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The Modified Direct Method: An Iterative Approach for Smoothing Planar Meshes☆

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Abstract

The Modified Direct Method (MDM) is an iterative method for smoothing planar triangular, quadrilateral and tri-quad meshes, which has been developed from the non-iterative smoothing method for planar meshes that was originated by Balendran [1]. Its simple aim is to make triangular elements as close to equilateral as possible and quadrilateral elements as close to square as possible. The MDM is the computationally simpler of the two methods. The performance of the MDM for both triangular and quadrilateral meshes is effectively identical to that of Laplacian smoothing; however, it outperforms Laplacian smoothing for tri-quad meshes. Test examples show that the MDM is always convergent for triangular, quadrilateral and tri-quad meshes.

Keywords: Finite element, Mesh smoothing, Iterative smoothing, Laplacian smoothing

1. Introduction

A mesh newly created usually needs to be improved. This improvement can be made using either (1) *mesh clear-up* methods, which insert or delete nodes as well as change the connectivity of the mesh elements, or (2) *mesh smoothing* methods, which leave the element connectivity unchanged and instead reposition the mesh nodes. This paper is chiefly concerned with smoothing methods.

These methods can be divided into four types: (1) geometry-based, (2) optimization-based, (3) physics-based, and (4) hybrid. Geometry-based methods obtain new node locations by using geometric rules or local optimization techniques [2, 3, 4]. Optimization-based methods obtain the smoothed node positions by minimizing some given distortion metric [5, 6, 7]. Physics-based methods operate by physical processing or by solving simple physics problems [8, 9]. Hybrid methods are the combinations of two or more basic methods [10, 11].

In this paper we introduce a novel smoothing method that has the same basic goal as the Direct Method (DM) [1], but is simper than the DM; we term this method the Modified Direct Method (MDM) which can be used to smooth planar triangular, quadrilateral and tri-quad meshes.

The main procedure of the MDM for smoothing planar meshes is relatively simple. Firstly element stiffness matrices are created based on type of elements. The modified forms of element stiffness matrices are simpler than those of DM. And then by assembling all element stiffness matrices, a system of Jacobi iteration equations can be formed. Finally, the smoothed positions can be generated by solving the system of Jacobi iteration equations.

☆A more general but unfinished preliminary version has been posted online at `http://arxiv.org/abs/1212.3133`.

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2. The Modified Direct Method (MDM)

The development of the MDM from the DM involves: (1) the use of different element stiffness matrices, (2) the use of a Jacobian iteration matrix instead of a global stiffness matrix, and (3) the replacement of the optimization equations with iteration equations. The mathematical steps involved in this development are broadly similar for the triangular mesh, the quadrilateral mesh and the tri-quad mesh.

2.1. The element stiffness matrices

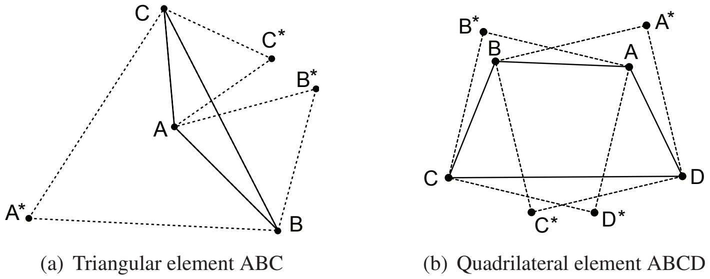


Fig. 1. The expected nodal positions. (This figure is redrawn according to the Figures. 2 and 3 in [1])

Consider a triangular element ABC shown in Fig.1(a). A^* is the position to which node A would have to be moved to make the element equilateral, assuming that nodes B and C were fixed; B^* and C^* are obtained similarly. The coordinates of A, B, C, A^*, B^* and C^* are then related by the following equations:

$$\begin{bmatrix} 0 & 0 & 1/2 & \sqrt{3}/2 & 1/2 & -\sqrt{3}/2 \\ 0 & 0 & -\sqrt{3}/2 & 1/2 & \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 & 0 & 0 & 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 & 0 & 0 & -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 & 1/2 & -\sqrt{3}/2 & 0 & 0 \\ -\sqrt{3}/2 & 1/2 & \sqrt{3}/2 & 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_A \\ Y_A \\ X_B \\ Y_B \\ X_C \\ Y_C \end{bmatrix} = \begin{bmatrix} X_A^* \\ Y_A^* \\ X_B^* \\ Y_B^* \\ X_C^* \\ Y_C^* \end{bmatrix} \tag{1}$$

Consider next a quadrilateral element ABCD shown in Fig.1(b). A^* and C^* are the positions to which nodes A and C would have to be moved to make the element square, assuming B and D were fixed. The coordinates of A, B, C, D, A^*, B^*, C^* and D^* are then related by the following equations:

$$\begin{bmatrix} 0 & 0 & 1/2 & 1/2 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & -1/2 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & -1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 & 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_A \\ Y_A \\ X_B \\ Y_B \\ X_C \\ Y_C \\ X_D \\ Y_D \end{bmatrix} = \begin{bmatrix} X_A^* \\ Y_A^* \\ X_B^* \\ Y_B^* \\ X_C^* \\ Y_C^* \\ X_D^* \\ Y_D^* \end{bmatrix} \tag{2}$$

The left-hand matrices in Eq.1 and Eq.2 are the element stiffness matrices for a planar triangular and a quadrilateral element, respectively; and they are both simpler in form comparing to those used in the DM.

2.2. The MDM for smoothing planar meshes

Just as in the DM, the element stiffness matrices can be assembled into a global matrix, also of size $2n \times 2n$ for a mesh of n nodes. This Jacobi iteration matrix needs to be solved iteratively, starting with the original node coordinates at step 0 and continuing until no node needs to be moved by more than the given tolerance.

The algorithm for implementing the MDM in 2D has three basic steps: (1) the searching for elements that share a node, for each node; (2) the assembling of the element stiffness matrices into the iteration matrix; (3) the solving of the iteration equations until the tolerance distance is reached.

3. Test applications of the MDM

3.1. Mesh quality

The simplest way to measure mesh quality is to calculate distortion values for each of the mesh elements separately, and then to compare the distributions of all those values. The distortion value for a triangular element should measure how close that triangle is to equilateral. One appropriate measure, α, was proposed by Lee and Lo [12]. The distortion value for a quadrilateral element should measure how close that quadrilateral is to square. In this paper we use the measure λ proposed by Li [13]. For the quadrilateral ABCD shown in Fig.1(b):

$$\lambda = 2\sqrt[4]{\frac{\|AB \times AD\| \cdot \|BC \times BA\| \cdot \|CD \times CB\| \cdot \|DA \times DC\|}{(\|AB\|^2 + \|AD\|^2)(\|BC\|^2 + \|BA\|^2)(\|CD\|^2 + \|CB\|^2)(\|DA\|^2 + \|DC\|^2)}}$$

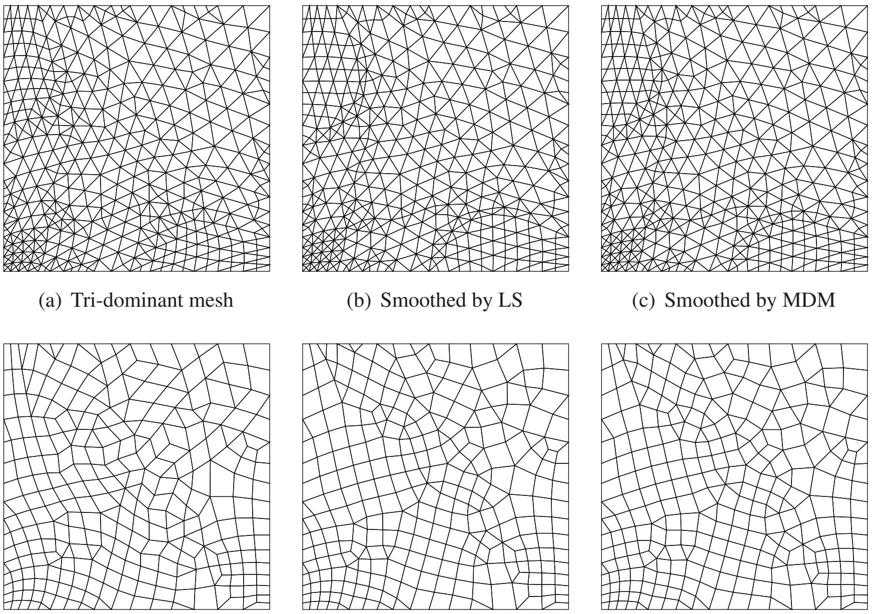
The value of λ also lies between 0 and 1. For a mesh with n elements, we have defined two indicators, mean quality (MQ) and mean square error (MSE):

$$MQ = \frac{1}{n}\sum_{i=1}^{n}\gamma_i, \quad MSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\gamma_i - MQ)^2} \tag{3}$$

, where $\gamma_i = \alpha_i$ when the element is triangle and $\gamma_i = \lambda_i$ when the element is quadrilateral. For a mixed mesh, i.e., tri-quad mesh, the indicators MQ and MSE need to be calculated for triangles and quadrilaterals separately.

3.2. The test applications

For smoothing planar meshes, a number of test meshes were created and smoothed by Laplacian smoothing (LS) [2] and the MDM separately. Smoothing results of tri-dominant and quad-dominant meshes are compared in Fig.2; and mesh quality results before and after smoothing are given in Table.1. We have not given the comparison of mesh qualities for triangular and quadrilateral meshes before and after smoothing, as that the results for a mesh with same type of element are exactly the same. The improvement of mesh quality can be seen clearly in Fig.3.



(a) Tri-dominant mesh (b) Smoothed by LS (c) Smoothed by MDM

(d) Quad-dominant mesh (e) Smoothed by LS (f) Smoothed by MDM

Fig. 2. Smoothing for planar tri-dominant and quad-dominant meshes

3.3. Tests assessment and summary

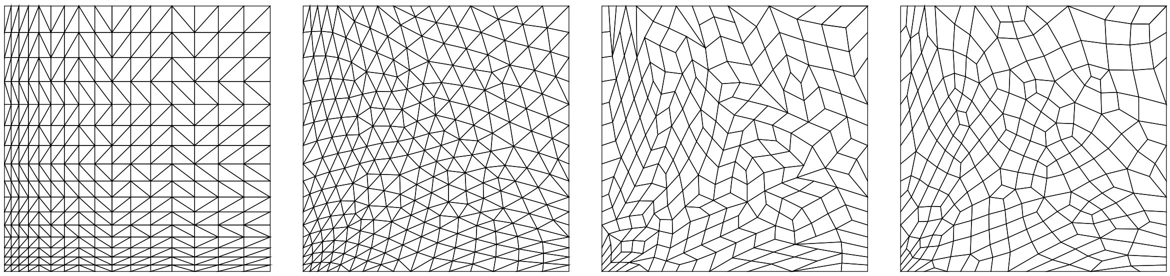
1) For the two topologically uniform test meshes, i.e., the meshes with the same type of element throughout, there is effectively no difference between LS and the MDM. Both these methods give smoothed meshes that are markedly better than the original unsmoothed meshes (Fig.3).

2) For the triangular elements in the planar tri-dominant mixed mesh (Fig.2(a)), the MDM outperforms LS; while for the quadrilateral elements, the reverse is true (Table 1). However, for planar quad-dominant mixed mesh (Fig.2(d)), MDM perfectly outperforms LS for both triangular and quadrilateral elements.

3) Test examples show that the MDM is convergent when applied to planar triangular, quadrilateral and tri-quad meshes. But we could not give the accurate mathematical proof for it in theory. Future work may focus on extending the MDM to be implemented for smoothing surface meshes and volumetric meshes.

Table 1. Mesh quality results for smoothing tri-quad meshes (Tri: triangle; Quad: quadrilateral; MQ and MSE are defined in Eq.3)

	Tri-dominant mesh ((a), (b), and (c) in Fig.2)				Quad-dominant mesh ((d), (e), and (f) in Fig.2)			
	MQ of tris	MSE of tris	MQ of quads	MSE of quads	MQ of tris	MSE of tris	MQ of quads	MSE of quads
Original	0.8688	0.1188	0.8211	0.0713	0.9079	0.1243	0.9194	0.0739
LS	0.8756	0.1083	0.8464	0.0571	0.9064	0.0936	0.9554	0.0568
MDM	0.8794	0.1083	0.8306	0.0583	0.9394	0.0737	0.9576	0.0568



(a) Original tri mesh (b) Smoothed by LS / MDM (c) Original quad mesh (d) Smoothed by LS / MDM

Fig. 3. Smoothing for triangular and quadrilateral meshes (Some degenerate quads that look like as triangles are intentionally created in (c))

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