposition 6(iv), then (iv) holds. The corollary is proved. □

The Foster graph and the Biggs–Smith graph are the only known examples of a distance-regular graph as in Corollary 10(iii) and (iv), respectively. A distance-regular graph as in Corollary 10(iii), (iv) with $b_{r+1} = a_1 + 1$ is the Foster graph and the Biggs–Smith graph, respectively, since it has valency $k = c_{r+1} + a_{r+1} + b_{r+1} = 3$ (see [3] or [4, Theorem 7.5.1]). A distance-regular graph as in Corollary 10(ii) with $b_m = a_1 + 1$ for some m is the doubled Odd graph (see [4, Section 9.1B, Section 9.3]).

Let Γ be a distance-regular graph of diameter d and $r = r(\Gamma)$ with $r + 2 \le d$. Let m be an integer with $r + 1 \le m \le d - 1$ such that $b_{m-1} > b_m$. If Γ satisfies the condition $(SC)_m$, then Γ satisfies the condition $(SC)_j$ for $j = 1, 2, \ldots, m$ by Proposition 9. Let Λ be a strongly closed subgraph of diameter m. Then Λ is distance-regular since $b_{m-1} > b_m$. Take any pair (x, y) of vertices in Λ and let $j = \partial_\Gamma(x, y)$. Then $\Delta(x, y)$ has diameter j since Γ satisfies the condition $(SC)_j$. By the definition of $\Delta(x, y)$ we have $\Delta(x, y) \subseteq \Lambda$. Hence $\Delta(x, y)$ is also a strongly closed subgraph in Λ of diameter j containing x and y. Therefore Λ also satisfies the condition $(SC)_j$ for all $j = 1, 2, \ldots, m$. This implies that Λ satisfies the condition $(*)$ in the introduction.

3. Designs

In this section we study a design obtained from strongly closed subgraphs in a distance-regular graph. In the end of this section we will remark that designs obtained from the Hamming graphs are trivial.

First we recall the following result proved in [7, Proposition 4.1].

Proposition 11. *Let Γ be a distance-regular graph of diameter $d \ge 2$, and h be an integer with $1 \le h \le d - 1$. Suppose $b_{h-1} > b_h$ and the condition $(SC)_h$ holds. Then the following hold.*

(i) *For any pair (x, y) of vertices at distance $h - 1$ there are exactly $\frac{b_{h-1}}{b_{h-1}-b_h}$ strongly closed subgraphs of diameter h containing x and y. In particular, $(b_{h-1} - b_h) \mid b_{h-1}$.*

(ii) *For any vertex u in Γ there are exactly $\prod_{i=0}^{h-1} \frac{b_i}{b_i-b_h}$ strongly closed subgraphs of diameter h containing u. In particular, $\prod_{i=0}^{h-1}(b_i - b_h) \mid \prod_{i=0}^{h-1} b_i$.*

Here we reprove the following generalization. Proposition 11 is the cases of $j = h - 1$ and $j = 0$.

Proposition 12. *Let Γ be a distance-regular graph of diameter $d \ge 2$. Let j and h be integers with $0 \le j < h \le d - 1$. Suppose $b_{h-1} > b_h$ and the condition $(SC)_h$ holds. Then for any pair (x, y) of vertices at distance j there are exactly*

$$\prod_{i=j}^{h-1} \frac{b_i}{b_i - b_h}$$

strongly closed subgraphs of diameter h containing x and y. In particular, $\prod_{i=j}^{h-1}(b_i - b_h) \mid \prod_{i=j}^{h-1} b_i$.

Proof. Let $\mathcal{S}$ be the set of strongly closed subgraphs of diameter h containing x and y. We count the size of the set

$$\{(z, \Lambda) \mid z \in \Gamma_h(x) \cap \Gamma_{h-j}(y), \Lambda \in \mathcal{S}, z \in \Lambda\}$$

in two ways. For any $z \in \Gamma_h(x) \cap \Gamma_{h-j}(y)$ there exists a unique strongly closed subgraph $\Delta(x, z)$ of diameter h containing x and z. Then $y \in \Delta(x, z)$ and thus $\Delta(x, z) \in \mathcal{S}$. Conversely each Λ in $\mathcal{S}$ is distance-regular with $c_i(\Lambda) = c_i$ and $b_i(\Lambda) = b_i - b_h$ for all i with $1 \le i \le h$. Hence we have

$$\frac{b_j \cdots b_{h-1}}{c_1 \cdots c_{h-j}} = \sum_{\Lambda \in \mathcal{S}} \frac{b_j(\Lambda) \cdots b_{h-1}(\Lambda)}{c_1(\Lambda) \cdots c_{h-j}(\Lambda)} = |\mathcal{S}| \frac{(b_j - b_h) \cdots (b_{h-1} - b_h)}{c_1 \cdots c_{h-j}}.$$

The desired result is proved. □

Let X and B be sets together with an incidence relation I. Then a structure (X, B, I) is called a t-(n, ℓ, λ) *design* if X has n elements, each element of B is incident with exactly ℓ elements of X and any t elements of X are incident with λ common elements of B. It is known as the *Fisher's inequality* that $\lambda(n - 1) \ge \ell(\ell - 1)$ holds for a 2-(n, ℓ, λ) design.

More information for designs can be found in [2] (see also [4, A6]).

Then we can prove the following fact as a direct consequence of the previous result.

Corollary 13. *Let Γ be a distance-regular graph of diameter $d \ge 3$. Let j and m be integers with $1 \le j < m \le d - 1$. Suppose $b_{j-1} > b_j$, $b_{m-1} > b_m$ and the condition $(SC)_m$ holds. Let (u, v) be a pair of vertices at distance $j - 1$. Define $\mathcal{P}$ (resp. $\mathcal{L}$) to be the set of strongly closed subgraphs of diameter j (resp. m) containing u and v. Then $\mathcal{D} := (\mathcal{P}, \mathcal{L}, \subseteq)$ is a 2-(n, ℓ, λ)-design, where*

$$n = \frac{b_{j-1}}{b_{j-1} - b_j}, \qquad \ell = \frac{b_{j-1} - b_m}{b_{j-1} - b_j} \quad \textit{and} \quad \lambda = \begin{cases} 1 & \textit{if } m = j + 1, \\ \displaystyle\prod_{i=j+1}^{m-1} \frac{b_i}{b_i - b_m} & \textit{if } m \ge j + 2. \end{cases}$$

In particular, $\lambda b_j(b_{j-1} - b_j) \ge (b_{j-1} - b_m)(b_j - b_m)$ holds.

Proof. There are exactly n elements in $\mathcal{P}$ by Proposition 11(i). Each element Λ of $\mathcal{L}$ is a distance-regular graph with $b_i(\Lambda) = b_i - b_m$ for $i = 1, \ldots, m$. Hence it contains exactly ℓ elements of $\mathcal{P}$ by applying Proposition 11(i) to Λ. Let P and P' be two distinct elements of $\mathcal{P}$. Take $x \in \Gamma_1(u) \cap \Gamma_j(v) \cap P$ and $y \in \Gamma_j(u) \cap \Gamma_1(v) \cap P'$. Then $P = \Delta(x, v)$ and $P' = \Delta(u, y)$. We have $\partial_\Gamma(x, y) = j + 1$ or otherwise $y \in C(x, v) \cup A(x, v) \subseteq P$ and $x \in C(y, u) \cup A(y, u) \subseteq P'$ which implies that $P' = \Delta(u, y) \subseteq P$ and $P = \Delta(x, v) \subseteq P'$. If an element Λ in $\mathcal{L}$ contains both of P and P', then Λ contains x and y. Conversely if Λ contains x and y, then $v \in C(x, y) \subseteq \Lambda$ and $u \in C(y, x) \subseteq \Lambda$. Hence $P = \Delta(x, v) \subseteq \Lambda$ and $P' = \Delta(u, y) \subseteq \Lambda$. It follows that an element Λ in $\mathcal{L}$ contains both P and P' if and only if Λ contains both x and y. Thus there are exactly λ elements in $\mathcal{L}$ containing both P and P' by Proposition 12. The last assertion follows by Fisher's inequality. □

Corollary 14. *Let Γ be a distance-regular graph of diameter $d \ge 3$. If $b_{d-3} > b_{d-2} > b_{d-1}$ and the condition $(SC)_{d-1}$ holds, then the following hold.*

(i) *$(b_{d-3} - b_{d-2})$ divides both $(b_{d-2} - b_{d-1})$ and b_{d-1}.*
(ii) *$(b_{d-2} - b_{d-1})^2 \le b_{d-1}(b_{d-3} - b_{d-2})$.*

Proof. Put $j = d - 2$ and $m = d - 1$ in Corollary 13. Then we have a 2-(n, ℓ, 1)-design, where

$$n = \frac{b_{d-3}}{b_{d-3} - b_{d-2}} \quad \text{and} \quad \ell = \frac{b_{d-3} - b_{d-1}}{b_{d-3} - b_{d-2}}.$$

Since both $\ell - 1$ and $n - \ell$ are integers, the first assertion is proved. The second assertion follows by Fisher's inequality $\ell(\ell - 1) \le n - 1$ for 2-(n, ℓ, 1)-designs. □

Corollary 15. *Let Γ be a distance-regular graph of diameter $d \ge 4$. Let m be an integer with $2 \le m \le d - 2$. If $b_{m-2} > b_{m-1} > b_m > b_{m+1}$ and the condition $(SC)_{m+1}$ holds, then the following hold.*

(i) $(b_{m-2} - b_{m-1})$ *divides both* $(b_{m-1} - b_m)$ *and* $(b_m - b_{m+1})$.
(ii) $(b_{m-1} - b_m)^2 \le (b_m - b_{m+1})(b_{m-2} - b_{m-1})$.

Proof. Let Λ be a strongly closed subgraph of diameter $m+1$. Then it is a distance-regular graph with $b_i(\Lambda) = b_i - b_{m+1}$ for $i = 0, \ldots, m$. The desired results follow by applying Corollary 14 to Λ. □

Take a pair (x, y) of vertices at distance $j - 1$ in the Hamming graph $H(d, q)$ and let $\mathcal{D} := (\mathcal{P}, \mathcal{L}, \subseteq)$ be the design defined in Corollary 13. Then $\mathcal{D}$ is a trivial design. In fact, there exists a one-to-one correspondence between $\mathcal{L}$ and the family of all $(m - j + 1)$-elements subsets of $\mathcal{P}$ as follows:

Let $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$ and $T = \{i \mid 1 \le i \le d, x_i \ne y_i\}$. Then we have $|T| = j - 1$. For any subset S of $(X \setminus T)$, where $s = |S|$, the induced subgraph on

$$\Upsilon(S) = \{z = (z_1, \ldots, z_d) \in X^d \mid z_i = x_i \text{ for all } i \in (X \setminus (S \cup T))\}$$

is the Hamming graph $H(s + j - 1, q)$ containing x and y which is strongly closed in $H(d, q)$. Conversely, for any strongly closed subgraph Λ of diameter $s + j - 1$ containing x and y, take $z \in \Lambda_{s+j-1}(x) \cap \Lambda_s(y)$. Let $T' = \{i \mid 1 \le i \le d, x_i \ne z_i\}$ and $S' = (T' \setminus T)$. Then we have $T \subseteq T'$ and $|T'| = s + j - 1$ by considering the distance in the Hamming graph. Thus $|S'| = s$ and $\Upsilon(S')$ is a strongly closed subgraph of diameter $s + j - 1$ containing x and z. Hence $\Lambda = \Delta(x, z) = \Upsilon(S')$. This implies that there exists a one-to-one correspondence between the family of all s-elements subsets of $(X \setminus T)$ and the set of all strongly closed subgraphs of diameter $(s + j - 1)$ containing x and y. Therefore there exists a one-to-one correspondence between $\mathcal{L}$ (resp. $\mathcal{P}$) and the family of all $(m - j + 1)$-elements subsets (resp. all 1-elements subsets) of $(X \setminus T)$.

Let Γ be a distance-regular graph of diameter d and $r = r(\Gamma)$ which satisfies the condition $(SC)_m$ for some integer m with $r + 1 \le m \le d - 1$. Suppose that $m - r$ is large. Then we obtain many designs and many restrictions for the intersection numbers of Γ by Corollaries 13 and 15. It may suggest that $m - r$ cannot be so large except for the case that almost all designs are trivial. This observation will be a step to solve our problem in the introduction.

4. Proof of the theorems

In this section we prove our main results. First we collect the following known results which we need to prove our main results.

Proposition 16. *Let Γ be a distance-regular graph of valency $k \ge 3$, diameter $d \ge 2$ and $r = r(\Gamma)$. Then the following hold.*

(i) *If Γ contains an induced quadrangle, then $c_i - b_i \geq c_{i-1} - b_{i-1} + a_1 + 2$ for $i = 1, \ldots, d$.*
(ii) *Suppose that Γ is of order (s, t) such that $c_i = i$ and $a_i = i(s-1)$ for all i with $1 \leq i \leq d$. Then Γ is the Hamming graph $H(t+1, s+1)$.*
(iii) *Suppose that Γ is of order $(s, 1)$ with $d = r + 1$. Then Γ is either the Hamming graph $H(2, a_1 + 2)$ or the collinearity graph of a generalized $2d$-gon of order $(s, 1)$ with $d \in \{3, 4, 6\}$. In particular, $c_{r+1} = 2$ and $a_{r+1} = 2a_1$.*
(iv) *Suppose that $a_1 \geq 1$, $c_{r+1} = 2$ and $a_{r+1} = 2a_1$. Then $c_{r+2} \geq 3$. Moreover if Γ is of order $(s, 2)$ with $c_{r+2} = 3$, $a_{r+2} = 3a_1$ and $d = r + 2$, then $r = 1$ and Γ is the Hamming graph $H(3, a_1 + 2)$.*

Proof. (i), (ii) These are proved in [12,6] (see also [4, Theorem 5.2.1, Corollary 9.2.5]).

(iii) [4, Theorem 4.3.4] implies that Γ is a line graph. It follows, by [4, Theorem 4.2.16], that Γ is a lattice graph (i.e., Hamming graph $H(2, a_1 + 2)$) or the collinearity graph of a generalized $2d$-gon of order $(s, 1)$ with $d \in \{3, 4, 6\}$ since Γ is of order $(s, 1)$ with $d = r + 1$.

(iv) This is proved in [10, Proposition 2, Proposition 5]. □

We recall that the Hamming graph $\Gamma = H(d, q)$ is a graph of order $(q-1, d-1)$. Moreover for any pair (x, y) of vertices in Γ the subgraph induced by $B(x, y)$ is a disjoint union of $d - j$ cliques of size $q - 1$, and the subgraph induced by $C(x, y)$ is a coclique of size j, where $j = \partial_\Gamma(x, y)$. In general the following holds.

Lemma 17. *Let Γ be a distance-regular graph of diameter $d \geq 2$. Suppose that the condition $(SC)_m$ holds for some integer m with $1 \leq m \leq d-1$. Then Γ is of order (s, t), where $s = a_1 + 1$ and $t = \frac{b_1}{a_1+1}$. Moreover, for any integer i with $1 \leq i \leq m$ the following hold.*

(i) *For any pair (x, y) of vertices at distance i the subgraph induced by $B(x, y)$ is a disjoint union of $\frac{b_i}{a_1+1}$ cliques of size $a_1 + 1$.*
(ii) *For any pair (x', z') of vertices at distance $i + 1$ the subgraph induced by $C(x', z')$ is a coclique of size c_{i+1}. In particular, $c_{i+1} a_1 \leq a_{i+1}$.*

Proof. The condition $(SC)_i$ holds for all i with $1 \leq i \leq m$ by Proposition 9. Then Lemma 7 shows that Γ is of order (s, t), where $s = a_1 + 1$ and $t = \frac{b_1}{a_1+1}$.

(i) Take any $z \in B(x, y)$. It is sufficient to see that $A(y, z) \subseteq B(x, y)$. Since the condition $(SC)_i$ holds, there exists a strongly closed subgraph Λ of diameter i containing x and y. Suppose that there exists $w \in A(y, z) \setminus B(x, y)$. Then $w \in A(x, y) \subseteq \Lambda$ and thus $z \in A(y, w) \subseteq \Lambda$ which is a contradiction as $i + 1 = \partial_\Gamma(x, z) \leq d(\Lambda) = i$. Hence the desired result is proved.

(ii) Take any $y' \in C(x', z')$. Using the same manner as in the proof of the statement (i) we obtain $A(y', z') \subseteq B(x', y')$. Hence $C(x', z')$ is a coclique of size c_{i+1} and

$$\left(\bigcup_{y' \in C(x', z')} A(y', z') \right) \subseteq A(x', z').$$

The left-hand side is a disjoint union and the desired result is proved. □

Lemma 18. *Let Γ be a distance-regular graph of valency $k \geq 3$ and diameter d. Let m be an integer with $2 \leq m \leq d - 1$. Suppose that $b_m = a_1 + 1$. If the condition $(SC)_m$ holds and a strongly closed subgraph of diameter m is the Hamming graph $H(m, a_1 + 2)$, then $d = m + 1$ and Γ is the Hamming graph $H(m + 1, a_1 + 2)$.*

Proof. Let Λ be a strongly closed subgraph of diameter m which is the Hamming graph $H(m, a_1+2)$. Then $c_j = c_j(\Lambda) = j$ and $a_j = a_j(\Lambda) = ja_1$ for all j with $1 \leq j \leq m$. By putting $i = m+1$ in Proposition 16(i) we have $c_{m+1} \geq c_m - b_m + a_1 + 2 = (m+1)$. It follows, by Lemma 17(ii), that

$$(m+1)(1+a_1) = c_m + a_m + b_m \geq c_{m+1} + a_{m+1} \geq c_{m+1}(1+a_1) \geq (m+1)(1+a_1).$$

This implies $c_{m+1} = m+1, a_{m+1} = (m+1)a_1$ and $b_{m+1} = 0$. Hence $d = m+1$ and Γ is the Hamming graph $H(m+1, a_1+2)$ by Lemma 17 and Proposition 16(ii). The lemma is proved. □

We will use an inductive argument by the previous lemma. So we consider the cases $m = r+1$ and $m = r+2$ in the next proposition. Some informations for the graphs in the statement can be found in [4, p 221, Section 12.3, Section 13.2A, Section 13.4].

Proposition 19. *Let Γ be a distance-regular graph of valency $k \geq 3$, diameter d and $r = r(\Gamma)$.*

(i) *Assume that $r+2 \leq d$. If the condition $(SC)_{r+1}$ holds and $b_{r+1} = a_1 + 1$, then Γ is either the Pappus graph, the Coxeter graph, the dodecahedron, the Desargues graph, the Foster graph, the Biggs–Smith graph, or the Hamming graph $H(3, a_1+2)$.*

(ii) *Assume that $r+3 \leq d$. If the condition $(SC)_{r+2}$ holds and $b_{r+2} = a_1 + 1$, then Γ is either the Desargues graph, the Foster graph, the Biggs–Smith graph, or the Hamming graph $H(4, a_1+2)$.*

Proof. Lemma 17 shows that Γ is of order (s, t) with $s = a_1 + 1$ and $t = \frac{b_1}{a_1+1}$.

(i) Put $h = r+1$ in Proposition 11(i). Then $(b_r - b_{r+1}) = (t-1)(a_1+1)$ divides $b_r = t(a_1+1)$, and thus $t = 2$. If $s = 1$, then Γ has valency 3, and thus Γ is either one of the graphs in the statement by using the classification of distance-regular graphs of valency three (see [3], [4, Theorem 7.5.1]). Suppose $s > 1$. Let Ξ be a strongly closed subgraph of diameter $r+1$. Then Ξ is distance-regular with $k(\Xi) = c_{r+1} + a_{r+1} = k - b_{r+1} = 2(a_1+1)$ and thus Ξ is of order $(s, 1)$ with $s > 1$ and $d(\Xi) = r+1$. Hence $c_{r+1} = c_{r+1}(\Xi) = 2$, and $a_{r+1} = a_{r+1}(\Xi) = 2a_1$ by Proposition 16(iii). Since

$$3(1+a_1) = (c_{r+1} + a_{r+1} + b_{r+1}) \geq c_{r+2} + a_{r+2} \geq 3(1+a_1)$$

by Lemma 17(ii) and Proposition 16(iv), we have $c_{r+2} = 3$, $c_{r+2} = 3a_1$, $b_{r+2} = 0$ and $d = r+2$. Hence Γ is the Hamming graph $H(3, a_1+2)$ by Proposition 16(iv).

(ii) Put $m = r+2$ in Corollary 10. Suppose that $b_{r+1} = b_{r+2} = a_1 + 1$. Then Γ is either the doubled Odd graph $2O_3$ (the Desargues graph), the Foster graph or the Biggs–Smith graph. We assume that $b_{r+1} > b_{r+2}$. Then there is an integer $n \geq 2$ such that $b_{r+1} = n(a_1+1)$ by Lemma 17(i). Put $h = r+2$ in Proposition 11(i). Then we have $n = 2$ and $b_{r+1} = 2(a_1+1)$. Let Λ be a strongly closed subgraph of diameter $r+2$. Then Proposition 6 implies that Λ is distance-regular with $b_i(\Lambda) = k(\Lambda) - c_i(\Lambda) - a_i(\Lambda) = (c_{r+2} + a_{r+2}) - c_i - a_i = b_i - b_{r+2}$ for all i with $0 \leq i \leq r+2$. Apply Proposition 11(i) to Λ with $h = r+1$. Then $(b_r(\Lambda) - b_{r+1}(\Lambda)) = (t-2)(a_1+1)$ divides $b_r(\Lambda) = (t-1)(a_1+1)$. Thus $t = 3$ and Γ is of order $(a_1+1, 3)$. Moreover, Λ is of order $(a_1+1, 2)$ as $k(\Lambda) = c_{r+2} + a_{r+2} = k - b_{r+2} = 3(a_1+1)$. If $a_1 = 0$, then Γ has valency 4 and hence Γ is the Hamming graph $H(4, 2)$ by using the classification of distance-regular graphs with valency four [5]. If $a_1 > 0$, then Λ is the Hamming graph $H(3, a_1+2)$ by (i) since $b_{r+1}(\Lambda) = b_{r+1} - b_{r+2} = (a_1+1)$. Therefore Γ is the Hamming graph $H(4, a_1+2)$ by Lemma 18. □

We remark that each graph in Proposition 19(i) satisfies the condition $(SC)_{r+1}$ and any strongly closed subgraph of diameter $r + 1$ is a polygon except the case that Γ is the Hamming graph. Each graph in Proposition 19(ii) satisfies the condition $(SC)_{r+2}$ (see [11, Theorem 1.4–1.5] and [9, Section 2]). The Desargues graph is the doubled Odd graph $2O_3$. The doubled Odd graph $2O_k$ with diameter $d = 2k - 1$ satisfies the condition $(SC)_{d-1}$ and any strongly closed subgraph of diameter $d - 1$ is a non-regular distance-biregular graph (see Proposition 6 and [9, Section 2]). However the dodecahedron, the Foster graph and the Biggs–Smith graph do not satisfy the condition $(SC)_{d-1}$ by Corollary 10.

Proposition 20. *Let Γ be a distance-regular graph of valency $k \geq 3$, diameter d and $r = r(\Gamma)$ with $r + 4 \leq d$. Let m be an integer with $r + 3 \leq m \leq d - 1$. If the condition $(SC)_m$ holds and $b_m = a_1 + 1$, then Γ is either the doubled Odd graph $2O_k$, or the Hamming graph $H(d, a_1 + 2)$.*

Proof. We prove the assertion by induction on m. If $b_{m-1} = b_m$, then Γ is the doubled Odd graph $2O_k$ by Corollary 10 as $m \geq r + 3$ and $b_m = a_1 + 1$. We assume that $b_{m-1} > b_m = a_1 + 1$. Then $b_{m-1} = n(a_1 + 1)$ for some integer n with $n \geq 2$ by Lemma 17, and $n = 2$ by putting $h = m$ in Proposition 11(i). Let Λ be a strongly closed subgraph of diameter m. Then Λ is distance-regular with $b_i(\Lambda) = b_i - b_m$ for all i with $0 \leq i \leq m$. In particular, $b_{m-1}(\Lambda) = a_1 + 1$. Suppose that $m = r + 3$. Then Λ is one of the graphs in Proposition 19(ii). If Λ has valency 3, then $a_1 = 0$ and Γ has valency $k = c_m + a_m + b_m = 4$, and hence Γ is the doubled Odd graph $2O_4$ by using the classification of distance-regular graphs with valency four [5]. If Λ is the Hamming graph $H(4, a_1 + 2)$, then Γ is the Hamming graph $H(5, a_1 + 2)$ by Lemma 18. Suppose that $m \geq r + 4$. Then Λ is either the doubled Odd graph or the Hamming graph $H(m, a_1 + 2)$ by the inductive hypothesis. If Λ is the doubled Odd graph, then it contains a non-regular distance-biregular graph Ξ of diameter $m - 1$ as a strongly closed subgraph. Then Γ is a distance-regular graph which satisfies the condition $(SC)_{m-1}$ and contains a non-regular distance-biregular graph Ξ of diameter $m - 1$ as a strongly closed subgraph. Hence Γ is the doubled Odd graph $2O_k$ by Proposition 8 as $b_m = a_1 + 1$. If Λ is the Hamming graph $H(m, a_1 + 2)$, then Γ is the Hamming graph $H(m + 1, a_1 + 2)$ by Lemma 18. □

Proof of Theorem 1. Let $r = r(\Gamma) = \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$. Suppose that $d = r + 1$. Then $k = c_r + a_r + b_r = 2(1 + a_1)$ and Γ is of order $(a_1 + 1, 1)$. Hence Γ is either the Hamming graph $H(2, a_1 + 2)$ or the collinearity graph of a generalized $2d$-gon of order $(a_1 + 1, 1)$ with $d \in \{3, 4, 6\}$ by Proposition 16(iii). Suppose that $d \in \{r + 2, r + 3\}$. Then Γ is either the Pappus graph, the Coxeter graph, the Hamming graph $H(3, a_1 + 2)$, the Desargues graph or the Hamming graph $H(4, a_1 + 2)$ by Proposition 19. Suppose that $d \geq r + 4$. Then Γ is either the doubled Odd graph $2O_k$ or the Hamming graph $H(d, a_1 + 2)$ by Proposition 20. The theorem is proved. □

Theorem 1 shows that a distance-regular graph which satisfies the conditions (i) and (ii) in the introduction is a Hamming graph since other graphs in Theorem 1 have trivial strongly closed subgraphs of diameter 2 which are not distance-regular.

Proof of Theorem 2. (iii) ⇒ (ii) ⇒ (i). These are straightforward.

(i) ⇒ (iii). The Pappus graph and the Coxeter graph have valency 3. Strongly closed subgraphs of diameter $d - 1$ in the doubled Odd graph $2O_k$ are non-regular distance-biregular graphs. The desired result is proved by Theorem 1. □

Finally we prove Theorem 3. We need the following lemma.

Lemma 21. *Let Γ be a distance-regular graph of diameter d and $r = r(\Gamma)$ such that $r + 3 \le d$. Suppose that $b_{d-2} = 2b_{d-1}$ and the condition $(SC)_{d-1}$ holds. Then $b_{d-j} = j\, b_{d-1}$ holds for all j with $1 \le j \le d - r$. In particular, $r = 1$.*

Proof. We write $b := b_{d-1}$ and $e := d - r - 1$. We prove the first assertion by induction on j. The cases $j = 1$ and $j = 2$ follow by our assumption. Corollary 14 shows that there exists a positive integer f such that $(b_{d-2} - b_{d-1}) = f(b_{d-3} - b_{d-2})$ and

$$b^2 = (b_{d-2} - b_{d-1})^2 \le b_{d-1}(b_{d-3} - b_{d-2}) = b\left(\frac{b}{f}\right).$$

Hence we have $f = 1$ and $b_{d-3} = 3b$. The first assertion is proved if $d = r + 3$. Suppose that $r + 4 \le d$. Let $3 \le j < d - r$ and assume that $b_{d-i} = ib$ for $i = 1, \dots, j$. Then there exists a positive integer n such that $(b_{d-j} - b_{d-j+1}) = n(b_{d-j-1} - b_{d-j})$ and

$$b^2 = (b_{d-j} - b_{d-j+1})^2 \le (b_{d-j+1} - b_{d-j+2})(b_{d-j-1} - b_{d-j}) = b\left(\frac{b}{n}\right)$$

by putting $m = d - j + 1$ in Corollary 15 and our inductive hypothesis. Hence we obtain $n = 1$ and $b_{d-j-1} = (j+1)b$. The first assertion is proved. Put $j = 1$ and $h = d - 1$ in Proposition 12. Then

$$\prod_{i=1}^{d-2} \frac{b_i}{(b_i - b_{d-1})} = \left(\frac{(e+1)b}{eb}\right)^r \left(\frac{eb}{(e-1)b}\right) \cdots \left(\frac{3b}{2b}\right)\left(\frac{2b}{b}\right) = \frac{(e+1)^r}{e^{r-1}}$$

is an integer. The second assertion is proved. $\square$

Proof of Theorem 3. Let $s = a_1 + 1$. Then there exists an integer n such that $b_{d-1} = ns$ and $b_{d-i} = ins$ for all i with $1 \le i \le d-1$ by Lemma 17(i) and Lemma 21. Put $j = 0$ and $h = d-1$ in Proposition 12. Then

$$\begin{aligned}\prod_{i=0}^{d-2} \frac{b_i}{(b_i - b_{d-1})} &= \left(\frac{(d-1)ns + s}{(d-1)ns}\right)\left(\frac{(d-1)ns}{(d-2)ns}\right) \cdots \left(\frac{3ns}{2ns}\right)\left(\frac{2ns}{ns}\right)\\ &= \left(\frac{(d-1)n + 1}{n}\right)\end{aligned}$$

is an integer. Hence $n = 1$ and $b_{d-1} = a_1 + 1$. The desired result follows by Theorem 1. $\square$

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