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Artificial boundary conditions for parabolic Volterra integro-differential equations on unbounded two-dimensional domains

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Abstract

In this paper we study the numerical solution of parabolic Volterra integro-differential equations on certain unbounded two-dimensional spatial domains. The method is based on the introduction of a feasible artificial boundary and the derivation of corresponding artificial (fully transparent) boundary conditions. Two examples illustrate the application and numerical performance of the method.

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Keywords: Partial Volterra integro-differential equation; Unbounded spatial domain; Artificial boundary conditions; Numerical solution

1. Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be a semi-infinite strip domain with boundary $\Gamma = \Gamma_i \cup \Gamma_U \cup \Gamma_L$ (as shown in Fig. 1). Γ_U and Γ_L are assumed to be parallel.

Consider the following initial-boundary-value problem for a parabolic equation with memory term

$$\frac{\partial u}{\partial t} + \int_0^t k(x, t-\tau)u(x,\tau)\,\mathrm{d}\tau = \nabla(\alpha(x)\nabla u) - \beta(x)u + f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

$$u = g(x,t), \quad (x,t) \in \Gamma \times (0,T], \tag{1.2}$$

$$u(x,0) = u_0(x) \quad x \in \Omega, \tag{1.3}$$

$$u(x,t) \to 0 \quad \text{as } x_1 \to +\infty. \tag{1.4}$$

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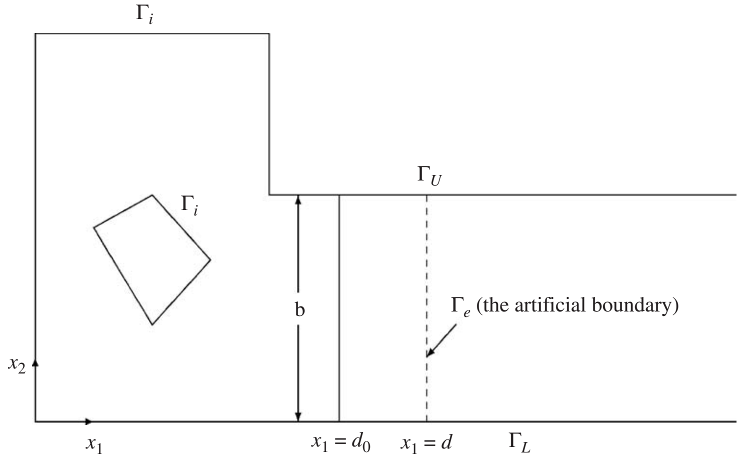

Fig. 1. Unbounded domain Ω and artificial boundary Γ_e.

We assume that:

(i) $\alpha(x) - 1 \geqslant 0$, $\beta(x) - \beta_0 \geqslant 0$ (β_0 is a non-negative constant), and $u_0(x)$ has compact support;
$\mathrm{Supp}\{\alpha(x) - 1\} \subset \bar{\Omega}_0 := \{x | x \in \bar{\Omega} \text{ and } x_1 \leqslant d_0\}$,

$\mathrm{Supp}\{\beta(x) - \beta_0\} \subset \bar{\Omega}_0$,

$\mathrm{Supp}\{u_0(x)\} \subset \bar{\Omega}_0$.

(ii) $f(x, t)$ and $g(x, t)$ have compact support:
$\mathrm{Supp}\{f\} \subset \bar{\Omega}_0 \times [0, T]$ and $\mathrm{Supp}\{g\} \subset \bar{\Omega}_0 \times [0, T]$.

(iii) $k(x, t) \equiv k_0(t)$ for $x_1 \geqslant d_0$.

In order to solve this problem numerically we introduce an artificial boundary $\Gamma_e \times [0, T]$ defined by

$$\Gamma_e := \{x = (x_1, x_2) \in \Omega : x_1 = d,\ 0 \leqslant x_2 \leqslant b,\ d \geqslant d_0\}.$$

This artificial boundary divides the domain $\bar{\Omega} \times [0, T]$ into two parts, the *bounded* part $\bar{\Omega}_i \times [0, T]$ and the *unbounded* part $\Omega_e \times [0, T]$

$$\Omega_i = \{x | x \in \Omega \text{ and } x_1 < d\}, \quad \Omega_e = \Omega \backslash \overline{\Omega}_i.$$

Our aim is to present a feasible and computationally effective numerical scheme for the approximate solution of the problem (1.1)–(1.4) on the bounded domain $\bar{\Omega}_i \times [0, T]$. This hinges on the derivation of a suitable artificial boundary condition on the given artificial boundary $\Gamma_e \times [0, T]$.

The artificial boundary method was introduced and analyzed for elliptic problems in [6,7]; see also [8,3]. In [4,5], these artificial boundary techniques were extended to the heat equation and related parabolic PDEs, and their approach was subsequently generalized [9] to one-dimensional "non-local" parabolic equations containing a memory term given by a (regular or weakly singular) Volterra integral operator.

The purpose of the present paper is to describe the computational form of the artificial boundary method for parabolic Volterra integro-differential equations of the form (1.1) on unbounded two-dimensional spatial domains given essentially by a semi-infinite strip, and to illustrate its numerical performance. It will be seen in Sections 2 and 3 that passing from one to two (or more) spatial dimensions is not trivial (compare also [7,8,4]).

The content of the paper is as follows. In Section 2 we derive the corresponding transparent artificial boundary condition on $\Gamma_e \times [0, T]$, significantly extending the approach in [9]. The reduction of the original problem (1.1)–(1.4) to the bounded domain $\Omega_i \times [0, T]$ is presented in Section 3. Here, we also state and prove a first result dealing with the

(L^2-)convergence of the numerical scheme. Section 4 contains two numerical examples illustrating the effectiveness and accuracy of our method.

The mathematical foundation (convergence analysis; a priori and a posteriori error estimates for the spatially semidiscretized problem and its temporally (fully) discretized counterpart) of the artificial boundary methods for one-dimensional and two-dimensional initial-boundary-value problems of the form (1.1)–(1.4), and resulting adaptive versions, will be presented in a forthcoming sequel to this paper (see also Section 5).

2. The artificial boundary conditions

We consider the restriction of the original problem (1.1)–(1.4) on the domain $\Omega_e \times [0, T]$. By the assumptions (i)–(iii) (cf. Section 1), $u(x, t)$ has to satisfy

$$\frac{\partial u}{\partial t} + \int_0^t k_0(t-\tau)u(x,\tau)\,\mathrm{d}\tau = \Delta u - \beta_0 u, \quad x \in \Omega_e, \ 0 \leqslant t \leqslant T, \tag{2.1}$$

$$u|_{t=0} = 0, \quad d \leqslant x_1 \leqslant +\infty, \ 0 \leqslant x_2 \leqslant b, \tag{2.2}$$

$$u = 0, \quad d \leqslant x_1 \leqslant +\infty, \ x_2 = b \text{ or } x_2 = 0, \tag{2.3}$$

$$u(x,t) \to 0 \quad \text{when } x_1 \to +\infty. \tag{2.4}$$

The problem (2.1)–(2.4) is an incompletely posed problem; it might have many solutions.

Let $u(x, t)$ be a solution of the problem (2.1)–(2.4) possessing the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x_1,t)\sin\left(\frac{n\pi}{b}x_2\right), \tag{2.5}$$

where u_n is given by

$$u_n(x_1,t) = \frac{2}{b}\int_0^b u(x_1,y_2,t)\sin\left(\frac{n\pi}{b}y_2\right)\mathrm{d}y_2. \tag{2.6}$$

Then $u_n(x_1, t)$ has to satisfy

$$\begin{aligned}
&\frac{\partial u_n}{\partial t} + \int_0^t k_0(t-\tau)u_n(x_1,\tau)\,\mathrm{d}\tau = \frac{\partial^2 u_n}{\partial x_1^2} - \beta_n u_n, \quad d < x_1 < +\infty, \ 0 < t \leqslant T,\\
&u_n|_{t=0} = 0, \quad d \leqslant x_1 \leqslant +\infty,\\
&u_n \to 0 \quad \text{as } x_1 \to +\infty,
\end{aligned}$$

where

$$\beta_n = \beta_0 + \left(\frac{n\pi}{b}\right)^2, \quad n = 1, 2, \ldots. \tag{2.7}$$

Let

$$u_n = \mathrm{e}^{-\beta_n t} v_n. \tag{2.8}$$

Then

$$\frac{\partial u_n}{\partial t} = \mathrm{e}^{-\beta_n t}\left(\frac{\partial v_n}{\partial t} - \beta_n v_n\right),$$

and

$$\mathrm{e}^{-\beta_n t}\left(\frac{\partial v_n}{\partial t} - \beta_n v_n\right) + \int_0^t k_0(t-\tau)\mathrm{e}^{-\beta_n \tau}v_n(x_1,\tau)\,\mathrm{d}\tau = \mathrm{e}^{-\beta_n t}\left(\frac{\partial^2 v_n}{\partial x_1^2} - \beta_n v_n\right).$$

This leads to

$$\frac{\partial \upsilon_n}{\partial t} + \int_0^t k_0(t-\tau) e^{\beta_n (t-\tau)} \upsilon_n(x_1,\tau)\, d\tau = \frac{\partial^2 \upsilon_n}{\partial x_1^2}, \quad d < x_1 < +\infty, \;\; 0 < t \leqslant T,$$
$$\upsilon_n|_{t=0} = 0, \quad x \in \Omega,$$
$$\upsilon_n \to 0 \quad \text{as } x_1 \to +\infty.$$

Setting $k_n(t) = k_0(t) e^{\beta_n t}$, we see that $\upsilon_n = \upsilon_n(x_1, t)$ satisfies

$$\frac{\partial \upsilon_n}{\partial t} + \int_0^t k_n(t-\tau) \upsilon_n(x_1,\tau)\, d\tau = \frac{\partial^2 \upsilon_n}{\partial x_1^2}, \quad d < x_1 < +\infty, \;\; 0 < t \leqslant T, \tag{2.9}$$

$$\upsilon_n|_{t=0} = 0, \quad d \leqslant x_1 \leqslant +\infty, \tag{2.10}$$

$$\upsilon_n \to 0 \quad \text{as } x_1 \to +\infty. \tag{2.11}$$

For given $k_n(t)$, the (one-dimensional) problem (2.9)–(2.11) has been studied in the paper by Han et al. [9]. Accordingly, let

$$\hat{\upsilon}_n(x_1, s) := \int_0^{+\infty} \exp(-st) \upsilon_n(x_1, t)\, dt$$

denote the Laplace transform of the unknown function $\upsilon_n(x_1, t)$. In view of the Eq. (2.9) and the initial condition (2.10), $\hat{\upsilon}_n(x_1, s)$ satisfies

$$(s + \hat{k}_n(s)) \hat{\upsilon}_n(x_1, s) = \frac{d^2 \hat{\upsilon}_n(x_1, s)}{dx_1^2}, \tag{2.12}$$

where $\hat{k}_n(s)$ is the Laplace transform of the kernel $k_n(t)$. It follows from a basic property of the Laplace transform, ($\mathscr{L}\left\{f(t)e^{at}\right\} = \hat{f}(s-a)$), that

$$\hat{k}_n(s) := \mathscr{L}\{k_n(t)\} = \mathscr{L}\{k_0(t) e^{\beta_n t}\} = \hat{k}_0(s - \beta_n), \quad n = 1, 2, \ldots \;. \tag{2.13}$$

Eq. (2.12) is a linear second-order differential equation with constant coefficients. Its general solution is given by

$$\hat{\upsilon}_n(x_1, s) = C_1(s) \exp\left\{-\sqrt{s + \hat{k}_n(s)}(x_1 - d)\right\} + C_2(s) \exp\left\{\sqrt{s + \hat{k}_n(s)}(x_1 - d)\right\},$$

where $x_1 \geqslant d$. Suppose that

$$\mathrm{Re}\left\{\sqrt{s + \hat{k}_n(s)}\right\} > 0.$$

The condition (2.11) implies that $C_2(s) \equiv 0$, and hence we have

$$\hat{\upsilon}_n(x_1, s) = C_1(s) \exp\left\{-\sqrt{s + \hat{k}_n(s)}(x_1 - d)\right\}, \quad x_1 \geqslant d. \tag{2.14}$$

This yields

$$\frac{d\hat{\upsilon}_n(x_1, s)}{dx_1} = -C_1(s)\sqrt{s + \hat{k}_n(s)} \exp\left\{-\sqrt{s + \hat{k}_n(s)}(x_1 - d)\right\}. \tag{2.15}$$

On the artificial boundary Γ_e, we obtain

$$\frac{d\hat{\upsilon}_n(d, s)}{dx_1} = -\sqrt{s + \hat{k}_n(s)}\, \hat{\upsilon}_n(d, s). \tag{2.16}$$

Define

$$H_n(t) = \sqrt{\pi t}\,\mathrm{e}^{-\beta_n t}\,\mathscr{L}^{-1}\left\{\frac{\sqrt{s+\hat{k}_n(s)}}{s}\right\}. \tag{2.17}$$

By (2.13), the explicit expression for the function H_n can be obtained by using the techniques in [9].

We deduce from Eq. (2.16) and the convolution theorem for the Laplace transform that

$$\left.\frac{\partial v_n}{\partial x_1}\right|_{x_1=d} = -\frac{1}{\sqrt{\pi}}\int_0^t \frac{H_n(t-\tau)}{\sqrt{t-\tau}}\mathrm{e}^{\beta_n(t-\tau)}\frac{\partial v_n(d,\tau)}{\partial\tau}\,\mathrm{d}\tau. \tag{2.18}$$

Using (2.8), we return to the unknown function $u_n(x_1, t)$ and its boundary conditions,

$$\begin{aligned}\left.\frac{\partial u_n}{\partial x_1}\right|_{x_1=d} &= -\frac{1}{\sqrt{\pi}}\int_0^t \frac{H_n(t-\tau)}{\sqrt{t-\tau}}\mathrm{e}^{-\beta_n\tau}\frac{\partial}{\partial\tau}(u_n(d,\tau)\mathrm{e}^{\beta_n\tau})\,\mathrm{d}\tau \\ &= -\frac{1}{\sqrt{\pi}}\int_0^t \frac{H_n(t-\tau)}{\sqrt{t-\tau}}\left[\frac{\partial u_n(d,\tau)}{\partial\tau}+\beta_n u_n(d,\tau)\right]\mathrm{d}\tau. \end{aligned} \tag{2.19}$$

It thus follows from (2.6) and (2.19) that

$$\begin{aligned}\left.\frac{\partial u}{\partial x_1}\right|_{x_1=d} &= \sum_{n=1}^{\infty}\left.\frac{\partial u_n}{\partial x_1}\right|_{x_1=d}\sin\left(\frac{n\pi}{b}x_2\right) \\ &= -\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left\{\int_0^t \frac{H_n(t-\tau)}{\sqrt{t-\tau}}\left[\frac{\partial u_n(d,\tau)}{\partial\tau}+\beta_n u_n(d,\tau)\right]\mathrm{d}\tau\,\sin\left(\frac{n\pi}{b}x_2\right)\right\} \\ &= -\frac{2}{b\sqrt{\pi}}\sum_{n=1}^{\infty}\left\{\int_0^t\int_0^b \frac{H_n(t-\tau)}{\sqrt{t-\tau}}\right. \\ &\quad\times\left.\left[\frac{\partial u(d,y_2,\tau)}{\partial\tau}+\beta_n u(d,y_2,\tau)\right]\sin\left(\frac{n\pi}{b}y_2\right)\sin\left(\frac{n\pi}{b}x_2\right)\,\mathrm{d}y_2\,\mathrm{d}\tau\right\} \\ &:= \mathscr{B}(u|_{x_1=d}, t). \end{aligned} \tag{2.20}$$

We see that these artificial boundary conditions are *non-local* with respect to the temporal and spatial variables. The condition (2.20) is the *fully transparent artificial boundary condition* on the given artificial boundary $\Gamma_e \times [0, T]$. On the right-hand side of (2.20), taking the first N terms, we obtain a series of approximate artificial boundary conditions on $\Gamma_e \times [0, T]$, namely

$$\begin{aligned}\left.\frac{\partial u}{\partial x_1}\right|_{x_1=d} &= -\frac{2}{b\sqrt{\pi}}\sum_{n=1}^{N}\int_0^t\int_0^b \frac{H_n(t-\tau)}{\sqrt{t-\tau}}\sin\left(\frac{n\pi}{b}y_2\right)\sin\left(\frac{n\pi}{b}x_2\right) \\ &\quad\times\left[\frac{\partial u(d,y_2,\tau)}{\partial\tau}+\beta_n u(d,y_2,\tau)\right]\mathrm{d}y_2\,\mathrm{d}\tau \\ &:= \mathscr{B}_N(u|_{x_1=d}, t), \quad N = 0, 1, 2, \ldots, \end{aligned} \tag{2.21}$$

with $u = u_N$.

3. The reduced problems on the bounded domain

By the artificial boundary condition (2.20), the initial-boundary-value problem (1.1)–(1.4) is equivalent to the following problem on the bounded domain $\Omega_i \times [0, T]$:

$$\frac{\partial u}{\partial t} + \int_0^t k(x, t-\tau)u(x,\tau)\,\mathrm{d}\tau = \nabla(\alpha(x)\nabla u) - \beta(x)u + f(x,t), \quad (x,t) \in \Omega_i \times (0, T], \tag{3.1}$$

$$u = g(x,t), \quad (x,t) \in (\Gamma \cap \partial\Omega_i) \times (0, T], \tag{3.2}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega_i, \tag{3.3}$$

$$\left.\frac{\partial u}{\partial x_1}\right|_{x_1=d} = \mathscr{B}(u|_{x_1=d}, t). \tag{3.4}$$

Using the approximate artificial boundary conditions (2.21), the problem (1.1)–(1.4) can be reduced to the following approximating problems on the bounded domain $\bar{\Omega}_i \times [0, T]$: denoting the approximation to u by u_N, these problems are given by

$$\frac{\partial u_N}{\partial t} + \int_0^t k(x, t-\tau)u_N(x,\tau)\,\mathrm{d}\tau$$
$$= \nabla(\alpha(x)\nabla u_N) - \beta(x)u_N + f(x,t), \quad (x,t) \in \Omega_i \times (0, T], \tag{3.5}$$

$$u_N = g(x,t), \quad (x,t) \in (\Gamma \cap \partial\Omega_i) \times (0, T], \tag{3.6}$$

$$u_N(x, 0) = u_0(x), \quad x \in \Omega_i, \tag{3.7}$$

$$\left.\frac{\partial u_N}{\partial x_1}\right|_{x_1=d} = \mathscr{B}_N(u_N|_{x_1=d}, t), \quad N = 0, 1, 2, \ldots\ . \tag{3.8}$$

The existence, uniqueness and the regularity properties of solutions to the reduced partial Volterra integro-differential equations on bounded spatial domains with non-local artificial boundary conditions can be derived by using for example the well-known energy method (or: variational method). Relevant details can be found in the monograph [2] by Chen and Shih (see also its bibliography for additional references on this use of the energy method). Although [2] does not explicitly deal with problems with non-local boundary conditions, the techniques described there are readily extended to encompass our reduced problems with the non-local artificial boundary conditions (2.15) and (2.16), since the boundary conditions contain only the lower-order terms.

The following theorem shows that sequence of (unique) solutions u_N to the approximate problems (3.5)–(3.8) converges in L_2-norm.

Theorem 3.1. *Both problem* (3.1)–(3.4) *and problem* (3.5)–(3.8) *have one, and only one, solution. Moreover, the solution of* (3.5)–(3.8) *converges to the solution of* (3.1)–(3.4), *i.e.*, $\lim_{N\to+\infty}\|u_N - u\|_{L_2} = 0$.

Proof. For ease of exposition we will assume that the initial function is $g \equiv 0$. The proof is based on the equivalent weak form of the problem (3.1)–(3.4): find $u(\cdot, t) \in V := \{v \in H^1(\Omega_i) : v = 0 \text{ on } \Gamma_i\}$ such that

$$(u_t, v) + a(u, v) = -\int_0^t (k(x, t-\tau)u, v)\,\mathrm{d}\tau - (\beta(x)u, v)$$
$$- \int_0^t \frac{1}{\sqrt{t-\tau}}[b_1(u_\tau, v) + b_2(u, v)]\,\mathrm{d}\tau + (f, v), \quad v \in V, \tag{3.9}$$

where

$$u_t := \frac{\partial u}{\partial t}, \quad (u, v) := \int_{\Omega_i} uv\,\mathrm{d}x, \quad a(u, v) := \int_{\Omega_i} a(x)\nabla u \nabla v\,\mathrm{d}x,$$

$$\begin{aligned} b_1(u, v) &:= b_1(u(x, \tau), v, t-\tau) \\ &= \frac{2}{b\sqrt{\pi}}\left(\sum_{n=1}^{\infty}\int_0^b\int_0^b H_n(t-\tau)\sin\left(\frac{n\pi v}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)u(d, v, \tau)v(d, x_2)\,\mathrm{d}v\,\mathrm{d}x_2\right), \end{aligned}$$

and

$$\begin{aligned} b_2(u, v) &:= b_2(u(x, \tau), v, t-\tau) \\ &= \frac{2}{b\sqrt{\pi}}\left(\sum_{n=1}^{\infty}\int_0^b\int_0^b \beta_n H_n(t-\tau)\sin\left(\frac{n\pi v}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)u(d, v, \tau)v(d, x_2)\,\mathrm{d}v\,\mathrm{d}x_2\right). \end{aligned}$$

The analogous equivalent weak form of (3.5)–(3.8) is given by: find $u_N \in V$ such that

$$\begin{aligned} (u_{N,t}, v) + a(u_N, v) = &-\int_0^t (k(x, t-\tau)u_N, v)\,\mathrm{d}\tau - (\beta(x)u_N, v) \\ &-\int_0^t \frac{1}{\sqrt{t-\tau}}[b_1^N(u_{N,\tau}, v) + b_2^N(u_N, v)]\,\mathrm{d}\tau + (f, v), \quad v \in V, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} b_1^N(u, v) &:= b_1^N(u(x, \tau), v, t-\tau) \\ &= \frac{2}{b\sqrt{\pi}}\left(\sum_{n=1}^{N}\int_0^b\int_0^b H_n(t-\tau)\sin\left(\frac{n\pi v}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)u(d, v, \tau)v(d, x_2)\,\mathrm{d}v\,\mathrm{d}x_2\right), \end{aligned}$$

and

$$\begin{aligned} b_2^N(u, v) &:= b_2^N(u(x, \tau), v, t-\tau) \\ &= \frac{2}{b\sqrt{\pi}}\left(\sum_{n=1}^{N}\int_0^b\int_0^b \beta_n H_n(t-\tau)\sin\left(\frac{n\pi v}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)u(d, v, \tau)v(d, x_2)\,\mathrm{d}v\,\mathrm{d}x_2\right). \end{aligned}$$

The following lemma contains the key to the proof.

Lemma 3.1. *The bilinear form $a(\cdot, \cdot)$ is symmetric, continuous and coercive, that is,*

$$a(u, v) = a(v, u), \quad |a(u, v)| \leqslant \mu^*\|u\|_{H^1(\Omega_i)}\|v\|_{H^1(\Omega_i)}, \quad \mu_*\|u\|^2_{H^1(\Omega_i)} \leqslant a(u, u) \quad \forall u, v \in V.$$

The bilinear forms $b_j(\cdot, \cdot)$ and $b_j^N(\cdot, \cdot)$ $(j = 1, 2)$ are symmetric, continuous and positive semi-definite, i.e., there exists a positive constant C which is independent of d, N, such that

$$b_j(u, v) = b_j(v, u), \quad b_j^N(u, v) = b_j^N(v, u) \quad \forall u, v \in V, \tag{3.11}$$

$$0 \leqslant b_j^N(u, u) \leqslant b_j(u, u) \leqslant C\|u\|^2_{H^1(\Omega_i)} \quad \forall u \in V, \tag{3.12}$$

$$|b_j(u, v)| + |b_j^N(u, v)| \leqslant C\|u\|_{H^1(\Omega_i)}\|v\|_{H^1(\Omega_i)} \quad \forall u, v \in V. \tag{3.13}$$

Proof. By observing that H_n and β_n are positive, the proofs of (3.11)–(3.13) can be carried out, with a minor modification, along the lines of the ones given in [3]. $\square$

Lemma 3.1 leads directly to the uniqueness of the solutions to (3.1)–(3.4) and to (3.5)–(3.8). To prove that $u_N \to u$ as $N \to \infty$ (in L_2), we subtract (3.10) from (3.9) and obtain

$$\begin{aligned}
&(u_t - u_{N,t}, v) + a(u - u_N, v)\\
&\quad = -\int_0^t (k(x, t-\tau)(u - u_N), v)\,\mathrm{d}\tau - (\beta(x)(u - u_N), v)\\
&\qquad - \int_0^t \frac{1}{\sqrt{t-\tau}}[b_1(u_\tau, v) + b_2(u, v)]\,\mathrm{d}\tau + \int_0^t \frac{1}{\sqrt{t-\tau}}[b_1^N(u_{N,\tau}, v) + b_2^N(u_N, v)]\,\mathrm{d}\tau\\
&\quad = -\int_0^t (k(x, t-\tau)(u - u_N), v)\,\mathrm{d}\tau - (\beta(x)(u - u_N), v)\\
&\qquad - \int_0^t \frac{1}{\sqrt{t-\tau}}[b_1(u_\tau - u_{N,\tau}, v) + b_2(u - u_{N,\tau}, v)]\,\mathrm{d}\tau - \int_0^t \frac{1}{\sqrt{t-\tau}}[b_1(u_{N,\tau}, v) + b_2(u_N, v)]\,\mathrm{d}\tau\\
&\qquad + \int_0^t \frac{1}{\sqrt{t-\tau}}[b_1^N(u_{N,\tau}, v) + b_2^N(u_N, v]\,\mathrm{d}\tau \quad \forall v \in V.
\end{aligned} \tag{3.14}$$

We now take the limit as $N \to \infty$ on both sides of (3.14): by observing that

$$-\int_0^t \frac{1}{\sqrt{t-\tau}}[b_1(u_{N,\tau}, v) + b_2(u_N, v)]\,\mathrm{d}\tau + \int_0^t \frac{1}{\sqrt{t-\tau}}[b_1^N(u_{N,\tau}, v) + b_2^N(u_N, v)]\,\mathrm{d}\tau \to 0$$

and setting $E := E(x,t) := u_t(x,t) - \lim_{N\to\infty} u_{N,t}(x,t)$, (3.14) becomes

$$\begin{aligned}
&(E_t, v) + a(E, v)\\
&\quad = -\int_0^t (k(x, t-\tau)E, v)\,\mathrm{d}\tau - (\beta(x)E, v)\\
&\qquad - \int_0^t \frac{1}{\sqrt{t-\tau}}[b_1(E, v) + b_2(E, v)]\,\mathrm{d}\tau, \quad v \in V.
\end{aligned} \tag{3.15}$$

Substituting $v = E$ in (3.15) and using the properties of $a(\cdot,\cdot)$, $b_j(\cdot,\cdot)$ $(j = 1, 2)$ and the positivity of k and β we obtain, noting that $E(x, 0) \equiv 0$, the desired result that $E = 0$ in the weak (L_2) sense. This completes our proof. $\square$

4. Numerical solution of the reduced problem

We will illustrate the effectiveness and the accuracy of the numerical solution of the two-dimensional problem (1.1)–(1.4) based on the artificial boundary conditions (3.8) by two examples. While the first example is a test problem with known analytic solution, the second one is more typical of practical applications where the solution is unknown.

Example 4.1. Consider the problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \int_0^t k(t-\tau)u(x,\tau)\,\mathrm{d}\tau = \Delta u - \beta_0 u + f(x,t),\\
&\quad x = (x_1, x_2) \in \Omega := [0, +\infty) \times [0, b], \quad t \in [0, T],
\end{aligned} \tag{4.1}$$

$$u(0, x_2, t) = x_2(b - x_2)t, \quad u(x_1, 0, t) = u(x_1, b, t) = 0, \quad t \in (0, T], \tag{4.2}$$

$$u(x, 0) = 0, \tag{4.3}$$

$$u(x,t) \to 0 \quad \text{as } x_1 \to +\infty, \tag{4.4}$$

where $k(t)=\mathrm{e}^{-\beta_0 t}$ and

$$f(x,t)=\left(1+\beta_0 t-\beta_0^2 t+\frac{t\beta_0+\mathrm{e}^{-\beta_0 t}-1}{\beta_0^2}\right)x_2(b-x_2)\mathrm{e}^{-\beta_0 x_1}+2t\mathrm{e}^{-\beta_0 x_1}.$$

The exact solution of (4.1)–(4.4) is $u(x,t)=x_2(b-x_2)t\mathrm{e}^{-\beta_0 x_1}$.

The reduced problem is given by

$$\frac{\partial u}{\partial t}+\int_0^t k(t-\tau)u(x,\tau)\,\mathrm{d}\tau=\Delta u-\beta_0 u+f(x,t)$$
$$x\in\Omega_i:=[0,d]\times[0,b],\quad t\in(0,T], \tag{4.5}$$
$$u(0,x_2,t)=x_2(b-x_2)t,\ u(x_1,0,t)=u(x_1,b,t)=0,\quad t\in(0,T], \tag{4.6}$$
$$u(x,0)=0, \tag{4.7}$$

$$\left.\frac{\partial u}{\partial x_1}\right|_{x_1=d}=-\frac{2}{b\sqrt{\pi}}\sum_{n=1}^{N}\int_0^t\int_0^b\frac{H_n(t-\tau)}{\sqrt{t-\tau}}\sin\left(\frac{n\pi}{b}y_2\right)\sin\left(\frac{n\pi}{b}x_2\right)\times\left[\frac{\partial u(d,y_2,\tau)}{\partial\tau}+\beta_n u(d,y_2,\tau)\right]\mathrm{d}y_2\,\mathrm{d}\tau, \tag{4.8}$$

where

$$\beta_n=\beta_0+\left(\frac{n\pi}{b}\right)^2,$$

$$H_n(t)=\sqrt{\pi t}\,\mathrm{e}^{-\beta_n t}\mathscr{L}^{-1}\left\{\frac{\sqrt{s+\hat{k}_n(s)}}{s}\right\}=\mathrm{e}^{-\beta_n t}\left\{1+\sqrt{t}\sum_{j=1}^{+\infty}\frac{\alpha_j}{\gamma_j j!}\int_0^t(t-s)^{j-1/2}s^{j-1}\mathrm{e}^{(n\pi/b)^2 s}\,\mathrm{d}s\right\}, \tag{4.9}$$

$$\gamma_j=(j-1/2)(j-3/2)\ldots(1/2),$$

and

$$\alpha_j:=\frac{(-1)^{j-1}(2j-3)!!}{2^j j!}\quad(\text{with }(-1)!!:=1).$$

This result was derived in Han et al. [9].

In order to discretize the above problem, we introduce a triangulation $\mathscr{T}_h$ of Ω_i, based on the mesh given by

$$0=x_1^0<x_1^1<x_1^2<\cdots<x_1^I=d,\quad 0=x_2^0<x_2^1<x_2^2<\cdots<x_2^J=b,$$

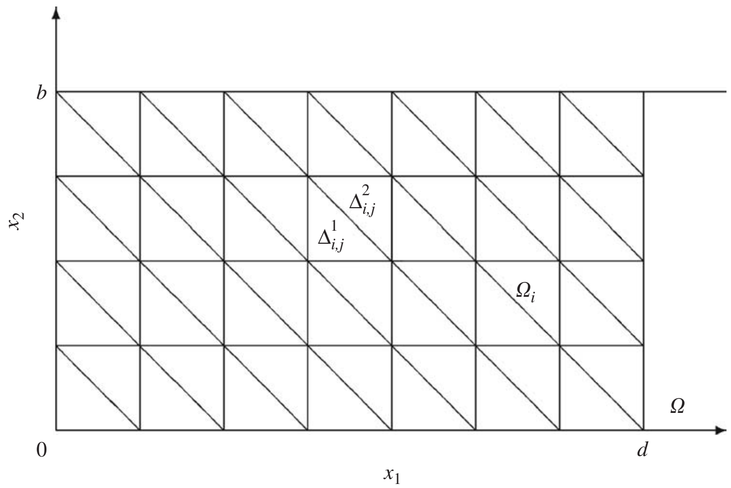

Fig. 2. Triangulation of Ω_i.

and employ a uniform mesh on the interval $[0, T]$,

$$0 = t_0 < t_1 < t_2 < \cdots < t_L = T$$

(see Fig. 2). Let $\tau = T/L$, $h = \max\{d/I, b/J\}$.

We will use the finite element (Galerkin) method for the spatial discretization of the problem (4.5)–(4.8). The underlying variational problem consists in finding $u \in U$ so that for any $v \in V$,

$$\left(\frac{\partial u}{\partial t}, v\right) + \int_0^t k(t-s)(u(x,s), v)\,\mathrm{d}s = -a(u,v) - \beta_0(u,v) + (f,v) + \int_0^b \frac{\partial u(d, y_2, t)}{\partial x_1} v(d, y_2)\,\mathrm{d}y_2, \tag{4.10}$$

where

$$(u, v) = \int_{\Omega_i} uv\,\mathrm{d}x,$$

$$a(u, v) = \int_{\Omega_i} \nabla u \cdot \nabla v\,\mathrm{d}x.$$

The spaces U and V are given by

$$U := \{u(x_1, x_2, t)|\ u(\cdot, \cdot, t) \in L^2(\Omega_i),\ u(x_1, 0, t) = 0, u(x_1, b, t) = 0, u(0, x_2, t) = x_2(b - x_2)t\},$$

$$V := \{v \in H^1(\Omega_i)|v(0, x_2) = 0, v(x_1, 0) = 0, v(x_1, b) = 0\}.$$

We define the corresponding finite element spaces U_h and V_h by

$$V_h := \{v \in C^0(\Omega_i)|v|_{\Delta^k_{i,j}} \text{ is a bilinear function of } x_1 \text{ and } x_2,\ 1 \leqslant i \leqslant I, 1 \leqslant j \leqslant J, k = 1, 2\},$$

$$
\begin{aligned}
U_h := \{u_h(x_1, x_2, t)| :\ & u_h(\cdot, \cdot, t) \in C^0(\Omega_i),\\
& u_h|_{\Delta^k_{i,j}}, \partial_t u_h|_{\Delta^k_{i,j}} \text{ is a bilinear function of } x_1 \text{ and } x_2, \text{ and}\\
& u(x_1, 0, t) = 0, u(x_1, b, t) = 0, u(0, x_2, t) = x_2(b - x_2)t\}.
\end{aligned}
$$

Here, $\Delta^k_{i,j}$ is the triangular element in Ω_i with vertices (A, B, C) given by $A = ((i-1) \cdot d/n, j \cdot b/m)$, $B = (i \cdot d/n, (j-1) \cdot b/m)$, $C = ((i-1) \cdot d/n, (j-1) \cdot b/m)$ when $k = 1$ and $A = ((i-1) \cdot d/n, j \cdot b/m)$, $B = (i \cdot d/n, (j-1) \cdot b/m)$, $C = (i \cdot d/n, j \cdot b/m)$ when $k = 2$ (compare Fig. 2).

This leads to the following approximation problem for (4.10): find $u_h \in U_h$, such that

$$
\begin{aligned}
\left(\frac{\partial u_h}{\partial t}, v\right) + \int_0^t k(t-s)(u_h(x, s), v)\,\mathrm{d}s = & -a(u_h, v) - \beta_0(u_h, v) + (f, v)\\
& + \int_0^b \frac{\partial u_h(d, y_2, t)}{\partial x_1} v(d, y_2)\,\mathrm{d}y_2,
\end{aligned}
\tag{4.11}
$$

for all $v \in V_h$. Let $\{\varphi_k(x)\}_{k=1}^K$ be a basis of V_h. We then can write

$$
u_h(x_1, x_2, t) = \sum_{k=1}^K X_k(t)\varphi_k(x_1, x_2). \tag{4.12}
$$

Substitution of (4.12) into (4.11) leads to

$$
\begin{aligned}
& \sum_{k=1}^K X'_k(t)(\varphi_k, \varphi_{k'}) + \sum_{k=1}^K \int_0^t k(t-s) X_k(s)(\varphi_k, \varphi_{k'})\\
& \quad = -\sum_{k=1}^K X_k(t) a(\varphi_k, \varphi_{k'}) + (f, \varphi_{k'}) - \beta_0 \sum_{k=1}^K X_k(t)(\varphi_k, \varphi_{k'})\\
& \qquad + \int_0^b \frac{\partial u_h(d, y_2, t)}{\partial x_1} \varphi_{k'}(d, y_2)\,\mathrm{d}y_2, \quad k' = 1, \ldots, K.
\end{aligned}
\tag{4.13}
$$

We will use the backward Euler method for the time-stepping in (4.13). This yields the numerical scheme

$$
\begin{aligned}
& \sum_{k=1}^K \left(\left[\beta_0 + \frac{1}{\tau}\right](\varphi_k, \varphi_{k'}) + a(\varphi_k, \varphi_{k'})\right) X_k(t_L)\\
& \quad = \sum_{k=1}^K \left(-\tau \sum_{l=0}^{L-1} k(t_L - t_l) X_k(t_l) + \frac{1}{\tau} X_k(t_{L-1})\right)(\varphi_k, \varphi_{k'})\\
& \qquad + (f(x_1, x_2, t_L), \varphi_{k'}) + \int_0^b \frac{\partial u_h(d, y_2, t)}{\partial x_1} \varphi_{k'}(d, y_2)\,\mathrm{d}y_2, \quad k' = 1, \ldots, K.
\end{aligned}
\tag{4.14}
$$

Remark 4.1. The coefficient matrix in the above system of linear algebraic equations is regular (see also the sequel to the present paper, for a detailed analysis). This result is a consequence of the fact that the diffusion term in (1.1) “dominates” the Volterra memory term (compare also [10]).

By (4.8) and (4.9) we obtain

$$
\begin{aligned}
&\int_0^b \frac{\partial u_h(d, y_2, t)}{\partial x} \varphi_{k'}(d, y_2)\,\mathrm{d}y_2 \\
&\quad = \int_0^b \left(-\frac{2}{b\sqrt{\pi}} \sum_{n=1}^{N} \int_0^t \int_0^b \frac{H_n(t-s)}{\sqrt{t-s}} \sin\left(\frac{n\pi}{b} r\right) \sin\left(\frac{n\pi}{b} y_2\right) \right. \\
&\qquad \left. \times \left[\frac{\partial u_h(d, r, s)}{\partial s} + \beta_n u_h(d, r, \pi) \right] \mathrm{d}r\,\mathrm{d}s \right) \varphi_{k'}(d, y_2)\,\mathrm{d}y_2 \\
&\quad = -\frac{2}{b\sqrt{\pi}} \sum_{n=1}^{N} \int_0^b \sin\left(\frac{n\pi}{b} y_2\right) \varphi_{k'}(d, y_2)\,\mathrm{d}y_2 \\
&\qquad \times \left(\int_0^t \int_0^b \frac{H_n(t-s)}{\sqrt{t-s}} \sin\left(\frac{n\pi}{b} r\right) \left[\frac{\partial u_h(d, r, s)}{\partial s} + \beta_n u_h(d, r, \pi) \right] \mathrm{d}r\,\mathrm{d}s \right) \\
&\quad = -\frac{2}{b\sqrt{\pi}} \sum_{n=1}^{N} \int_0^b \sin\left(\frac{n\pi}{b} y_2\right) \varphi_{k'}(d, y_2)\,\mathrm{d}y_2 \\
&\qquad \times \left(\sum_{k=1}^{K} \int_0^b \sin\left(\frac{n\pi}{b} r\right) \varphi_k(d, r)\,\mathrm{d}r \int_0^t \frac{H_n(t-s)}{\sqrt{t-s}} (X_k'(s) + \beta_n X_k(s))\,\mathrm{d}s \right) \\
&\quad = -\frac{2}{b\sqrt{\pi}} \sum_{n=1}^{N} \int_0^b \sin\left(\frac{n\pi}{b} y_2\right) \varphi_{k'}(d, y_2)\,\mathrm{d}y_2 \\
&\qquad \times \left\{ \sum_{k=1}^{K} \int_0^b \sin\left(\frac{n\pi}{b} r\right) \varphi_k(d, r)\,\mathrm{d}r \left[\sum_{l=0}^{L-1} \int_{t_l}^{t_{l+1}} \frac{H_n(t_L - s)}{\sqrt{t_L - s}}\,\mathrm{d}s \right.\right. \\
&\qquad \left.\left. \times \left(\frac{X_k(t_{l+1}) - X_k(t_l)}{\tau} + \beta_n X_k(t_{l+1}) \right) \right] \right\}.
\end{aligned}
$$

The explicit expressions for the integrals $\int_{t_l}^{t_{l+1}} H_n(t_L - s)/\sqrt{t_L - s}\,\mathrm{d}s$ can be found in [9].

In order to illustrate performance of the above numerical scheme, we choose $\beta_0 = 5$, $b = 1$, $d = 2$, $L = 10$, $T = 0.5$, $N = 5$. A selection of numerical results is shown in Figs. 3, 4 and Table 1.

Example 4.2. We now turn to another example. Its analytical solution cannot be obtained exactly; moreover, its value on the artificial boundary is not close to 0. This initial-boundary-value problem is

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \int_0^t k(t-\tau) u(x, \tau)\,\mathrm{d}\tau = \Delta u - \beta_0 u + f(x, t) \\
&\quad x \in \Omega := [0, +\infty) \times [0, b], \quad t \in [0, T], \\
&u(0, x_2, t) = x_2(b - x_2)t,\ u(x_1, 0, t) = u(x_1, b, t) = 0, \quad t \in (0, T], \\
&u(x, 0) = 0, \quad x \in \Omega, \\
&u(x, t) \to 0 \quad \text{as } x_1 \to +\infty,
\end{aligned}
$$

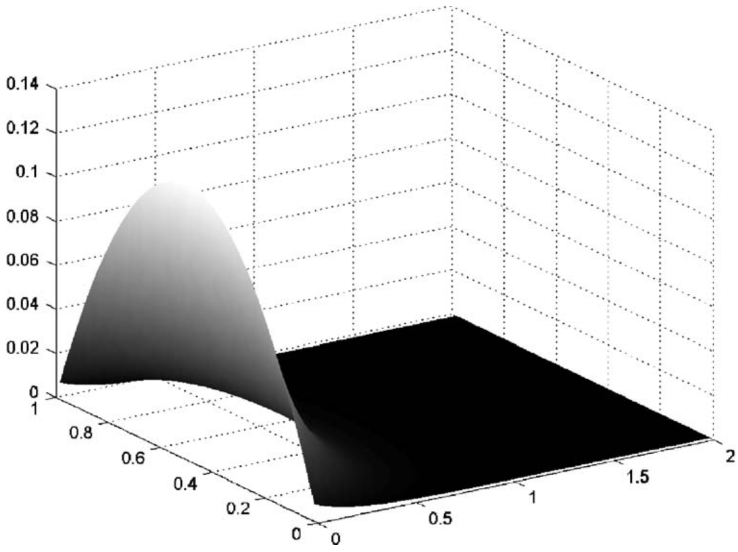

Fig. 3. The numerical solution at $T = 0.5$ when $J \times I = 64 \times 128$.

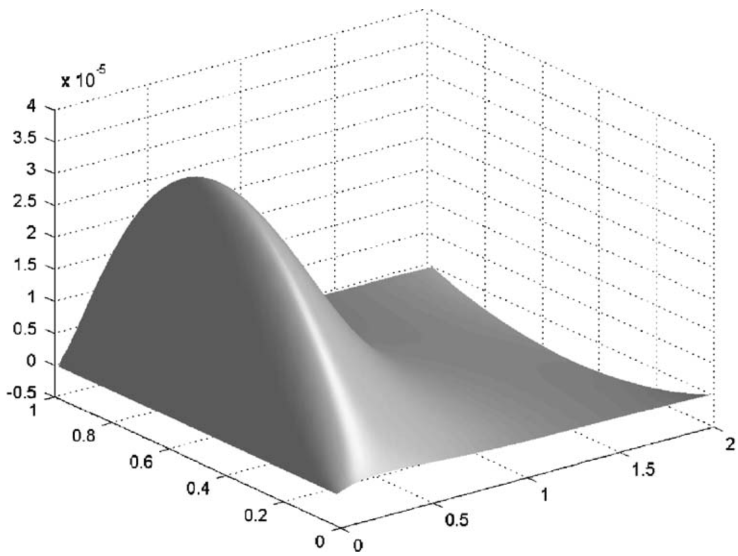

Fig. 4. The error at $T = 0.5$ when $J \times I = 64 \times 128$.

Table 1
The results for Example 1

h	$J \times I$	$\frac{\|u_h-u\|_{L_2}}{\|u\|_{L_2}}$	$\frac{\|u_h-u\|_\infty}{\|u\|_\infty}$
1/4	4×8	1.0754e − 1	1.1613e − 1
1/8	8×16	3.0232e − 2	3.7789e − 2
1/16	16×32	8.1801e − 3	1.0796e − 2
1/32	32×64	2.3516e − 3	2.8978e − 3
1/64	64×128	5.4289e − 4	2.8291e − 4

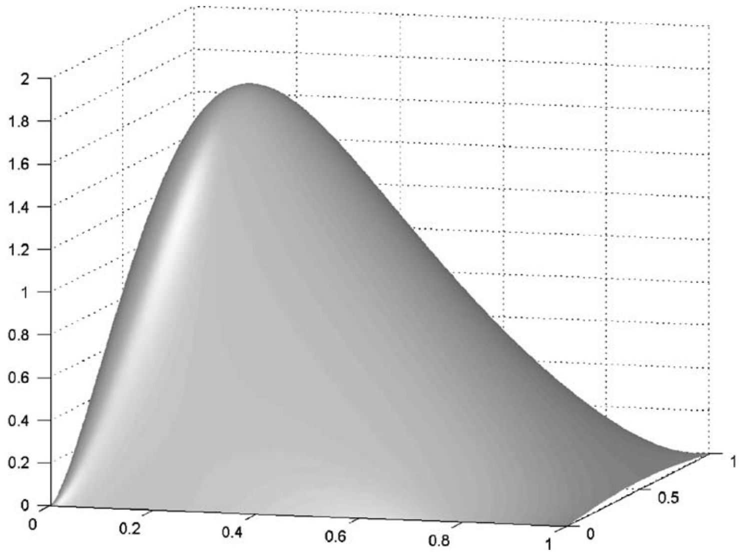

Fig. 5. The numerical solution at $T = 0.5$ when $J \times I = 128 \times 128$.

Table 2
The results for Example 2

h	$J \times I$	$\frac{\|u_h-u\|_{L_2}}{\|u\|_{L_2}}$	$\frac{\|u_h-u\|_\infty}{\|u\|_\infty}$
1/4	4×4	2.4205e − 1	3.5468e − 1
1/8	8×8	7.0059e − 2	1.2047e − 1
1/16	16×16	1.8347e − 2	3.4713e − 2
1/32	32×32	4.6363e − 3	9.3144e − 3
1/64	64×64	1.1431e − 3	2.4125e − 3
1/128	128×128	2.6185e − 4	6.1377e − 4

where

$$k(t) = \mathrm{e}^{-\beta_0 t},$$

$$f(x,t) = \begin{cases} 100x_2(b-x_2)\mathrm{e}^{-5x_1} + 200\mathrm{e}^{-5x_1} & \text{if } x_1 \leqslant d, \\ 0 & \text{if } x_1 > d. \end{cases}$$

We employ the same numerical method as for Example 4.1 and select the values $\beta_0 = 1$, $b = d = 1$, $L = 10$, $T = 0.5$, $N = 5$ for the parameters. The numerical solution corresponding to $J \times I = 256 \times 256$ is used as the "exact" reference solution. Fig. 5 and Table 2 illustrate the accuracy and the order of convergence of the scheme. Note that in this example we have $\|u\|_{\infty,\Gamma_e} = 3.9633\mathrm{e} - 2$.

5. Conclusion

In this paper we have described the artificial boundary method for the approximate (numerical) solution of partial Volterra integro-differential equations on certain (strip-like) unbounded two-dimensional domains, thus answering a question raised at the end of [9]. The foregoing analysis suggests that the artificial boundary method can be readily extended to doubly-infinite strip-like domains (see also [9]). We leave the details to the reader.

As we mentioned at the end of the Introduction, in a forthcoming sequel to the present paper we shall study the derivation of (a priori and a posteriori) error estimates depending on the numbers d (cf. Fig. 1 and (2.2), (2.3)) and N

(cf. (2.20) and (3.8)) and present alternative, more accurate, time-stepping methods based on discontinuous Galerkin methods, thus extending the approaches of Larsson et al. [10], Ma [11], Ma and Brunner [12], and Brunner and Schötzau [1]. These results will form the basis for adaptive time-stepping.

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