On the domain invariance of countably condensing vector fields

In-Sook Kim

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, South Korea

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Abstract

Using homotopy theory, we give the domain invariance theorem for countably condensing vector fields, where the notion of countably condensing maps is due to Väth. A starting point of this investigation is that there is a symmetric characteristic set for a countably condensing map.

Keywords: Domain invariance; Countably condensing vector fields

1. Introduction

The famous domain invariance theorem for the Euclidean spaces $\mathbb{R}^n$ originates from Brouwer [1]. Schauder [13] extended this theorem to completely continuous vector fields in Banach spaces as follows:

Let $\Omega$ be a domain in a Banach space $E$ and $F: \Omega \to E$ a completely continuous map. If the vector field $f = id - F$ is injective, then its range $f(\Omega)$ is a domain in $E$.

There are two approaches to prove this: degree theory and homotopy theory. Granas [4] first established homotopy theory for compact vector fields in Banach spaces. With the aid of homotopy results stated in [7], Hahn [6] obtained the domain invariance theorem.
for condensing vector fields in quasicomplete metrizable locally convex spaces; see also [3,11].

It is known that the invariance of domain plays an important role in solving nonlinear equations; see [14]. In this point of view, it is natural to investigate this problem for a large class of countably condensing maps, roughly speaking, condensing on countable subsets of the space. The use of such countable subsets in fixed point theory was initiated by Daher [2] and developed by Mönch [10]. The notion of a countably condensing map with respect to a measure of noncompactness which will be introduced in the present paper is due to Väth [15]. A starting point of our investigation is that there is a symmetric characteristic set for a countably condensing map.

Our goal is to prove the domain invariance theorem for countably condensing vector fields in the strong sense, by using homotopy theory, where we mainly follow the basic idea of [6]. To this end we first give fundamental properties of countably condensing maps and then a homotopy extension theorem for countably condensing maps. Next, we show when the vector field of a countably condensing map is an open map. It is remarkable that our result implies many known results, including Schauder and Hahn; see [1,6,11,13].

For arbitrary topological vector spaces $X$ and $Y$, a continuous map $F : X \to Y$ is said to be compact if its range $F(X)$ is contained in a compact subset of $Y$. $F$ is said to be completely continuous if it is continuous and maps bounded subsets of $X$ into relatively compact sets.

For a subset $K$ of a topological vector space $E$, the closure, the boundary, the convex hull, and the closed convex hull of $K$ in $E$ are denoted by $\overline{K}$, $\partial K$, $\text{co } K$, and $\overline{\text{co }} K$, respectively.

We introduce the concepts of countably condensing maps due to Väth [15,16] and countably $k$-condensing maps in the strong sense for our purpose; see also [6–8].

**Definition 1.1.** Let $E$ be a topological vector space and $\mathcal{M}$ a collection of nonempty subsets of $E$ containing all precompact subsets of $E$ with the property that for any $M, N \in \mathcal{M}$, the sets $\overline{\text{co }} M$, $M \cup N$, $M + N$, $tM$ ($t \in \mathbb{R}$) and every subset of $M$ belong to $\mathcal{M}$. A function $\psi : \mathcal{M} \to [0, \infty]$ is called a measure of noncompactness on $E$ provided that it satisfies the following properties:

1. $\psi(\overline{\text{co }} M) = \psi(M)$;
2. (regularity) $\psi(M) = 0$ if and only if $M$ is precompact;
3. (semi-additivity) $\psi(M \cup N) = \max\{\psi(M), \psi(N)\}$;
4. (subadditivity) $\psi(M + N) \leq \psi(M) + \psi(N)$;
5. (homogeneity) $\psi(tM) = |t|\psi(M)$ for any real number $t$.

**Definition 1.2.** Let $E$ be a topological vector space, $Q$ a topological space, $X$ a nonempty subset of $E$, $\psi$ a measure of noncompactness on $E$, and $k$ a nonnegative real number. An upper semicontinuous set-valued map $H : X \times Q \rightrightarrows E$ is said to be countably condensing (with respect to $\psi$) provided that $H(X \times Q) \in \mathcal{M}$ and if $C$ is any countable subset of $X$ such that $\psi(C) \leq \psi(H(C \times Q))$, then $H(C \times Q)$ is relatively compact. $H$ is said to be countably $k$-condensing (with respect to $\psi$) if $H(X \times Q) \in \mathcal{M}$ and $\psi(H(C \times Q)) \leq k\psi(C)$ for each countable set $C$ in $X$. 
In particular, a continuous map \( F: X \to E \) is said to be countably \( k \)-condensing (with respect to \( \psi \)) if \( F(X) \in \mathcal{M} \) and \( \psi(F(C)) \leq k\psi(C) \) for each countable set \( C \) in \( X \). Analogously, \( F \) is defined to be countably condensing.

Moreover, a continuous map \( F: X \to E \) is said to be countably \( k \)-condensing in the strong sense if \( F(X) \in \mathcal{M} \) and \( \psi(F(\overline{C})) \leq k\psi(C) \) for each countable set \( C \) in \( X \) with \( \overline{C} \subset X \).

A special case of countably condensing maps \( F: X \to E \) can be found in [2], where the Hausdorff measure of noncompactness on \( E \) was considered.

An additional condition \( \overline{C} \subset X \) may be automatically dropped whenever \( X \) is a closed convex subset of \( E \). In this case, every countably \( k \)-condensing map \( F: X \to E \) in the strong sense is countably \( k \)-condensing. The above definitions reduce to the usual definitions of condensing and \( k \)-condensing maps if the condition holds for all sets in \( X \), in place of all countable sets \( C \); see [6–8].

In what follows we suppose that \( E \) is a quasicomplete metrizable locally convex topological vector space and \( \psi \) is a measure of noncompactness on \( E \).

2. Countably condensing maps

In this section we present a few of important facts concerning countably condensing maps which are mathematically interesting.

We begin with the following result on countably condensing maps which will be often needed. The corresponding statement for condensing maps is given in [6, Anmerkung 4].

**Proposition 2.1.** Let \( X \) be a closed subset of \( E \), and \( F: X \to E \) a countably condensing map. Then the set \( M := \{ z \in E : z = x - F(x), x \in X \} \) is closed in \( E \).

**Proof.** Fix \( z \in \overline{M} \). Let \( (z_n) \) be a sequence in \( M \) that converges to \( z \). For each \( n \in \mathbb{N} \), there exists an \( x_n \in X \) such that \( z_n = x_n - F(x_n) \). Since \( \{z_n : n \in \mathbb{N}\} \) is precompact and \( \psi \) is subadditive and regular, it follows that

\[
\psi(\{x_n : n \in \mathbb{N}\}) \leq \psi(\{z_n : n \in \mathbb{N}\}) + \psi(\{F(x_n) : n \in \mathbb{N}\}) = \psi(\{F(x_n) : n \in \mathbb{N}\}).
\]

Since \( F \) is countably condensing, the set \( \{F(x_n) : n \in \mathbb{N}\} \) is relatively compact which implies that \( \psi(\{x_n : n \in \mathbb{N}\}) = 0 \). Note that in a quasicomplete Hausdorff topological vector space every precompact set is relatively compact; see [12]. Since \( \{x_n : n \in \mathbb{N}\} \) is relatively compact and \( X \) is closed in \( E \), the sequence \( (x_n) \) has a subsequence \( (x_{n_k}) \) that converges to some point \( x \in X \). From the continuity of \( F \), we obtain that \( z = x - F(x) \in M \). Therefore, \( M \) is closed in \( E \). This completes the proof. \( \square \)

Let \( K \) be a closed convex subset of \( E \) and \( X \) a closed subset of \( K \). Let \( P_0(X, K) \) denote the collection of all countably condensing maps \( F: X \to K \) such that \( F \) has no fixed points, and \( Ph_0(X, K) \) the collection of all countably condensing homotopy \( H: X \times [0, 1] \to K \) such that \( H(x, t) \neq x \) for all \( (x, t) \in X \times [0, 1] \), respectively. In particular, if \( F \) and
$H$ are replaced by compact maps, we write $C_0(X, K)$ and $Ch_0(X, K)$ for $P_0(X, K)$ and $P_{h0}(X, K)$, respectively.

**Definition 2.2.** Two maps $F, G : X \to K$ are said to be homotopic in $P_0(X, K)$ if there exists a homotopy $H \in P_{h0}(X, K)$ such that $H(x, 0) = F$ and $H(x, 1) = G$ for all $x \in X$. It is denoted by $F \sim G$ in $P_0(X, K)$. In the same way, we say that $F$ and $G$ are homotopic in $C_0(X, K)$, denoted by $F \sim G$ in $C_0(X, K)$.

We give the following basic properties of countably condensing maps which will be used later. For the case of condensing maps, see [6, Anmerkung 5] and [8, Satz 5.2].

**Proposition 2.3.** Let $X$ be a closed subset of $E$. Then the following statements hold:

1. If $Q_1, Q_2$ are two closed subsets of a topological space $Q$ with $Q = Q_1 \cup Q_2$ and $H_i : X \times Q_i \to E$ ($i = 1, 2$) are two countably condensing maps such that $H_1(x, t) = H_2(x, t)$ for each $(x, t) \in X \times (Q_1 \cap Q_2)$, then the map $H : X \times Q \to E$ defined by
   $$H(x, t) := \begin{cases} H_1(x, t) & \text{for } (x, t) \in X \times Q_1, \\ H_2(x, t) & \text{for } (x, t) \in X \times Q_2, \end{cases}$$
   is countably condensing.
2. Let $F \in P_0(X, E)$ be a countably condensing map and $V$ a starshaped neighborhood of $0$ in $E$ such that $[x - F(x)] \cap V = \emptyset$ for all $x \in X$. Then any countably condensing map $G : X \to E$ satisfying $G(x) \in F(x) + V$ for $x \in X$ is homotopic to $F$ in $P_0(X, E)$.

**Proof.** (1) Let $C$ be any countable subset of $X$ such that $\psi(C) \leq \psi(H(C \times Q))$. Without loss of generality we may suppose that
   $$\max \{ \psi(H_i(C \times Q_i)) : i = 1, 2 \} = \psi(H_1(C \times Q_1)).$$
   From the semi-additivity of $\psi$ it follows that
   $$\psi(C) \leq \psi(H_1(C \times Q_1) \cup H_2(C \times Q_2)) = \max \{ \psi(H_1(C \times Q_1)), \psi(H_2(C \times Q_2)) \} = \psi(H_1(C \times Q_1)).$$
   Since $H_1$ is countably condensing, the set $H_1(C \times Q_1)$ is relatively compact and hence $\psi(H_1(C \times Q_1)) = 0 = \psi(H_2(C \times Q_2))$. Since $\psi$ is regular and $E$ is quasicomplete, $H_2(C \times Q_2)$ is relatively compact. Consequently, $H(C \times Q)$ is relatively compact. Therefore, $H$ is countably condensing.

(2) Let $F \in P_0(X, E)$ be a countably condensing map and $V$ a starshaped neighborhood of $0$ in $E$ such that
   $$[x - F(x)] \cap V = \emptyset \quad \text{for all } x \in X.$$ 
   Let $G : X \to E$ be any countably condensing map such that $G(x) \in F(x) + V$ for each $x \in X$. Define a map $H : X \times [0, 1] \to E$ by
   $$H(x, t) := tF(x) + (1 - t)G(x) \quad \text{for } (x, t) \in X \times [0, 1].$$
Since $V$ is starshaped with respect to 0, we have
\[ H(x, t) \in tF(x) + (1-t)(F(x) + V) \subset F(x) + V \quad \text{for all } (x, t) \in X \times [0, 1]. \]

To prove that $H$ is countably condensing, let $C$ be any countable set in $X$ such that $\psi(C) \subseteq \psi(H(C \times [0, 1]))$. We may suppose that
\[
\max \{ \psi(F(C)), \psi(G(C)) \} = \psi(F(C)).
\]
The definition of $\psi$ implies that
\[
\psi(C) \subseteq \psi(H(C \times [0, 1])) \leq \psi(\text{co}(F(C) \cup G(C))) \leq \max \{ \psi(F(C)), \psi(G(C)) \} = \psi(F(C)).
\]
Since $F$ is countably condensing, the set $F(C)$ is relatively compact and so $\psi(F(C)) = 0$. From $\psi(H(C \times [0, 1])) = 0$ it follows that $H(C \times [0, 1])$ is relatively compact. Thus, $H$ is countably condensing. Moreover, we have $x \neq H(x, t)$ for all $(x, t) \in X \times [0, 1]$. In fact, if $x_0 = H(x_0, t_0)$ for some $(x_0, t_0) \in X \times [0, 1]$, then $x_0 \in F(x_0) + V$, which contradicts the choice of $V$. We conclude that $F$ and $G$ are homotopic in $P_0(X, E)$. This completes the proof. \( \square \)

Recall that $S$ is called a characteristic set for $H : X \times [0, 1] \to E$ if $S$ is a closed convex subset of $E$, $X \cap S \neq \emptyset$, $H((X \cap S) \times [0, 1]) \subset S$ and $H((X \cap S) \times [0, 1])$ is relatively compact.

For completeness, we prove the following result due to Váth [15,16] which says that there is a symmetric characteristic set for a countably condensing map. A similar argument is used in the proof of Lemma 3.1 in [9].

**Proposition 2.4.** Let $X$ be a closed, symmetric and nonempty subset of $E$. If $H : X \times [0, 1] \to E$ is a countably condensing map, there exists a symmetric, convex and compact subset $S$ of $E$ such that $X \cap S \neq \emptyset$, $0 \in S$, and $H((X \cap S) \times [0, 1])$ is a subset of $S$.

**Proof.** Fix $x_0 \in X$. Let
\[ \Sigma := \{ B \subset E : B = \overline{\text{co}}B, B = -B, \{ x_0, -x_0, 0 \} \subset B, \ H((X \cap B) \times [0, 1]) \subset B \}. \]

Then $\Sigma$ is nonempty because $E \in \Sigma$. Set $S := \bigcap_{B \in \Sigma} B$ and
\[ S_1 := \overline{\text{co}} \left[ H \left( (X \cap S) \times [0, 1] \right) \cup -H \left( (X \cap S) \times [0, 1] \right) \cup \{ x_0, -x_0, 0 \} \right]. \]

Since $S \in \Sigma$, we have $S_1 \subset S$ and so $H((X \cap S_1) \times [0, 1]) \subset H((X \cap S) \times [0, 1]) \subset S_1$ and therefore $S_1 \in \Sigma$. It follows from definition of $S$ that $S \subset S_1$ and hence
\[ S \subseteq \overline{\text{co}} \left[ H \left( (X \cap S) \times [0, 1] \right) \cup -H \left( (X \cap S) \times [0, 1] \right) \cup \{ x_0, -x_0, 0 \} \right]. \]

Thus, $S$ is a closed, convex and symmetric subset of $E$ and $H((X \cap S) \times [0, 1]) \subset S$. It remains to show that $S$ is compact. To prove this, we use Theorem 3.1 of [16]. Consider a set-valued map $G : X \times [0, 1] \to E$ given by $G(x, t) := H(x, t) \cup -H(x, t)$ for $(x, t) \in X \times [0, 1]$. Then $G$ is upper semicontinuous because $H$ and $-H$ are continuous. Let $C$ be any countable subset of $X \cap S$ such that $X \cap \text{co}(G(C \times [0, 1]) \cup \{ x_0, -x_0, 0 \}) \subset \overline{C} \subset X \cap \overline{\text{co}}(G(C \times [0, 1]) \cup \{ x_0, -x_0, 0 \})$. Then the definition of $\psi$ implies...
\[ \psi(C) \leq \psi(G(C \times [0,1]) \cup \{x_0, -x_0, 0\}) \]
\[ \leq \psi(H(C \times [0,1]) \cup -H(C \times [0,1]) \cup \{x_0, -x_0, 0\}) \]
\[ = \max\{\psi(H(C \times [0,1]), \psi(-H(C \times [0,1]), 0)\} \]
\[ = \psi(H(C \times [0,1])). \]

Since \( H \) is countably condensing, the set \( H(C \times [0,1]) \) is relatively compact. From \( \psi(C) = 0 \) it follows that \( C \) is precompact. Hence the compactness of \( S \) follows from Theorem 3.1 of [16]. This completes the proof. \( \square \)

We show that a result of Hahn does hold for countably condensing maps in the strong sense, by following basic lines of proof in [6, Hilfssatz 2].

**Proposition 2.5.** Let \( W \) be a closed, convex, symmetric and bounded neighborhood of \( 0 \) in \( E \) and \( k \) a nonnegative real number. If \( F : W \rightarrow E \) is a countably \( k \)-condensing map in the strong sense, then the map \( H : \partial W \times [0,1] \rightarrow E \) defined by
\[ H(x,t) := F\left(\frac{1}{1+t} x \right) - F\left(\frac{-t}{1+t} x \right) \]
for \( x \in \partial W \) and \( t \in [0,1] \)
is countably \( k \)-condensing.

**Proof.** Let \( C \) be any countable subset of \( \partial W \). Let \( \varepsilon \) be an arbitrary positive real number. Then we have
\[ \psi(H(C \times [0,1])) = \max\{\psi(H(C \times \left[\frac{j}{n}, \frac{j+1}{n}\right])) : j = 0, 1, \ldots, n-1\} \]
and
\[ H\left(C \times \left[\frac{j}{n}, \frac{j+1}{n}\right]\right) \subset F\left(\overline{G}\left(\frac{n}{n+j} C \cup \{0\}\right)\right) - F\left(-\overline{G}\left(\frac{1}{n+j+1} C \cup \{0\}\right)\right). \]
Since \( F \) is countably \( k \)-condensing in the strong sense, it follows from the definition of \( \psi \) that
\[ \psi(H(C \times \left[\frac{j}{n}, \frac{j+1}{n}\right])) \leq k\left(\frac{n}{n+j} + \frac{j+1}{n+j+1}\right) \psi(C) \leq \psi(C) < k(1 + \varepsilon) \psi(C) \]
for \( j = 0, 1, \ldots, n-1 \).

As the inequality \( \psi(H(C \times [0,1])) < (1 + \varepsilon)k \psi(C) \) holds for all \( \varepsilon > 0 \), we obtain that \( \psi(H(C \times [0,1])) \leq k \psi(C) \). Therefore, \( H \) is countably \( k \)-condensing. This completes the proof. \( \square \)

### 3. Homotopy theory

This section is devoted to homotopy theory for countably condensing maps. Granas [4] considered homotopy of compact vector fields in Banach spaces.
**Definition 3.1.** Let \( X \) and \( Y \) be two closed subsets of \( E \) such that \( X \) is a subset of \( Y \). Let \( F : X \rightarrow E \) be a countably condensing map which is fixed-point-free. If \( S \) is a characteristic set for the map \( F \), then \( F \) is said to be *approximately inessential* with respect to \((X \cap S, Y \cap S, S)\) provided that for every neighborhood \( V \) of 0 in \( E \) there exist a finite-dimensional subspace \( E_V \) of \( E \) with \( S \cap E_V \neq \emptyset \) and a compact fixed-point-free map \( F_V : X \cap S \rightarrow S \cap E_V \) such that \( F_V(x) \in F(x) + V \) for all \( x \in X \cap S \) and the restriction \( F_V|_{X \cap S \cap E_V} \) has a compact fixed-point-free extension \( \tilde{F}_V : Y \cap S \cap E_V \rightarrow S \cap E_V \).

For condensing maps, the following statement is known in [7, Satz 1].

**Lemma 3.2.** Let \( E, X, Y \) and \( F \) be as in Definition 3.1. If \( F \) has a countably condensing fixed-point-free extension \( \tilde{F} : Y \rightarrow E \), then \( F \) is approximately inessential with respect to \((X \cap S, Y \cap S, S)\) for every characteristic set \( S \) for \( \tilde{F} \) with \( X \cap S \neq \emptyset \).

**Proof.** Let \( S \) be a characteristic set for \( \tilde{F} \) with \( X \cap S \neq \emptyset \) and \( V \) a neighborhood of 0 in \( E \). Since \( \tilde{F}|_{X \cap S} : Y \cap S \rightarrow S \) is a compact fixed-point-free map, there exist a finite-dimensional subspace \( E_V \) of \( E \) with \( S \cap E_V \neq \emptyset \) and a compact fixed-point-free map \( \tilde{F}_V : Y \cap S \rightarrow S \cap E_V \) such that \( \tilde{F}_V(x) \in \tilde{F}(x) + V \) for all \( x \in Y \cap S \); see [5, Hilfssatz 5]. Hence \( \tilde{F}_V(x) \in F(x) + V \) for all \( x \in X \cap S \) and the restriction \( \tilde{F}_V|_{X \cap S \cap E_V} \) is a compact fixed-point-free extension of \( \tilde{F}_V|_{X \cap S \cap E_V} \). Since \( S \) is also a characteristic set for \( F, F \) is approximately inessential with respect to \((X \cap S, Y \cap S, S)\). This completes the proof. \( \square \)

Now we give the following homotopy extension theorem for countably condensing maps, where we proceed the proof by the same argument as in [7, Satz 2].

**Lemma 3.3.** Let \( X \) and \( Y \) be two closed subsets of \( E \) such that \( X \) is a subset of \( Y \). Let \( F_1, F_2 : X \rightarrow E \) be two countably condensing maps which are homotopic in \( P_0(X, E) \) and \( S \) a characteristic set for the homotopy \( H \). If \( F_1 \) is approximately inessential with respect to \((X \cap S, Y \cap S, S)\), then \( F_2 \) is also approximately inessential with respect to \((X \cap S, Y \cap S, S)\).

**Proof.** Suppose that \( F_1 \) is approximately inessential with respect to \((X \cap S, Y \cap S, S)\). Let \( U \) be any neighborhood of 0 in \( E \). In view of Proposition 2.1, we can choose a starshaped symmetric neighborhood \( V \) of 0 in \( E \) with \( V \subset U \) such that

\[
[x - F_i(x)] \cap V = \emptyset \quad \text{for all } x \in X \cap S \text{ and } i = 1, 2.
\]

Since \( F_1 \) is approximately inessential with respect to \((X \cap S, Y \cap S, S)\), there exist a finite-dimensional subspace \( E_1 \) of \( E \) with \( S \cap E_1 \neq \emptyset \) and a compact map \( G_1 \in C_0(X \cap S, S \cap E_1) \) such that \( G_1(x) \in F_1(x) + V \) for all \( x \in X \cap S \) and \( G_1|_{X \cap S \cap E_1} \) has an extension \( \tilde{G}_1 \in C_0(Y \cap S \cap E_1) \). As in the proof of Lemma 3.2, it follows from \( F_2 \in C_0(X \cap S, S) \) that there are a finite-dimensional subspace \( E_2 \) of \( E \) with \( S \cap E_2 \neq \emptyset \) and a compact map \( G_2 \in C_0(X \cap S, S \cap E_2) \) such that \( G_2(x) \in F_2(x) + V \) for all \( x \in X \cap S \).

It is obvious that \( F_1 \sim G_1 \) and \( F_2 \sim G_2 \) in \( C_0(X \cap S, S) \) and by hypothesis \( F_1 \sim F_2 \) in \( C_0(X \cap S, S) \); see [7, Hilfssatz 1]. Since \( G_1 \sim G_2 \) in \( C_0(X \cap S, S) \) and \( G_1(X \cap S) \subset E_1 \) and
$G_2(X \cap S) \subset E_2$, there exists a finite-dimensional subspace $E_0$ of $E$ with $(E_1 \cup E_2) \subset E_0$ such that

$$G_1|_{X \cap S \cap E_0} \sim G_2|_{X \cap S \cap E_0} \text{ in } C_0(X \cap S \cap E_0, S \cap E_0);$$

see [7, Hilfssatz 1]. Since $G_1|_{X \cap S \cap E_1}$ has an extension $\bar{G}_1 \in C_0(Y \cap S \cap E_1, S \cap E_1)$, we know that $G_1|_{X \cap S \cap E_0}$ has an extension $\bar{G}_1 \in C_0(Y \cap S \cap E_0, S \cap E_0)$; see [7, Hilfssatz 4]. The homotopy extension theorem for finite-dimensional spaces given in [7, Hilfssatz 2] implies that $G_2|_{X \cap S \cap E_0}$ has an extension $\bar{G}_2 \in C_0(Y \cap S \cap E_0, S \cap E_0)$. Moreover, $G_2: X \cap S \to S \cap E_0$ is a compact fixed-point-free map and $G_2(x) \in F(x) + U$ for all $x \in X \cap S$. Therefore, $F_2$ is approximately inessential with respect to $(X \cap S, Y \cap S, S)$. This completes the proof.

Applying Borsuk’s theorem, we show that an odd fixed-point-free countably condensing map is not approximately inessential, which extends Hilfssatz 1 of [6].

**Lemma 3.4.** Let $W$ be an open, symmetric and bounded neighborhood of 0 in $E$. Let $F: \partial W \to E$ be an odd countably condensing map which is fixed-point-free. If $S$ is a symmetric characteristic set for $F$ with $0 \in S$, then $F$ is not approximately inessential with respect to $(\partial W \cap S, \overline{W} \cap S, S)$.

**Proof.** Assume that there is a symmetric characteristic set $S$ for $F$ with $0 \in S$ such that $F$ is approximately inessential with respect to $(\partial W \cap S, \overline{W} \cap S, S)$. Choose a neighborhood $V$ of 0 in $E$ such that

$$x - F(x) \notin V \quad \text{for all } x \in \partial W \cap S.$$

Let $U$ be a symmetric convex neighborhood of 0 in $E$ such that $U + U \subset V$. By our assumption, there exist a finite-dimensional subspace $E_0$ of $E$ with $S \cap E_0 \neq \emptyset$ and a compact fixed-point-free map $F_U: \partial W \cap S \to S \cap E_0$ such that $F_U(x) \in F(x) + U$ for all $x \in \partial W \cap S$ and the restriction $F_U|_{\partial W \cap S \cap E_0}$ has a compact fixed-point-free extension $\tilde{F}_U: \overline{W} \cap S \cap E_0 \to S \cap E_0$.

Set $W_0 := W \cap E_0$, $S_0 := S \cap E_0$, and $\partial_0 W_0 := \partial W_0 \cap E_0$. Consider a map $F_0 : \partial_0 W_0 \cap S_0 \to S_0$ defined by

$$F_0(x) := \frac{1}{2} \left[ F_U(x) - F_U(-x) \right] \quad \text{for } x \in \partial_0 W_0 \cap S_0.$$

Then $F_0$ is clearly odd and compact. For every $x \in \partial_0 W_0 \cap S_0$, it follows from $F(x) + F(-x) = 0$ that

$$F_0(x) - F_U(x) = \frac{1}{2} \left[ F(x) - F_U(x) \right] + \frac{1}{2} \left[ F(-x) - F_U(-x) \right] \in \frac{1}{2} \overline{U} + \frac{1}{2} \overline{U} = U.$$

Moreover, we have $x - F_U(x) \notin U$ for all $x \in \partial_0 W_0 \cap S_0$. Indeed, if $x_0 \in \partial_0 W_0 \cap S_0$, then $x_0 - F(x_0) = (x_0 - F_U(x_0)) + (F_U(x_0) - F(x_0)) \in U + U \subset V$, which contradicts the choice of $V$. Obviously, $F_U|_{\partial_0 W_0 \cap S_0}$ and $F_0$ are homotopic in $C_0(\partial_0 W_0 \cap S_0, S_0)$. As in the proof of Lemma 3.3, since $\tilde{F}_U : \overline{W}_0 \cap S_0 \to S_0$ is a compact fixed-point-free extension of $F_U|_{\partial_0 W_0 \cap S_0}$, there exists a continuous fixed-point-free extension $\tilde{F}_0 : \overline{W}_0 \cap S_0 \to S_0$ of $F_0$, where $W_0$ is an open, symmetric and bounded neighborhood
of 0 in $E_0$ and $S_0$ is a closed, symmetric and convex subset of $E_0$ with $0 \in S_0$. This contradicts Borsuk’s theorem for finite-dimensional spaces because $F_0$ is odd on $\partial_0W_0 \cap S_0$; see, e.g., [3, Corollary 4.1]. This completes the proof. □

4. The domain invariance theorem

Using several results in the previous sections, we can prove the following result concerning countably $k$-condensing maps which is a generalization of Satz 4 of [6].

**Theorem 4.1.** Let $E$ be a quasicomplete metrizable locally convex topological vector space and $W$ an open, convex, symmetric and bounded neighborhood of 0 in $E$. Let $F : \overline{W} \to E$ be a countably $k$-condensing map in the strong sense, with $k \in [0, 1)$. Suppose that $f : \overline{W} \to E$ is the vector field of $F$, that is, $f(x) = x - F(x)$ for $x \in \overline{W}$, such that for $x_1, x_2 \in \overline{W}$,

$$f(x_1) = f(x_2) \implies x_1 - x_2 \notin \partial W.$$  

Then $f(W)$ contains a neighborhood of $f(0)$.

**Proof.** Let a map $F_0 : \overline{W} \to E$ be defined by

$$F_0(x) := F(x) - F(0) \quad \text{for } x \in \overline{W}.$$  

Then $F_0$ is countably $k$-condensing and $x - F_0(x) = f(x) - f(0) \neq 0$ for all $x \in \partial W$, by assumption on $f$. By Proposition 2.1, there exists a starshaped symmetric neighborhood $V$ of 0 in $E$ such that

$$\{x - F_0(x)\} \cap V = \emptyset \quad \text{for all } x \in \partial W.$$  

Now we will claim that $f(0) + V \subset f(W)$. Let $y$ be an arbitrary point of $f(0) + V$. Consider a countably $k$-condensing map $F_y : \partial W \to E$ given by

$$F_y(x) := F(x) + y \quad \text{for } x \in \partial W.$$  

Then $F_y(x) - F_0(x) = y - f(0) \in V$ for each $x \in \partial W$. Notice that in a quasicomplete Hausdorff topological vector space every countably $k$-condensing map with $k \in [0, 1)$ is countably condensing. In view of Proposition 2.3, since $F_y, F_0$ are countably condensing maps, there is a countably condensing map $H_1 : P h_0(\partial W, E)$ such that $H_1(x, 0) = F_y(x)$ and $H_1(x, 1) = F_0(x)$ for all $x \in \partial W$. Let $H_2 : \partial W \times [0, 1] \to E$ be defined by

$$H_2(x, t) := F\left(\frac{1}{1 + t}\cdot x\right) - F\left(-\frac{t}{1 + t}\cdot x\right) \quad \text{for } x \in \partial W \text{ and } t \in [0, 1].$$  

By Proposition 2.5, $H_2$ is a countably $k$-condensing map and has no fixed points on $\partial W$. In fact, if $H_2(x_0, t_0) = x_0$ for some $x_0 \in \partial W$ and $t_0 \in [0, 1]$, then $f(\frac{1}{1 + t_0}\cdot x_0) = f(-\frac{t_0}{1 + t_0}\cdot x_0)$ implies $x_0 \notin \partial W$ by assumption on $f$, which is impossible. Now define a homotopy $H : \partial W \times [0, 1] \to E$ by

$$H(x, t) := \begin{cases} H_1(x, 2t) & \text{for } x \in \partial W \text{ and } t \in [0, \frac{1}{2}], \\ H_2(x, 2t - 1) & \text{for } x \in \partial W \text{ and } t \in [\frac{1}{2}, 1]. \end{cases}$$
Then it is clear that $H$ is countably condensing by Proposition 2.3 and $H$ has no fixed points. Proposition 2.4 says that there is a symmetric characteristic set $S$ for $H$ with $0 \in S$. Let $G : \partial W \to E$ be defined by

$$G(x) := F\left(\frac{x}{2}\right) - F\left(-\frac{x}{2}\right)$$

for $x \in \partial W$.

Then $G$ is an odd countably condensing map and $G$ is homotopic to $Fy$ in $P_0(\partial W, E)$. Lemma 3.4 implies that $G$ is not approximately inessential with respect to $(\partial W \cap S, W \cap S, S)$. By Lemma 3.3, $Fy$ is also not approximately inessential with respect to $(\partial W \cap S, W \cap S, S)$. The map $\tilde{F}_y : W \to E$ given by $\tilde{F}_y(x) = F(x) + y$ for $x \in W$ is a countably condensing extension of $Fy$ and $S$ is a characteristic set for $\tilde{F}_y$. Since $Fy$ has no fixed points on $\partial W$, it follows from Lemma 3.2 that there exists a point $x_0 \in W$ such that $\tilde{F}_y(x_0) = x_0$. We conclude that $y = f(x_0) \in f(W)$. This completes the proof.

**Corollary 4.2.** Under the hypotheses of Theorem 4.1, if $F : W \to E$ is a $k$-condensing map, the same conclusion follows.

**Proof.** For each countable set $C$ in $W$, it follows from $co C \subset W$ that

$$\psi(F(co C)) \leq k\psi(co C) = k\psi(C).$$

Thus, $F$ is a countably $k$-condensing map in the strong sense. Apply Theorem 4.1.

Now we apply Theorem 4.1 to find conditions sufficient for the vector field of a countably $k$-condensing map to be open, that is, to map open sets onto open sets. For the case of condensing maps, we refer to Hahn [6, Satz 5]; see also [3, Theorem 9.5].

Recall that a map $f : \Omega \to E$ is said to be locally injective if for every $x \in \Omega$ there exists a neighborhood $U$ of $x$ such that the restriction $f|_U$ is injective.

**Theorem 4.3.** Let $E$ be a quasicomplete metrizable locally convex topological vector space, $\Omega$ an open subset of $E$, and $k \in [0, 1)$. Let $F : \Omega \to E$ be a countably $k$-condensing map in the strong sense. If the corresponding vector field $f$ of $F$ is locally injective, then $f$ is an open map.

**Proof.** It suffices to show that for every $x_0 \in \Omega$ the set $f(\Omega)$ contains a neighborhood of $y_0 = f(x_0)$. Passing to $\Omega - x_0$ and $f(x) = f(x + x_0) - f(x_0)$ for $x \in \Omega - x_0$, if necessary, we may suppose that $x_0 = 0$ and $f(0) = 0$. We can choose an open ball $B_0 = B(0; r)$ with center $0$ and radius $r > 0$ such that $\overline{B}_0 \subset \Omega$ and $f|_{\overline{B}_0}$ is injective. For $x_1, x_2 \in \overline{B}_0$,

$$f(x_1) = f(x_2) \quad \text{implies} \quad x_1 - x_2 = 0 \notin \partial B_0.$$  

Applying Theorem 4.1 to the restriction $f|_{\overline{B}_0}$ of $f$, the set $f(B_0)$ contains a neighborhood of $f(0)$. This completes the proof.

**Corollary 4.4.** Let $\Omega$ be an open subset of a Banach space $E$, $k \in [0, 1)$, and $F : \Omega \to E$ a $k$-condensing map. If $f = \text{id} - F$ is locally injective, then $f$ is an open map.
**Proof.** As in the proof of Theorem 4.3, a similar argument proves the case of \(-k\)-condensing maps by using Corollary 4.2, instead of Theorem 4.1.

**Corollary 4.5.** Let \( \Omega \) be an open set in a Banach space \( E \) and \( F : \Omega \to E \) a completely continuous map. If \( f = \text{id} - F \) is locally injective, then \( f \) is an open map.

**Proof.** Note that the Kuratowski measure \( \alpha \) of noncompactness defined by

\[
\alpha(A) = \inf\{d > 0 : A \text{ admits a finite cover by sets of diameter } \leq d\}
\]

for bounded sets \( A \) in \( E \) is a measure of noncompactness on \( E \) in the sense of Definition 1.1; see [3]. For each bounded set \( A \) in \( \Omega \) and \( k \in [0, 1) \), we have

\[
\alpha(F(A)) = 0 \leq k\alpha(A)
\]

because \( F(A) \) is relatively compact. Thus, \( F \) is a \( k \)-condensing map with respect to \( \alpha \). Corollary 4.4 is now applicable.

**Corollary 4.6.** Let \( \Omega \) be an open set in \( \mathbb{R}^n \) and \( F : \Omega \to \mathbb{R}^n \) a continuous map. If \( f = \text{id} - F \) is locally injective, then \( f \) is an open map.

**Proof.** Observe that for finite-dimensional Banach spaces continuous and completely continuous maps are the same whenever the domain of definition is closed. This means that the restriction of \( F \) to the closed bounded set is completely continuous and hence \( k \)-condensing with respect to \( \alpha \), with \( k \in [0, 1) \). As in the proof of Theorem 4.3, a similar argument proves the case of completely continuous maps by using Corollary 4.2.

Finally we give the domain invariance theorem for countably \( k \)-condensing maps in the strong sense which is an immediate consequence of Theorem 4.3.

**Theorem 4.7.** Let \( \Omega \) be a domain in a quasicomplete metrizable locally convex topological vector space \( E \), \( k \in [0, 1) \), and \( F : \Omega \to E \) a countably \( k \)-condensing map in the strong sense. If the vector field \( f = \text{id} - F \) is locally injective, then \( f(\Omega) \) is a domain in \( E \).

**Corollary 4.8.** Let \( \Omega \) be a domain in a Banach space \( E \) and \( F : \Omega \to E \) a completely continuous map. If \( f = \text{id} - F \) is locally injective, then \( f(\Omega) \) is a domain in \( E \).

**Corollary 4.9.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and \( F : \Omega \to \mathbb{R}^n \) a continuous map. If \( f = \text{id} - F \) is locally injective, then \( f(\Omega) \) is a domain in \( \mathbb{R}^n \).

Thus, our result implies many known results, including Schauder and Brouwer; see [1,3,6,13].

**References**