Topological ordered C- (resp. I-)spaces and generalized metric spaces

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The following result due to Hanai, Morita, and Stone is well known: Let f be a closed continuous map of a metric space X onto a topological space Y. Then the following statements are equivalent: (i) Y satisfies the first countability axiom; (ii) for each y ∈ Y, f⁻¹{y} has a compact boundary in X; (iii) Y is metrizable.

In this article we obtain several related results in the setting of topological ordered spaces. In particular we investigate the upper and lower topologies of metrizable topological ordered spaces which are both C- and I-spaces in the sense of Priestley.

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1. Introduction

The following result due to Hanai, Morita, and Stone is well known (see [6, Theorem 4.4.17] or compare [25, Theorem 1]): Let f be a closed continuous map of a metric space X onto a topological space Y. Then the following statements are equivalent: (i) Y satisfies the first countability axiom; (ii) for each y ∈ Y, f⁻¹{y} has a compact boundary in X; (iii) Y is metrizable.

In this article we investigate the upper topology τ♮ and the lower topology τ♭ of those metrizable topological ordered spaces (X, τ, ≤) which are both C- and I-spaces in the sense of Priestley [21]. Because—in some respect—the spaces (X, τ♮) and (X, τ♭) of a topological ordered C-space (resp. topological ordered I-space) (X, τ, ≤) behave like closed (resp. open) images of (X, τ), where i(x) and d(x) (with x ∈ X) correspond to the fibers of the map, the following result due to Balachandran [1] seems to be of interest in our context.

If f : X → Y is a closed and open continuous map from a bounded metric space (X, e) onto a topological space Y, then Y is metrizable by the Hausdorff metric r; that is, r defined by

\[ r(y_1, y_2) = \max \left\{ \sup_{y_1' \in f^{-1}(y_1)} e(y_1', f^{-1}(y_2)), \sup_{y_2' \in f^{-1}(y_2)} e(f^{-1}(y_1), y_2') \right\} \]

whenever y₁, y₂ ∈ Y.

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2 For example, compare [6, Problem 1.7.16] with the definition of the maps f, and f' discussed at the end of this section. Note however that the analogy cannot be pushed too far. While each fiber of a map is open or compact under the hypotheses mentioned in Balachandran’s result (compare [25, p. 700]), the sets i(x) and d(x) (x ∈ X) of a metrizable topological ordered C- and I-space X need neither be open nor compact, as Example 1 below will show.

3 Here e(A, B) = inf{e(a, b); a ∈ A, b ∈ B} for given nonempty subsets A, B ⊆ X.
Similarly, one might hope that if \((X, \tau, \leq)\) is a topological ordered \(C\) and \(I\)-space and \(e\) is a (bounded) metric inducing the topology \(\tau\), then
\[
s(x, y) = \max \left\{ \sup_{x' \in d(x)} e(x', d(y)), \sup_{y' \in d(y)} e(i(x), y') \right\}
\]
whenever \(x, y \in X\) yields a quasi-pseudometric inducing the topologies of the bitopological space \((X, \tau^0, \tau^1)\) associated with \((X, \tau, \leq)\). Unfortunately the authors do not have a counterexample to refute this doubtful conjecture. (Indeed, note that in any case \(s\), as defined above, is not the standard Hausdorff quasi-pseudometric (compare e.g. [16]), since, of course, given \(x \in X\), in general \(d(x) \neq i(x)\).) Below (see Corollary 3) however we shall show that at least the conjecture holds provided that both \(d(x)\) and \(i(x)\) are totally bounded whenever \(x \in X\). In fact in this article we are mainly concerned with the topological version underlying our problem, which does not ask for an explicit formula of the quasi-pseudometric in terms of the starting metric and should be formulated as follows:

**Problem 1.** If \((X, \tau, \leq)\) is a topological ordered \(C\) and \(I\)-space such that the topology \(\tau\) is metrizable, is the associated bitopological space \((X, \tau^0, \tau^1)\) quasi-pseudometrizable?

Hence Problem 1 for instance asks whether from a \(\tau\)-compatible uniformity with a countable base a \(\tau^0-\tau^1\)-compatible quasi-uniformity \(U\) (that is, \(\tau(U) = \tau^1\) and \(\tau(U^{-1}) = \tau^0\)) with a countable base can be constructed. Note that the difficulty of the problem stems from the cardinality restriction, since it is well known that if \((X, \tau, \leq)\) is a topological ordered \(C\) and \(I\)-space such that \(\tau\) is completely regular, then the associated bitopological space \((X, \tau^0, \tau^1)\) is pairwise completely regular ([3, p. 64] and [14, Proposition 2]); which means in other words that, if \((X, \tau, \leq)\) is a topological ordered \(C\) and \(I\)-space such that \(\tau\) is uniformizable, then the bitopological space \((X, \tau^0, \tau^1)\) is quasi-uniformizable (compare [17]). The corresponding construction can be achieved (compare [3,14]) by transforming under our conditions a continuous function \(f : X \to [0, 1]\) from \(X\) into the unit interval \([0, 1]\) (via the operation \(f^*(x) = \sup\{f(y) : y \in d(x)\}\) \((x \in X)\), resp. via the operation \(f_\circ(x) = \inf\{f(y) : y \in i(x)\}\) \((x \in X)\)) into an increasing continuous map, where a map \(f : X \to Y\) between two topological ordered spaces \(X\) and \(Y\) is said to be increasing provided that \(f(x) \leq f(y)\) whenever \(x, y \in X\) and \(x \leq y\). In this context it may be interesting to remark that quasi-pseudometrizability of a bitopological space \(X\) has been characterized in terms of the existence of special families of real-valued functions on \(X\) [24].

2. Preliminaries

We assume that the reader is familiar with the basic results about quasi-uniform spaces (see e.g. [7]). In particular for any quasi-uniformity \(U\) we shall denote by \(U^\circ\) the coarsest uniformity finer than \(U\). Furthermore a quasi-pseudometric \(d\) on a set \(X\) is a map \(d : X \times X \to [0, \infty]\) such that \(d(x, x) = 0\) and \(d(x, z) \leq d(x, y) + d(y, z)\) whenever \(x, y, z \in X\). We now recall some pertinent definitions from the theory of topological ordered spaces. Given a topological ordered space \((X, \tau, \leq)\), a subset \(A\) of \(X\) is said to be an upper set of \(X\) if \(x \leq y\) and \(x \in A\) imply that \(y \in A\). Similarly, we say that a subset \(A\) of \(X\) is a lower set of \(X\) if \(y \leq x\) and \(x \in A\) imply that \(y \in A\). In this article we shall consider the bitopological space\(^6\) (in the following often more briefly called bispace) \((X, \tau^0, \tau^1)\) associated with a given topological ordered space \((X, \tau, \leq)\) where \(\tau^0\) denotes the collection of \(\tau\)-open lower sets of \(X\) and \(\tau^1\) denotes the collection of \(\tau\)-open upper sets of \(X\). As usual for any subset \(E\) of \(X\), \(i(E)\) (resp. \(d(E)\)) will denote the intersection of all upper (lower) sets of \(X\) containing \(E\). Following Priestley [21], we shall call a topological ordered space \(X\) a \(C\)-space if \(d(F)\) and \(i(F)\) are closed whenever \(F\) is a closed subset of \(X\). Similarly, a topological ordered space \((X, \tau, \leq)\) is called an \(I\)-space if \(d(G)\) and \(i(G)\) are open whenever \(G\) is an open subset of \(X\). It is for instance known that each compact topological \(T_2\)-ordered space\(^7\) \((X, \tau, \leq)\) is a \(C\)-space (compare e.g. [7, Proposition 4.3]) and that each topological lattice is an \(I\)-space (see e.g. [14, p. 291]). We also recall that a topological ordered space \((X, \tau, \leq)\) is called convex provided that \(\tau^0 \vee \tau^1 = \tau\). Since each normal topological \(T_1\)-ordered \(C\)-space is convex [15, Lemma 2], in particular each metrizable topological ordered \(C\)-space is convex.

It is easy to see that Problem 1 has a positive answer for separable metric spaces:

**Proposition 1.** If \((X, \tau, \leq)\) is a topological ordered \(C\) and \(I\)-space such that \(\tau\) is a separable metric topology, then \((X, \tau^0, \tau^1)\) is quasi-pseudometrizable.

**Proof.** First observe that the bitopological space \((X, \tau^0, \tau^1)\) is (pairwise) completely regular by the result cited above (see [3,14]). Furthermore let \(B\) be a (countable) base for \(\tau\). Then note that \(i(G) : G \in B\) (resp. \(d(G) : G \in B\)) is a (countable) base for the upper topology \(\tau^1\) (resp. lower topology \(\tau^0\)) of \(X\), since \(X\) is an \(I\)-space. The statement follows, because

\(^4\) Here \([0, 1]\) means the real unit interval equipped with its usual Euclidean topology and order.

\(^5\) We do not assume that there holds any relation between the partial order \(\leq\) and the topology \(\tau\) on \(X\) without stating explicitly otherwise.

\(^6\) For basic concepts of the theory of bitopological spaces we refer the reader to [10,17].

\(^7\) As usual, a topological ordered space \((X, \tau, \leq)\) is called \(T_2\)-ordered if the relation \(\leq\) is \(\tau \times \tau\)-closed in \(X \times X\), and it is called \(T_1\)-ordered if \(d(x)\) and \(i(x)\) are \(\tau\)-closed whenever \(x \in X\).
by a result of Kelly [10, Theorem 2.8] each pairwise regular bispace with two second countable topologies is quasi-pseudometrizable. □

The method used to establish the preceding proposition still yields two other results of interest:

**Proposition 2.** Let \((X, \tau, \preceq)\) be a first-countable topological ordered 1-space. Then the two topologies \(\tau^\flat\) and \(\tau^\natural\) are first countable.

**Proof.** If \(\{B_n; n \in \mathbb{N}\}\) is a countable base of (open) neighborhoods for the topology \(\tau\) at \(x\), then \(\{d(B_n); n \in \mathbb{N}\}\) is a countable base of neighborhoods for the topology \(\tau^\flat\) at \(x\). □

**Proposition 3.** Let \((X, \tau, \preceq)\) be a metrizable topological ordered 1-space such that \(i(x)\) is \(\tau\)-compact whenever \(x \in X\). Then \(\tau^\flat\) has a \(\sigma\)-point-finite base and thus is quasi-pseudometrizable.

**Proof.** By the Nagata–Smirnov Theorem the topology \(\tau\) has a base \(\bigcup_{n \in \mathbb{N}} B_n\) such that each collection \(B_n\) is locally finite. For each \(n \in \mathbb{N}\) set \(d(B_n) = \{d(B); B \in B_n\}\). Then \(\bigcup_{n \in \mathbb{N}} d(B_n)\) is a \(\sigma\)-point-finite base for \(\tau^\flat\): Indeed it is obvious that it is a base for \(\tau^\flat\) and for each \(n \in \mathbb{N}\) and \(x \in X, x \in d(B)\) for only finitely many \(B \in B_n\), since otherwise \(i(x) \cap B \neq \emptyset\) for infinitely many \(B \in B_n\), which contradicts the facts that \(i(x)\) is \(\tau\)-compact and \(B_n\) is locally finite. □

3. Topological ordered C-spaces and pairwise stratifiability

In this section we shall show that indeed Problem 1 can be reduced to the question whether both the upper and the lower topology of a metrizable topological ordered C- and 1-space are quasi-pseudometrizable. We next recall that a bitopological space \((X, \tau_1, \tau_2)\) is called pairwise stratifiable [8,19] if and only if, for \(i, j \in \{1, 2\}\) with \(i \neq j\), we can assign to each \(\tau_i\)-closed set \(H \subseteq X\) a sequence \((H_n)_{n \in \mathbb{N}}\) of \(\tau_j\)-open sets satisfying the conditions

(i) if \(H \subseteq K\), then \(H_n \subseteq K_n\) whenever \(n \in \mathbb{N}\), and (ii) \(H = \bigcap_{n \in \mathbb{N}} \text{cl}_{\tau_j} H_n\).

Hence in particular for a topological ordered space \((X, \tau, \preceq)\), the bitopological space \((X, \tau^\flat, \tau^\natural)\) is pairwise stratifiable if for each closed upper set \(H\) there is a sequence \((H_n)_{n \in \mathbb{N}}\) of open upper sets such that

(i) if \(H \subseteq K\), then \(H_n \subseteq K_n\) whenever \(n \in \mathbb{N}\), and (ii) \(H = \bigcap_{n \in \mathbb{N}} \text{cl}_{\tau_j} H_n\),

and similarly for each closed lower set \(H\) there exists a sequence \((H_n)_{n \in \mathbb{N}}\) of open lower sets such that

(i) if \(H \subseteq K\), then \(H_n \subseteq K_n\) whenever \(n \in \mathbb{N}\), and (ii) \(H = \bigcap_{n \in \mathbb{N}} \text{cl}_{\tau_j} H_n\).

A topological space \((X, \tau)\) is called stratifiable if and only if \((X, \tau, \tau)\) is pairwise stratifiable. It is known that each stratifiable space \((X, \tau)\) is monotonically normal (compare e.g. [9]), that is, to each pair \((H, K)\) of disjoint closed subsets of \(X\) one can assign an open set \(D(H, K)\) such that

(i) \(H \subseteq D(H, K) \subseteq \text{cl}_{\tau} D(H, K) \subseteq X \setminus K\),

(ii) if pairs \((H, K), (H', K')\) satisfy \(H \subseteq H'\) and \(K' \subseteq K\), then \(D(H, K) \subseteq D(H', K')\).

Such a \(D\) is called a monotone normality operator on \(X\). Of course, each metrizable space is stratifiable. The following result is crucial for our investigations.

**Theorem 1.** Let \((X, \tau, \preceq)\) be a topological ordered C-space with a stratifiable topology \(\tau\). Then \((X, \tau^\flat, \tau^\natural)\) is pairwise stratifiable.

**Proof.** Since \(\tau\) is stratifiable, for each closed set \(F\) of \(X\) there is a sequence \((F_n)_{n \in \mathbb{N}}\) of open sets such that (i) \(\bigcap_{n \in \mathbb{N}} \text{cl}_{\tau} F_n = F\) and (ii) for all pairs \(H, F\) of closed sets such that \(H \subseteq F\), we have \(H_n \subseteq F_n\) whenever \(n \in \mathbb{N}\). Then we can define (compare e.g. [2,19]) a monotone normality operator \(D\) on \(X\) by setting \(D(H, K) = \bigcap_{n \in \mathbb{N}} (H_n \setminus \text{cl}_{\tau} K_n)\) whenever \((H, K)\) is a pair of disjoint closed sets of \(X\).

Let \(F\) be a closed upper set of \(X\). Put \(F^\flat = X \setminus d(X \setminus D(F, d(X \setminus F)))\). Note that \(F^\flat\) is an open upper set. Furthermore \(F \subseteq F^\flat \subseteq D(F, d(X \setminus F_n)) \subseteq \text{cl}_{\tau} D(F, d(X \setminus F_n)) \subseteq X \setminus d(X \setminus F_n)\). Therefore \(i(\text{cl}_{\tau} D(F, d(X \setminus F_n))) \subseteq X \setminus d(X \setminus F_n)\) and so \(\text{cl}_{\tau} D(F, d(X \setminus F_n)) \subseteq X \setminus d(X \setminus F_n) \subseteq F_n\), since \(X\) is a C-space. We conclude that \(F = \bigcap_{n \in \mathbb{N}} \text{cl}_{\tau} F_n\), because \(\text{cl}_{\tau} F_n \subseteq F_n\) whenever \(n \in \mathbb{N}\). Moreover if \(H\) and \(F\) are closed upper sets such that \(H \subseteq F\), then \(H^\flat \subseteq F^\flat\) whenever \(n \in \mathbb{N}\): Indeed given \(n \in \mathbb{N}\) we have \(H_n \subseteq F_n\), thus \(X \setminus F_n \subseteq X \setminus H_n\) and \(d(X \setminus F_n) \subseteq d(X \setminus H_n)\). Consequently, \(H \subseteq X \setminus d(X \setminus H_n)\), \(d(X \setminus F_n) \subseteq X \setminus F\) and \(d(X \setminus F_n) \subseteq d(X \setminus H_n)\). Hence \(D(H, d(X \setminus H_n)) \subseteq D(F, d(X \setminus F_n))\). Therefore \(X \setminus D(F, d(X \setminus F_n)) \subseteq X \setminus D(H, d(X \setminus H_n))\) and
of compact spaces yields a similar result under the condition that

\[ \text{Remark 1.} \] According to Proposition 4 and Remark 2 below under the conditions of Corollary 1 the topologies \( \tau^0 \) and \( \tau^1 \) are quasi-pseudometrizable provided that \( i(x) \) and \( d(x) \) are compact whenever \( x \in X \); furthermore Corollary 3 below yields a similar result under the condition that \( X \) is an \( I \)-space (see also [15, p. 131] for related results).

4. Compactness of the boundaries

In the light of the Hanai–Morita–Stone Theorem cited in the introduction one might wonder whether the hypotheses of Problem 1 imply \( \tau \)-compactness of the boundaries of the sets \( d(y) \) and \( i(y) \) whenever \( y \in X \). Indeed the answer to this question is positive, as our next result implies (compare Proposition 2). Unfortunately we do not know (even under the additional condition that \( X \) is an \( I \)-space) whether 'first-countable' can be replaced by 'quasi-pseudometrizable' in Theorem 2 (compare Corollary 1).

**Theorem 2.** Let \( X, \tau, \leq \) be a topological ordered \( C \)-space with metrizable topology \( \tau \). Then both the lower topology \( \tau^0 \) and the upper topology \( \tau^1 \) are first countable if and only if only if for each \( y \in X \), \( bd_\tau d(y) \) and \( bd_\tau i(y) \) are compact in \( (X, \tau) \).

**Proof.** Let \( r \) be a compatible metric on \( X \) and for each \( n \in \mathbb{N} \) let \( B_{r,2^n} = \{(x, y) \in X \times X : r(x, y) < 2^{-n}\} \). Suppose first that both the lower topology \( \tau^0 \) and the upper topology \( \tau^1 \) are first countable. Furthermore let \( y \in X \) and let \( (x_n)_{n \in \mathbb{N}} \) be a sequence in \( \partial d(y) \). Observe that \( i(y) \) is \( \tau \)-closed, since \( X \) is a metrizable \( C \)-space. Moreover let \( I_n \in \mathbb{N} \) be a \( \tau \)-neighborhood base at \( y \) consisting of \( \tau \)-open sets. Fix \( n \in \mathbb{N} \). Note that \( I_n \cap B_{r,2^{-\frac{1}{2n}}} = (x_n)_{n \in \mathbb{N}} \) is a \( \tau \)-neighborhood of \( x_n \), hence \( I_n \cap B_{r,2^{-\frac{1}{2n}}} \not\supseteq \{i(y) \neq \emptyset\} \). Choose \( x_n \in \{I_n \cap B_{r,2^{-\frac{1}{2n}}} \not\supseteq \{i(y) \neq \emptyset\} \} \cap \{i(y) \neq \emptyset\} \). Assume first that \( cl_\tau \{x_n : n \in \mathbb{N} \} \cap \{i(y) \neq \emptyset\} \). Since \( X \) is a \( C \)-space, \( d(cl_\tau \{x_n : n \in \mathbb{N} \} \cap i(y) \neq \emptyset\} \) is \( \tau \)-closed and obviously disjoint from \( i(y) \). Therefore there is \( m \in \mathbb{N} \) such that \( I_m \cap d(cl_\tau \{x_n : n \in \mathbb{N} \} \cap i(y) \neq \emptyset\} \) is a contradiction. Hence we conclude that there is \( a \in cl_\tau \{x_n : n \in \mathbb{N} \} \cap i(y) \). Then \( a \) is a \( \tau \)-cluster point of the sequence \( (x_n)_{n \in \mathbb{N}} \) and therefore belongs to the \( \tau \)-closed set \( bd_\tau i(y) \). We have shown that each sequence \( (x_n)_{n \in \mathbb{N}} \) in \( bd_\tau i(y) \) has a \( \tau \)-cluster point. Hence \( bd_\tau i(y) \) is countably compact and thus \( \tau \)-compact, since countably compact metrizable spaces are compact. Similarly, it can be shown that \( bd_\tau d(y) \) is \( \tau \)-compact in \( X \).

For the converse suppose that for each \( y \in X \), \( bd_\tau d(y) \) and \( bd_\tau i(y) \) are compact in \( (X, \tau) \). Fix \( x \in X \). Then for each \( m \in \mathbb{N} \) set \( H_m = X \setminus \{x \setminus \{i(x) \cup B_{r,2^{-\frac{1}{2m}}} - bd_\tau i(x)\} \} \). We claim that \( H_m : m \in \mathbb{N} \) is a neighborhood base at \( x \) for the upper topology \( \tau^1 \). Fix \( n \in \mathbb{N} \). Clearly \( i(x) \subseteq bd_\tau i(x) \cup B_{r,2^{-\frac{1}{2m}}} - bd_\tau i(x) \). Hence \( H_m \) is an open upper set containing \( x \), since \( X \) is a \( C \)-space. Let \( G \) be any open upper set such that \( x \in G \). Because \( G \) is an open upper set containing the \( \tau \)-closed set \( i(x) \), by compactness of \( bd_\tau i(x) \) there is \( p \in \mathbb{N} \) such that \( B_{r,2^{-\frac{1}{2p}}} - bd_\tau i(x) \subseteq G \). Thus \( bd_\tau i(x) \cup \partial bd_\tau i(x) \subseteq G \). We conclude that \( \tau \subseteq G \). Therefore we have established our claim and the upper topology \( \tau^1 \) is proven to be first countable. Analogously, one shows that the lower topology \( \tau^0 \) is first countable.

**Example 1.** Consider the subspace \( T = \{(u, v) \in \mathbb{R}^2 : 0 \leq \frac{1}{2}u \leq v \leq 3u\} \) of the plane \( \mathbb{R}^2 \), where the plane is equipped with the Euclidean product topology and the product order \((x_1, x_2) \leq (y_1, y_2) \) provided that \( x_1 \leq y_1 \) and \( x_2 \leq y_2 \).

Note first that \( T \) is an \( I \)-space. It clearly suffices to show that for each \( x \in T \) and open disk \( B(x) \) of radius \( \epsilon \) around \( x \) in \( \mathbb{R}^2 \), \( \partial T_{x} \) and \( \partial T_{x} \) are open in \( T \), which however is readily seen. We then recall the following fact that holds in any topological \( T_{2} \)-ordered space, so also in \( T \): If \( K \) is compact in \( T \), then \( \partial T_{x} \) and \( \partial T_{x} \) are open in \( T \) (compare e.g. [7, Proposition 4.3]). Let us now assume that \( \tau \neq \emptyset \) is compact in \( T \). If \( F \) is bounded in the usual metric on \( \mathbb{R}^2 \), then \( \partial T_{F} \) and \( \partial T_{F} \) are closed, because \( F \) is compact in \( T \). If \( F \) is unbounded, then \( \partial T_{F} \) is open in \( F \) and \( \partial T_{F} \) is closed, respectively. Hence \( F \) is bounded in the usual metric on \( \mathbb{R}^2 \), then \( \partial T_{F} \) and \( \partial T_{F} \) are closed, because \( F \) is compact in \( T \). If \( F \) is unbounded, then \( \partial T_{F} \) is open in \( T \) and \( \partial T_{F} \) is closed, respectively. Hence \( F \) is bounded in the usual metric on \( \mathbb{R}^2 \), then \( \partial T_{F} \) and \( \partial T_{F} \) are closed, because \( F \) is compact in \( T \).

Note that in a topological ordered \( I \)-space \( (X, \tau, \leq) \), \( \partial T_{x} \) is \( \tau \)-open whenever \( x \in X \), since \( i(\partial T_{x}) \subseteq i(x) \) (resp. \( d(\partial T_{x}) \subseteq d(x) \) and \( i(\partial T_{x}) \subseteq i(x) \) and \( d(\partial T_{x}) \subseteq d(x) \)).

8 Note that in a topological ordered \( I \)-space \( (X, \tau, \leq) \), \( i(\partial T_{x}) \subseteq i(x) \) (resp. \( d(\partial T_{x}) \subseteq d(x) \) and \( i(\partial T_{x}) \subseteq i(x) \) and \( d(\partial T_{x}) \subseteq d(x) \)).

9 Recall that \( R^2 \) with this topology and order is not a \( C \)-space (compare e.g. [23, p. 139]), because for instance \( F = \{(-n, \frac{1}{n}) : n \in \mathbb{N}\} \) is closed, but \( (0, 0) \in cl_\tau \{F\} \).
5. Quasi-metrizability in topological ordered spaces

In this section we construct a counterexample to that modification of Problem 1, in which we assume that the starting topology \( \tau \) is only quasi-metrizable (instead of metrizable). The following classical results seemed to suggest that this modified version of our problem could have had a positive solution: The perfect image of a quasi-metrizable space is quasi-metrizable \([11]\).\(^{10}\) In fact, a first-countable image of a quasi-metrizable space under a closed continuous map is quasi-metrizable \([12]\). In particular, the image of a quasi-metrizable space under a continuous map which is both open and closed is quasi-metrizable.

Example 2. A quasi-metrizable topological ordered space \((X, \tau, \leq)\) is constructed which is a \(C\) - and \(I\)-space, but such that neither the lower topology \(\tau^\leq\) nor the upper topology \(\tau^\geq\) are quasi-pseudometrizable: To this end we choose a fixed partition of the open real unit interval \([0, 1]\] equipped with the Euclidean topology into two sets \(A\) and \(B\) such that \(A \cap J \neq \emptyset\) and \(B \cap J\) are of second category in \([0, 1]\) whenever \(J\) is a (nonempty) open interval in \([0, 1]\] (see e.g. \([4]\)). We equip \(X = [0, 1]\] with its usual linear order \(\leq\). As a base for our topology \(\tau\) we take the union of the usual Euclidean topology on \([0, 1]\] and the set of all singletons \([a]\) with \(a \in A\). In this way we obtain a quasi-metrizable topology, because \(\tau\) clearly has a \(\sigma\)-interior-preserving base (compare with the construction of the Michael line \([7, p. 4]\), since for each metrizable topology there is a \(\sigma\)-locally finite base by the Nagata–Smirnov Theorem.

We first prove that our space \((X, \tau, \leq)\) is a \(C\) - and \(I\)-space. Let \(F\) be a \(\tau\)-closed set in \(X\). Evidently it suffices to consider the case that \(i(F) \neq \emptyset\) and \(i(F) \neq [0, 1]\]. If \(i(F) \in F\), then \(i(F) = \{\text{inf} F, 1\} = \tau\)-closed in \(X\). If \(i(F) \notin F\), then \(i(F) = A\) and \(i(F) = \text{inf} F, 1\] is \(\tau\)-closed. An analogous argument shows that \(d(F)\) is \(\tau\)-closed in \(X\) whenever \(F\) is \(\tau\)-closed in \(X\). We conclude that \(X\) is a \(C\)-space. Let \(G\) be \(\tau\)-open in \(X\). Similarly as above, it suffices to consider the case that \(i(G) \neq \emptyset\) and \(i(G) \neq [0, 1]\]. If \(i(G) \in G\), then \(i(G) \in A\), and thus \(i(G) = \{\text{inf} G, 1\} = \tau\)-open. Otherwise \(i(G) \notin G\), in which case \(i(G) = \text{inf} G, 1\] is \(\tau\)-open. We conclude that \(i(G)\) is \(\tau\)-open whenever \(G\) is \(\tau\)-open in \(X\). Analogously one shows that \(d(G)\) is \(\tau\)-open whenever \(G\) is \(\tau\)-open in \(X\). Hence \(X\) is an \(I\)-space.

We finally show that the two topologies of \((X, \tau^\leq, \tau^\geq)\) are not quasi-pseudometrizable. The closure of a subset \(D\) of \([0, 1]\] with respect to the Euclidean topology on \([0, 1]\] will be denoted by \(\overline{D}\). In order to reach a contradiction, suppose that \(r\) is a compatible quasi-pseudometric on \((X, \tau^\geq)\). We let \(B_{\gamma} = \{x, y\} \times X \times X: (x, y, \gamma) < 2^{-\gamma}\) whenever \(\gamma \in \mathbb{N}\). For each \(\gamma \in \mathbb{N}\) set \(A_{\gamma} = \{x \in A: B_{\gamma-\delta}(x) = \{x, 1\}\}\), where we note that for each \(x \in A\) the interval \([x, 1]\] is open in the upper topology \(\tau^\geq\). Since \(A = \bigcup_{\gamma \in \mathbb{N}} A_{\gamma}\) is of second category in \([0, 1]\], there are \(n_{\gamma} \in \mathbb{N}\) and a nonempty Euclidean open interval \(J\) of \([0, 1]\] such that \(J \subseteq \overline{A_{n_{\gamma}}}\). For each \(m \in \mathbb{N}\) set \(C_{m} = \{x \in B \cap J: |x - 2^{-m}| \in B_{\gamma - (n_{\gamma} + 1)}(x)\}\). Since \(B \cap J = \bigcup_{m \in \mathbb{N}} C_{m}\) and \(B \cap J\) is of second category in \([0, 1]\], there are \(n_{\gamma} \in \mathbb{N}\) and a nonempty Euclidean open interval \(J\) of \([0, 1]\] such that \(J \subseteq \overline{C_{n_{\gamma}}}\). In order to reach a contradiction, suppose that \(J \cap A_{n_{\gamma}} = \emptyset\). Then \(J \cap A_{n_{\gamma}} = \emptyset\). It follows that \(J \cap C_{n_{\gamma}} = \emptyset\), which implies that \(J \cap C_{n_{\gamma}} = \emptyset\) and hence \(J \cap C_{n_{\gamma}} = \emptyset\). It follows that \(J \cap A_{n_{\gamma}} = \emptyset\). Thus \(J \cap C_{n_{\gamma}} = \emptyset\), which implies that \(J \cap C_{n_{\gamma}} = \emptyset\) and hence \(J \cap C_{n_{\gamma}} = \emptyset\). Therefore there exists \(a \in J \cap A_{n_{\gamma}}\). Moreover there is \(\delta > 0\) such that \(|a, a + \delta| \subseteq J \subseteq \overline{C_{n_{\gamma}}}\). We conclude that there is a strictly decreasing sequence \((c_k)_{k \in \mathbb{N}}\) of elements of \(C_{n_{\gamma}}\) converging to \(a\) with respect to the Euclidean topology on \([0, 1]\]. Consequently \(c_k \in [a, 1] = B_{\gamma - (n_{\gamma} + 1)}(a)\) whenever \(k \in \mathbb{N}\) by compatibility of \(r\). Furthermore \(|c_k - 2^{-m_{\gamma}}| \in B_{\gamma - (n_{\gamma} + 1)}(c_k)\) whenever \(k \in \mathbb{N}\). Thus \(|c_k - 2^{-m_{\gamma}}| \in B_{\gamma - (n_{\gamma} + 1)}(a)\) whenever \(k \in \mathbb{N}\). We have reached a contradiction, since \((c_k)_{k \in \mathbb{N}}\) converges to \(a\) with respect to the Euclidean topology on \([0, 1]\]. Hence \((X, \tau^\geq)\) is not quasi-pseudometrizable. Similarly one shows that \((X, \tau^\leq)\) is not quasi-pseudometrizable.

6. A positive partial result using a uniform approach

In this section we consider the approach to Problem 1 that is based on the Hausdorff hyperspace idea outlined in the introduction. Since the following construction does not use that the studied uniformity has a countable base, we state the corresponding result without that restriction.

Proposition 4. Let \((X, \tau, \leq)\) be a completely regular topological ordered space. Furthermore let \(U\) be a compatible uniformity on \((X, \tau)\). We shall define a quasi-uniformity \(U_1\) on \(X\) having the base \(\{U_1: U \in U\} \) where \(U_1 = \{(x, y) \in X \times X: i(y) \subseteq U(i(x))\}\) whenever \(U \in U\).

(a) Let \((X, \tau, \leq)\) be a \(C\)-space. Then \(\tau(U_1) \subseteq \tau^\geq\). We have \(\tau(U_1) = \tau^\leq\) if \(i(x)\) is \(\tau\)-compact whenever \(x \in X\).

(b) Let \((X, \tau, \leq)\) be an \(I\)-space. Then \(\tau_i^\leq \subseteq \tau(U_1)^{-1}\). If \(i(x)\) is totally bounded whenever \(x \in X\), then \(\tau_i(U_1)^{-1} = \tau^\leq\).

Proof. (a) One readily checks that \(U_1\) is indeed a quasi-uniformity on \(X\). Note that \(\{U_1: U \in U\} \) is open whenever \(x \in X\) also yields a base of \(U_1\). For \(U \in U\) such that \(U(x)\) is open whenever \(x \in X\), we see that \(U_1(x) = \{y \in X: i(y) \subseteq U(i(x))\}\) is equal to the open upper set \(X \backslash U(i(x))\) whenever \(x \in X\), because \(X\) is a \(C\)-space. Therefore the topology induced by \(U_1\) is coarser than the upper topology \(\tau^\geq\). Suppose that \(x \in X, i(x)\) is compact and \(G\) is an open upper set of \(X\)

\(^{10}\) A continuous map \(f: X \to Y\) between topological spaces \(X\) and \(Y\) is called perfect provided that it is closed (that is, closed sets are mapped to closed sets) and all the fibers \(f^{-1}(y)\) where \(y \in Y\) are compact.
containing $x$. Then $i(x) \subseteq G$ and there is $U \in \mathcal{U}$ such that $U(i(x)) \subseteq G$. Thus $U_1(x) \subseteq G$. Altogether we see that $\tau(U_1) = \tau^3$ provided that $i(x)$ is compact whenever $x \in X$.

(b) Note next that for each $x \in X$ and $U \in \mathcal{U}$ with $U = U^{-1}$ we have that $(U_1)^{-1}(x) = \bigcap_{x \in (x)} d(U(x'))$, as a straightforward computation reveals. The first statement is now obvious. Fix $x \in X$ and $U \in \mathcal{U}$ such that $U = U^{-1}$. We want to show that if $i(x)$ is totally bounded, then $\bigcap_{x \in (x)} d(U(x'))$ is a neighborhood of $x$ in the lower topology: Indeed let $V \in \mathcal{U}$ be such that $V(x)$ is open whenever $x \in X$, $V = V^{-1}$, and $V^2 \subseteq U$. Then there is some finite subset $F_V$ of $i(x)$ such that $i(x) \subseteq V(F_V)$. It follows that for any $x' \in i(x)$ there is $f \in F_V$ such that $x' \in V(f)$, and thus $d(V(f)) \subseteq d(V^2(x'))$. Therefore $\bigcap_{f \in F_V} d(V(f)) \subseteq \bigcap_{x \in (x)} d(V^2(x')) \subseteq (U_1)^{-1}(x)$, where the first set is evidently a neighborhood of $x$ with respect to the lower topology $\tau^3$, since $X$ is an $I$-space. Hence the topology induced by $(U_1)^{-1}$ on $X$ is clearly the lower topology. □

Remark 2. Analogously under the conditions of Proposition 4, one defines the quasi-uniformity $U_{i}^{1}$ having the base $\{U_{i}^{1} : U \in \mathcal{U}\}$ where

$$U_{i}^{1} = \{ (x, y) \in X \times X : d(y) \subseteq U(d(x)) \}.$$ 

By an analogous argument one shows that if $X$ is a $C$-space, then $\tau(U_{i}^{1})$ is coarser than the lower topology $\tau^3$ of $X$, and equality for these two topologies holds provided that $d(x)$ is $\tau$-compact whenever $x \in X$. In case that $X$ is an $I$-space, $\tau(U_{i}^{1})^{-1}$ is finer than the upper topology $\tau^5$ of $X$. These two topologies are equal provided that $d(x)$ is totally bounded whenever $x \in X$.

Example 3. In the light of Proposition 4(b) the following observation may also be of interest: Let $(X, \tau, \leq)$ be a topological lattice equipped with a compatible uniformity $\mathcal{U}$. Moreover let $x \in X$ and suppose that the family $\{j_{a} : a \in i(x)\}$ of maps from $(X, \tau(f))$ to $(X, \mathcal{U})$ (defined by $j_{a}(y) = a \lor y$ whenever $y \in X$) is equicontinuous at $x$. Then for each $U \in \mathcal{U}$, $\bigcap_{a \in i(x)} d(U(a))$ is a $\tau^3$-neighborhood at $x$. □

Proof. Recall that $(X, \tau, \leq)$ is an $I$-space (compare e.g. [14, p. 291]). Let $U \in \mathcal{U}$. By assumption there is an (open) neighborhood $N$ of $x$ such that $j_{a}(N) \subseteq U(j_{a}(x))$ whenever $a \geq x$. Let $y \in N$. Then $a \lor y \in U(a \lor x) = U(a)$ whenever $a \geq x$, and consequently $y \in \bigcap_{a \in i(x)} d(U(a))$. Thus $N \subseteq \bigcap_{a \in i(x)} d(U(a))$ and $d(N) \subseteq \bigcap_{a \in i(x)} d(U(a))$. □

Corollary 2. Let $(X, \tau, \leq)$ be a completely regular (Hausdorff) topological ordered $C$- and $I$-space and $\mathcal{U}$ a compatible separated uniformity on $(X, \tau)$ such that both $i(x)$ and $d(x)$ are totally bounded whenever $x \in X$. Then $U_{i}^{1} = U_{i} \lor (U_{i}^{-1})$ is a compatible quasi-uniformity on the bispace $(X, \tau^{3}, \tau^{5})$. In case that the topology $\tau$ is convex, the uniformity $U_{i}^{1}$ determines the topological ordered space $(X, \tau, \leq)$ in the sense of Nachbin (see [7, p. 81] and [20]), that is, $\tau((U_{i}^{1})^{3}) = \tau$ and $\bigcap U_{i}^{1} = \leq$.

Proof. By Proposition 4 and Remark 2 we conclude that for the quasi-uniformity $U_{i}^{1} = U_{i} \lor (U_{i}^{-1})$ the topology $\tau((U_{i}^{1})^{-1})$ is the lower topology $\tau^3$ and the topology $\tau(U_{i}^{1})$ is the upper topology $\tau^5$ of $X$. Furthermore $\bigcap U_{i}^{1} = \leq$. Let $x \in X$. First note that for each $y \geq x$, $y \in \bigcap_{U \in \mathcal{U}} U_{i}$. Furthermore if $y \notin i(x)$, then there is a symmetric $U \in \mathcal{U}$ such that $y \notin U(i(x))$ because $i(x)$ is $\tau(U)$-closed, since $\tau(U) = \tau_{1}$-topology and $X$ is a $C$-space. Consequently $i(y) \notin U(i(x))$ and then $y \notin U_{i}(x)$. Similarly one also verifies that $\bigcap U_{i} = \geq$. Obviously then $\tau((U_{i}^{1})^{3}) = \tau^{3} \lor \tau^{5}$, which is equal to $\tau$ according to the definition of convexity of $\tau$. Moreover $\bigcap U_{i} = \bigcap(U_{i}^{1})^{-1} = \bigcap U_{i}^{1} = \leq$. □

Corollary 3. Let $(X, m, \leq)$ be a metric topological ordered $C$- and $I$-space such that $i(x)$ and $d(x)$ are totally bounded whenever $x \in X$. Then the bitopological space $(X, (\tau(m))^{3}, (\tau(m))^{5})$ is quasi-pseudometrizable.

Proof. Note that in the proof of Corollary 2 the quasi-uniformity $\mathcal{U}_{i}^{1}$ has a countable base provided that the uniformity $\mathcal{U}$ has a countable base, and apply Corollary 2 to the metric uniformity $\mathcal{U}_{m}$. □

We are now going to present an interesting application of the preceding result. We shall call a uniform space $(X, \mathcal{U})$ uniformly locally connected (compare [5]) provided that for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V \subseteq U$ and $V(x)$ is connected whenever $x \in X$.

Proposition 5. Let $(X, \tau, \leq)$ be a completely regular topological ordered $C$- and $I$-space equipped with a $\tau$-compatible uniformly locally connected separated uniformity $\mathcal{U}$ possessing a countable base. Then the bispace $(X, \tau^{3}, \tau^{5})$ is quasi-pseudometrizable.

Proof. Let us suppose that for some $V \in \mathcal{U}$, $V(x)$ is connected whenever $x \in X$ and that $V \subseteq U$ for some given symmetric $U \in \mathcal{U}$. We first show that for each $x \in X$, $V(i(x)) \subseteq U^{2}(bd_{x}(i(x))) \cup i(x)$: Fix $x \in X$. Observe that $cl_{x}(i(x)) = i(x)$. It clearly

\[\text{Note that the latter condition is satisfied in a uniform lattice, that is, in a lattice equipped with a uniformity such that the lattice operations } \land \text{ and } \lor \text{ are uniformly continuous.}\]
suffices to verify that $V(y) \subseteq U^2(\text{bd}_t(i(x))) \cup i(x)$ whenever $y \in \text{int}_t(i(x))$. So let us consider an arbitrary $y \in \text{int}_t(i(x))$.

If $V(y) \subseteq i(x)$, then we are finished. So we can suppose that $V(y) \setminus i(x) \neq \emptyset$. By connectedness of $V(y)$, we deduce that there is $a \in V(y) \cap \text{bd}_t(i(x))$. Then $y \in V^{-1}(a)$, and hence $V(y) \subseteq U(U^{-1}(a)) = U^2(a)$. Therefore the stated inclusion is established.

We now verify that the quasi-uniformity $\mathcal{U}_t$ defined in Proposition 4 induces the upper topology $\tau^1$ on $X$. Let $x \in X$ and $U$ be an open upper set containing $x$. According to Theorem 2 and Proposition 2, $\text{bd}_t(i(x))$ is a compact subset of $G$, and hence there is a symmetric entourage $U \in \mathcal{U}$ such that $U^2(\text{bd}_t(i(x))) \cup i(x) \subseteq G$. By the preceding arguments we conclude that by our assumption there is $V \in \mathcal{U}$ such that $V(i(x)) \subseteq U^2(\text{bd}_t(i(x))) \cup i(x)$. Thus $V(x) \subseteq G$ and therefore $\tau^1 \subseteq \tau(\mathcal{U}_t)$. We deduce that $\tau(\mathcal{U}_t) = \tau^1$ by Proposition 4. Hence $\tau^1$ is quasi-pseudometrizable, since $\mathcal{U}_t$ has a countable base. Similarly one shows with the help of Remark 2 that $\tau^2$ is quasi-pseudometrizable. By Corollary 1 we conclude that the bispaces $(X, \tau^1, \tau^2)$ is quasi-pseudometrizable. □

**Problem 2.** Let $(X, m, \leq)$ be a metric topological ordered $C$- and $I$-space such that the metric uniformity $\mathcal{U}_m$ is uniformly locally connected. With the aid of $m$, is there a simple way to construct a compatible quasi-uniformity with a countable base on the bispaces $(X, \tau(m)^1, \tau(m)^2)$?

**Corollary 4.** Let $(X, \tau, \leq)$ be a metrizable locally connected topological ordered $C$- and $I$-space. Then $(X, \tau^3, \tau^4)$ is quasi-pseudometrizable.

**Proof.** By Proposition 5 it suffices to show that $X$ admits a uniformly locally connected (separated) uniformity with a countable base. To this end assume that $U_n = U_n$ is a base for a compatible uniformity on $(X, \tau)$ such that $U^2_n \subseteq U_n$ whenever $n \in \mathbb{N}$. Suppose that for some $n \in \mathbb{N}$, $H_n$ is defined as a neighborhood of the diagonal of $X$. Since $X$ is paracompact (compare e.g. [18, Corollary 2.8]), there is a symmetric neighborhood $U$ of the diagonal of $X$ such that $U^4 \subseteq (H_n \cap U_{n+1})$. For each $x \in X$, find a connected neighborhood $C_x$ of $x$ such that $C_x \subseteq U(x)$. Set $H_{n+1} = \bigcup_{x \in X} (C_x \times C_x)$. Since $H_{n+1} \subseteq U_{n+1}$ and $H_{n+1} \subseteq H_n$ whenever $n \in \mathbb{N}$, we see that $\{H_n : n \in \mathbb{N}\}$ is a countable base for a compatible uniformity $\mathcal{U}$ on $(X, \tau)$. Furthermore $\mathcal{U}$ is uniformly locally connected, because for each $x \in X$ and $n \in \mathbb{N}$, $H_{n+1}(x)$ is connected as the union of connected sets intersecting at $x$ (compare [5, proof of Lemma 1]). □

7. Uniformities with friendly partial orders

In this section we finally discuss a natural, but strong compatibility condition between uniformity and partial order. Indeed we shall consider commutativity of composition under the order (quasi-uniformity) and the uniformity (see the next paragraph for the precise definition). A similar, but stronger condition appears in a metric form in [23] under the name of ball transitivity. A variant of our condition can also be found in a uniform form in [20, p. 72]. Again our discussion does not rely on a countable base of the studied uniformity and in the following work without that restriction. Recall finally that for two binary relations $A$ and $B$ on a set $X$ the composition $B \circ A$ is defined as the relation $(a, c) \in X \times X :$ there is $b \in X$ such that $(a, b) \in A$ and $(b, c) \in B$ on $X$.

Let $(X, \mathcal{U})$ be a uniform space and $\leq$ be a partial order on $X$. Because of lack of a better name, we shall say that $\leq$ is a $\mathcal{U}$-friendly partial order on $(X, \mathcal{U})$ provided that for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V(i(x)) \subseteq i(U(x))$ and $V(d(x)) \subseteq d(U(x))$ whenever $x \in X$. Note that these two conditions can be written as $V \circ i \leq \circ U$ and $V \circ i \geq \circ U$. Indeed let $x \leq t$ and $(t, y) \in W$. Then $(x, y) \in W$ and $(t, y) \in W$, which implies that $(x \wedge t, x \wedge y) \in V$. Consequently $(x, x \wedge y) \in V$ and $x \wedge y \leq y$, which establishes the first inclusion. Similarly let $x \geq y$ and $(y, t) \in W$. Then $(x, y) \in W$ and $(y, t) \in W$, which implies that $(x \vee y, x \vee t) \in V$. It follows that $(x, x \vee t) \in V$ and $x \vee t \geq t$, which verifies the second inclusion. □

A uniform space $(X, \mathcal{U})$ will be called uniformly locally order convex provided that for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \subseteq U$ and $V(x)$ is order convex whenever $x \in X$. (Note that such uniformities are called convex in [7, p. 84].)

**Example 4.** Let $(X, \mathcal{U})$ be a uniformly locally order convex uniform space equipped with a linear order $\leq$ (on $X$). Then $\leq$ is a $\mathcal{U}$-friendly partial order (compare with [7, Theorem 4.20]).

**Proof.** By a result of Redfield (compare [22, Proposition 2.2]) $(X, \mathcal{U}, \leq)$ is a uniform lattice so that the statement follows from Proposition 6. □
Since the property of friendliness is productive, we deduce from Example 4 that the usual product order $\leq$ on $\mathbb{R}^2$ is $\mathcal{U}_b$-friendly where $\mathcal{U}_b$ denotes the Euclidean product uniformity on $\mathbb{R}^2$, although, as observed above, the corresponding topological ordered space is not a C-space.

**Remark 3.** If $(X,\mathcal{U})$ is a uniform space with a $\mathcal{U}$-friendly partial order $\leq$, then $(X,\tau(\mathcal{U}),\leq)$ is an $L$-space (compare [23, Proposition 2]).

**Proof.** Let $G$ be $\tau(\mathcal{U})$-open and $x \in d(G)$. Then there are $y \in G$ and $U \in \mathcal{U}$ such that $x \leq y$ and $U(y) \subseteq G$. By friendliness there is $V \in \mathcal{U}$ such that $V(d(y)) \subseteq d(U(y))$. Then $V(d(y)) \subseteq d(U(y)) \subseteq d(G)$ and thus $V(x) \subseteq d(G)$. Therefore $d(G)$ is $\tau(\mathcal{U})$-open in $X$. Similarly, one shows that $i(G)$ is $\tau(\mathcal{U})$-open in $X$ whenever $G$ is $\tau(\mathcal{U})$-open in $X$. Hence $X$ is an $L$-space. \hfill $\square$

**Proposition 7.** Let $(X,\mathcal{U})$ be a uniform space with a $\mathcal{U}$-friendly partial order $\leq$ on $X$. Then the bitopological space $(X,\tau(\mathcal{U}))^2$, $(\tau(\mathcal{U}))^3$ is quasi-uniformizable by a quasi-uniformity of a weight smaller than or equal to the weight of $\mathcal{U}$.

**Proof.** We show that the filter $\mathcal{U}_f$ on $X \times X$ generated by the base $\{U_\varnothing := \bigcup_{x \in X}([x] \times d(U(x))): U \in \mathcal{U}\}$ is a quasi-uniformity on $X$: Clearly each described generating relation is reflexive. Fix $U \in \mathcal{U}$. Then there is $V \in \mathcal{U}$ such that $V \subseteq U$. Furthermore by friendliness there is $W \in \mathcal{U}$ such that $W \subseteq V$ and for each $y \in X$, $W(U(y)) \subseteq U(V(y))$. Consequently $i(W(U(X))) = \bigcup_{y \in X} W(U(y)) \subseteq \bigcup_{y \in X} i(U(V(y)))$ whenever $x \in X$. We conclude that $\mathcal{U}_f$ is a quasi-uniformity on $X$. Since by Remark 3 $(X,\tau(\mathcal{U}),\leq)$ is an $L$-space, it follows from the definition of $\mathcal{U}_f$ that $\tau(\mathcal{U}_f) = (\tau(\mathcal{U}))^2$. One readily checks that the family of all relations $\bigcup_{x \in X}([x] \times d(U(x)))(U \in \mathcal{U})$ generates on $X \times X$ the conjugate quasi-uniformity $(\mathcal{U}_f)^{-1}$. Obviously then $\tau((\mathcal{U}_f)^{-1}) \subseteq (\tau(\mathcal{U}))^2$, since by friendliness for symmetric $U \in \mathcal{U}$ there is symmetric $V \in \mathcal{U}$ such that $V \subseteq U$, and by conjugation thus for each $x \in X$, $d(V(x)) \subseteq U(d(x))$. Therefore for each $x \in X$, $d(U(x))$ is a $\tau^2$-neighborhood at $x$.

We now set $\mathcal{U}_g := \mathcal{U}_f \vee (\mathcal{U}_f)^{-1}$. Of course, here $\mathcal{U}_g$ is the quasi-uniformity on $X$ generated by the family of all relations $U_g = \bigcup_{x \in X}([x] \times d(U(X)))$ where $U \in \mathcal{U}$. Since by an argument similar to the one given above, $\tau((\mathcal{U}_g)^{-1}) = (\tau(\mathcal{U}))^3$ and $\tau((\mathcal{U}_g)^{-1}) \subseteq (\tau(\mathcal{U}))^3$, we conclude that $\tau(\mathcal{U}_g) = (\tau(\mathcal{U}))^2$ and similarly $\tau((\mathcal{U}_g)^{-1}) = (\tau(\mathcal{U}))^3$. Hence we have shown that the bitopological space $(X, (\tau(\mathcal{U}))^2, (\tau(\mathcal{U}))^3)$ is quasi-uniformizable by $\mathcal{U}_g$. Clearly if $\mathcal{U}$ has a base of cardinality $\kappa$, then $\mathcal{U}_g$ has a base of cardinality $\kappa$. \hfill $\square$

**Corollary 5.** Let $(X, m)$ be a metric space carrying a $\mathcal{U}_m$-friendly partial order $\leq$. Then the bispace $(X, (\tau(m))^2, (\tau(m))^3)$ is quasi-pseudometrizable.

**Remark 4.** We note that in the proof of Proposition 7 if $\mathcal{U}$ is a uniformly locally order convex uniformity, then $\mathcal{U} = (\mathcal{U}_g)^2$. Indeed consider any $U \in \mathcal{U}$ such that $U(x)$ is order convex whenever $x \in X$. Then for $x \in X$, $(U_g \cap U_g)(x) = i(U(x)) \cap d(U(x)) = U(x)$, since $U(x)$ is order convex, and thus $U_g \cap U_g = U$; therefore $[U_g \cap U_g]^{-1} \cap [(U_g)^{-1} \cap U_g] = (U_g \cap U_g) \cap (U_g \cap U_g)^{-1} = U \cup U^{-1}$. The statement follows, since $\mathcal{U}$ has a base consisting of such entours $U$.

We conclude this article by formulating a challenging generalization of Problem 1.

**Problem 3.** Characterize those (pairwise completely regular) bispaces $(X, \tau_1, \tau_2)$ for which there exists a metrizable topological ordered $C$- and $I$-space $(X, \tau, \leq)$ such that $\tau_1 = \tau^3$ and $\tau_2 = \tau^2$.

**References**


