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COHERENCE IN CLOSED CATEGORIES

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§ 1. Introduction

For the purposes of this paper we understand by a *closed category* the following collection of data:

(i) a category $\underline{V}$;

(ii) functors $\otimes$: $\underline{V} \times \underline{V} \to \underline{V}$ and $[\ ,\ ] : \underline{V}^{\text{op}} \times \underline{V} \to \underline{V}$;

(iii) an object I of $\underline{V}$;

(iv) natural isomorphisms

$$a = a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)\ ,$$

$$b = b_A : A \otimes I \to A\ ,$$

$$c = c_{AB} : A \otimes B \to B \otimes A\ ;$$

(v) natural transformations (in the generalized sense of [1])

$$d = d_{AB} : A \to [B, A \otimes B]\ ,$$

$$e = e_{AB} : [A, B] \otimes A \to B\ .$$

The axioms to be satisfied by these data are that, for all $A, B, C, D \in \underline{V}$, the following diagrams should commute:

C1

$$\begin{array}{ccccc}
((A\otimes B)\otimes C)\otimes D & \xrightarrow{a} & (A\otimes B)\otimes(C\otimes D) & \xrightarrow{a} & A\otimes(B\otimes(C\otimes D)) \\
{\scriptstyle a\otimes 1}\downarrow & & & & \uparrow{\scriptstyle 1\otimes a} \\
(A\otimes(B\otimes C))\otimes D & & \xrightarrow{\qquad\quad a \quad\qquad} & & A\otimes((B\otimes C)\otimes D)
\end{array}$$

C2

$$\begin{array}{ccc}
(A\otimes B)\otimes I & \xrightarrow{a} & A\otimes(B\otimes I) \\
{\scriptstyle b}\searrow & & \swarrow{\scriptstyle 1\otimes b} \\
& A\otimes B &
\end{array}$$

C3

$$\begin{array}{ccc}
A\otimes B & \xrightarrow{c} & B\otimes A \\
& {\scriptstyle 1}\searrow & \downarrow{\scriptstyle c} \\
& & A\otimes B
\end{array}$$

C4

$$\begin{array}{ccccc}
(A\otimes B)\otimes C & \xrightarrow{a} & A\otimes(B\otimes C) & \xrightarrow{c} & (B\otimes C)\otimes A \\
{\scriptstyle c\otimes 1}\downarrow & & & & \downarrow{\scriptstyle a} \\
(B\otimes A)\otimes C & \xrightarrow[a]{} & B\otimes(A\otimes C) & \xrightarrow[1\otimes c]{} & B\otimes(C\otimes A)
\end{array}$$

C5

$$\begin{array}{ccc}
[B, A] & \xrightarrow{d} & [B, [B, A]\otimes B] \\
& {\scriptstyle 1}\searrow & \downarrow{\scriptstyle [1, e]} \\
& & [B, A]
\end{array}$$

C6

$$\begin{array}{ccc}
A\otimes B & \xrightarrow{d\otimes 1} & [B, A\otimes B]\otimes B \\
& {\scriptstyle 1}\searrow & \downarrow{\scriptstyle e} \\
& & A\otimes B
\end{array}$$

Such a closed category, which we denote by the single letter $\underline{V}$, is not essentially different from what was called in [2] a "symmetric monoidal closed category". In particular we have a natural isomorphism

$$\pi : \underline{V}(A \otimes B, C) \to \underline{V}(A, [B, C])$$

where $\pi(f)$ is the composite

$$(1.1) \qquad A \xrightarrow[d]{} [B, A \otimes B] \xrightarrow[[1,f]]{} [B, C]$$

and $\pi^{-1}(g)$ is the composite

$$(1.2) \qquad A \otimes B \xrightarrow[g \otimes 1]{} [B, C] \otimes B \xrightarrow[e]{} C;$$

indeed the commutativity of C5 and C6 is exactly the condition that the natural transformations π and π^{-1} defined by (1.1) and (1.2) should be mutually inverse.

If we omit [,], d, and e from the data and C5 and C6 from the axioms, we obtain the description of what we shall call a *monoidal category*. (This was called a "symmetric monoidal category" in [2], but we shall consider no other kind.) The axioms C1–C4 are exactly (see [9] and [5]) what is needed to ensure that the natural isomorphisms a, b, c are *coherent* in the sense of [9]. Roughly speaking, this means that any diagram will commute if (as in the diagrams C1–C4) each arrow is a natural isomorphism manufactured from $1, a, b, c, a^{-1}, b^{-1}, c^{-1}$ by taking repeated $\otimes$-products. Another example of such a diagram would be

$$\begin{array}{ccccc}
(I \otimes A) \otimes B & & \xrightarrow{\quad a \quad} & & I \otimes (A \otimes B) \\
{\scriptstyle c \otimes 1}\downarrow & & & & \downarrow{\scriptstyle c} \\
(A \otimes I) \otimes B & \xrightarrow[b \otimes 1]{} & A \otimes B & \xrightarrow[b^{-1}]{} & (A \otimes B) \otimes I
\end{array}$$

Note that coherence asserts equality of *natural transformations*, and not of morphisms in $\underline{V}$ except insofar as these are components of natural transformations; thus it does not assert that $c: A \otimes A \to A \otimes A$ and $1: A \otimes A \to A \otimes A$ coincide, these being components of quite different natural transformations $c: A \otimes B \to B \otimes A$ and $1: A \otimes B \to A \otimes B$.

The question naturally arises whether the analogous coherence result holds for a *closed* category: does a diagram commute if each arrow is a natural transformation manufactured from $1, a, b, c, a^{-1}, b^{-1}, c^{-1}, d, e$ by the use of $\otimes$ and [,]? Evidence that *something* of this kind is true was provided by the partial results in this direction due to Epstein [3] (cf. also MacDonald [8]), and by the mass of diagrams proved to be commutative in [2]. Nevertheless the answer to the question as asked is negative. Write $k_A : A \to [[A, I], I]$ for the natural transformation given by the composite

$$A \xrightarrow[d]{} [[A, I], A \otimes [A, I]] \xrightarrow[[1,c]]{} [[A, I], [A, I] \otimes A] \xrightarrow[[1,e]]{} [[A, I], I] \; ;$$

then it is easy to see that the diagram

$$\begin{array}{ccc} [A, I] & \xrightarrow{k_{[A,I]}} & [[[A, I], I], I] \\ & \overset{1}{\searrow} & \Big\downarrow {\scriptstyle [k_A, 1]} \\ & & [A, I] \end{array} \tag{1.3}$$

commutes; however the diagram

$$\begin{array}{ccc} [[[A, I], I], I] & \xrightarrow{[k_A, 1]} & [A, I] \\ & \overset{1}{\searrow} & \Big\downarrow {\scriptstyle k_{[A,I]}} \\ & & [[[A, I], I], I] \end{array} \tag{1.4}$$

does not commute in general. For if (1.4) commuted as well as (1.3), $k_{[A, I]}$ would be an isomorphism; but this is not so when $\underline{V}$ is the category of real vector spaces with the usual $\otimes$ and $[\ ,\ ]$, in which case k_A is the usual embedding of a vector space into its double dual.

It is the chief purpose of the present paper to show that we do get a coherence result of the desired kind provided that we impose a restriction on the functors which form the "vertices" of the diagram: in the formation of these functors we must never write $[T, S]$ where S (like I in the example above) is a *constant* functor, unless T too is a constant functor. Both of the diagrams (1.3) and (1.4), then, escape this modified coherence result, as they stand; but (1.3) can equally well be written with a variable B in place of I, and then the result applies. Diagram (1.4) ceases to make sense, *as a diagram of natural transformations*, if we replace I by a variable B, and in fact as we have seen does not commute in general; but we could replace A in (1.4) by the constant I, and then our result applies and in this special case (1.4) commutes. The full statement of our results is given in §2 below.

The method we have used is inspired by the work of Lambek [6, 7], who deals with a similar problem for a structure closely related to a closed category but differing from it in certain essential ways. (In particular, Lambek's structures lack the "symmetry isomorphism" c, and this does seem to make an essential difference.) From his work we have learnt the possibility of replacing *composition* of morphisms in a closed category by other processes of combination more adapted to proofs by induction. By his own account, Lambek himself came to recognize this possibility by generalizing the work of Gentzen, whose scheme for eliminating the "cut" in certain logical systems (see [4]) is essentially a special case of the above elimination of composition. An essential step in our §6 below, the proof that what we there call "constructible morphisms" are closed under composition, does not yield to a direct inductive argument – one must go round about and prove instead our Proposition 6.4; and this trick too we learnt from Lambek's work. These essential insights leave us heavily in Lambek's debt. For the rest, however, our results differ considerably from those of Lambek, being expressed in the context of the generalized natural transformations introduced in [1], with which the reader is supposed to be familiar.

§ 2. Statement of results

For a particular closed category $\underline{V}$, the functors $T, S: \underline{V} \times \underline{V}^{op} \times \underline{V} \to \underline{V}$, given by $T(A, B, C) = A \otimes [B, C]$ and $S(A, B, C) = [[A, B], C]$, might fortuitously coincide; they do so, in fact, if $\underline{V}$ is the category with one object and one morphism (and also in less trival cases). Since for our inductive arguments it is essential that each such functor be assigned a *rank*, and since the above two functors are to have different ranks, it is clear that rank should be an attribute not of the functor as such but of its formal expression. We proceed to introduce these formal expressions under the name of *shapes*.

We define shapes, without reference to any particular closed category, by the following inductive rules:

S1 I is a shape.

S2 1 is a shape.

S3 If T and S are shapes there is a shape $T \otimes S$.

S4 If T and S are shapes there is a shape $[T, S]$.

Shapes, therefore, are formal expressions involving I, 1, $\otimes$, and $[\ ,\ ]$, with parentheses where necessary; for instance $[1, I] \otimes [1, (1 \otimes I) \otimes 1]$ is a shape.

We define a *variable-set* to be a totally-ordered finite set X, provided with a function called *variance* from X to the two-element set {covariant, contravariant}. Define the *ordinal sum* $X \dotplus Y$ of two variable-sets X and Y to be the disjoint union $X + Y$ of X and Y, so ordered that X and Y retain their orders and that every $x \in X$ precedes every $y \in Y$, and with the variance of $t \in X \dotplus Y$ being its variance in X or in Y as the case may be. Define the *twisted sum* $X \tilde{\dotplus} Y$ to be the same totally-ordered set as $X \dotplus Y$, but with the variance of $t \in X \tilde{\dotplus} Y$ being its variance in Y when $t \in Y$ and the opposite of its variance in X when $t \in X$.

With each shape T is associated a variable-set $v(T)$ called the *set of variables* of T. This is defined inductively by the rules:

V1 $v(I)$ is the empty set.

V2 $v(1)$ is a chosen one-element set $\{*\}$, with $*$ covariant.

V3 $v(T \otimes S) = v(T) \dotplus v(S)$.

V4 $v([T, S]) = v(T) \tilde{\dotplus} v(S)$.

In many contexts it is convenient to suppose that $v(T)$, if it has n elements, is actually the set $\{1, 2, ..., n\}$. We can accomplish this under the above conventions if we take $\{*\}$ to be $\{1\}$, and if we agree that the disjoint union of $\{1, ..., n\}$ and $\{1, ..., m\}$ is $\{1, ..., n+m\}$ with the given sets embedded as the complementary sets $\{1, ..., n\}$ and $\{n+1, ..., n+m\}$. We also, however, want to speak of $v(T)$ and $v(S)$ as being disjoint complementary subsets of $v(T) + v(S)$; the reader will recognize that we are then speaking of the *images* of $v(T)$ and $v(S)$ in $v(T) + v(S)$.

If T and S are shapes we define a *graph* $\xi: T \to S$ to be a fixed-point-free involution on the disjoint union $v(T) + v(S)$, with the property that mates under ξ have opposite variances in the *twisted* sum $v(T) \tilde{\dotplus} v(S)$. Given graphs $\xi: T \to S$ and $\eta: S \to R$, we define a composite graph $\eta\xi: T \to R$ as follows: different elements

$x, y \in v(T) + v(R)$ are mates under $\eta\xi$ if and only if there is a sequence $x = t_0, t_1, \ldots, t_r = y$, with each $t_i \in v(T) + v(S) + v(R)$, such that, for each i, t_{i-1} and t_i are mates either under ξ or under η. Then $\eta\xi$ is indeed a graph, and this law of composition is associative. Moreover for each shape T there is an evident identity graph $1: T \to T$, so that shapes and graphs form a category $\underline{G}$.

Two graphs $\xi: T \to S$, $\eta: S \to R$ are said to be *compatible* if there is no sequence $t_1, t_2, \ldots, t_{2r}$ $(r \geq 1)$ of elements of $v(S)$ such that t_{2i-1} and t_{2i} are mates under ξ for $1 \leq i \leq r$, t_{2i} and t_{2i+1} are mates under η for $1 \leq i \leq r-1$, and t_{2r} and t_1 are mates under η.

The definitions of composition of graphs and of compatibility of graphs become more perspicuous if we consider a graph $\xi: T \to S$ to be a graph in the literal sense, with the disjoint union $v(T) + v(S)$ as its vertex-set, and with one edge (or *linkage*) joining each pair of mates under ξ. The linkages in the graph $\eta\xi$ are then what we get by following alternately the linkages of ξ and of η, ignoring any closed loops that may arise; and ξ and η are compatible when in fact no closed loops do arise. All this is treated in detail in [1].

If $\xi: T \to T'$ and $\eta: S \to S'$ are graphs, we can define a graph $\xi \otimes \eta: T \otimes S \to T' \otimes S'$ by taking the linkages in $v(T) + v(S) + v(T') + v(S')$ to be those of ξ together with those of η. Similarly we can define a graph $[\xi, \eta]: [T', S] \to [T, S']$. It is easy to verify that $\otimes$ and $[\,,\,]$ are thereby made into functors $\underline{G} \times \underline{G} \to \underline{G}$ and $\underline{G}^{\mathrm{op}} \times \underline{G} \to \underline{G}$ respectively.

For any shapes T, S, R there are evident graphs

$$\alpha: (T \otimes S) \otimes R \to T \otimes (S \otimes R),$$

$$\beta: T \otimes I \to T,$$

$$\gamma: T \otimes S \to S \otimes T,$$

$$\delta: T \to [S, T \otimes S],$$

$$\epsilon: [T, S] \otimes T \to S;$$

it is easy to verify that these are natural transformations (natural isomorphisms in the case of α, β, γ), and that $\underline{G}$ becomes a closed category if we take these as its a, b, c, d, e.

Now let $\underline{V}$ be any closed category. For each shape T, with $v(T) = \{i_1, \ldots, i_n\}$ say (as an *ordered* set), we define a functor $|T|: \underline{V}_{i_1} \times \underline{V}_{i_2} \times \ldots \times \underline{V}_{i_n} \to \underline{V}$, where $\underline{V}_{i_r}$ is $\underline{V}$ or $\underline{V}^{\mathrm{op}}$ according as i_r is covariant or contravariant in $v(T)$. (If $v(T)$ is empty then $n = 0$ and we understand $\underline{V}_{i_1} \times \ldots \times \underline{V}_{i_n}$ to mean the unit category $\underline{I}$ with one object and one morphism.) The inductive definition of $|T|$ is the following:

F1 $|I|$ is the constant functor $I: \underline{I} \to \underline{V}$.

F2 $|1|$ is the identity functor $1: \underline{V} \to \underline{V}$.

F3 $|T \otimes S|$ is the composite functor

$$\underline{V}_{i_1} \times \ldots \times \underline{V}_{i_n} \times \underline{V}_{j_1} \times \ldots \times \underline{V}_{j_m} \xrightarrow[|T| \times |S|]{} \underline{V} \times \underline{V} \xrightarrow[\otimes]{} \underline{V},$$

where $v(T) = \{i_1, ..., i_n\}$ and $v(S) = \{j_1, ..., j_m\}$.

F4 $|[T, S]|$ is the composite functor

$$\underline{V}^{\mathrm{op}}_{i_1} \times ... \times \underline{V}^{\mathrm{op}}_{i_n} \times \underline{V}_{j_1} \times ... \times \underline{V}_{j_m} \xrightarrow[|T|^{\mathrm{op}} \times |S|]{} \underline{V}^{\mathrm{op}} \times \underline{V} \xrightarrow[[\,,\,]]{} \underline{V},$$

where $v(T)$ and $v(S)$ are as in F3.

Let T and S be shapes, with $v(T) = \{i_1, ..., i_n\}$ and $v(S) = \{j_1, ..., j_m\}$. Then, as in [1], a *natural transformation* $f: |T| \to |S|$ consists of a graph $\xi: T \to S$, called the *graph* Γf of f, and morphisms

$$f(A_{i_1}, ..., A_{i_n}, A_{j_1}, ..., A_{j_m}): |T|(A_{i_1}, ..., A_{i_n}) \to |S|(A_{j_1}, ..., A_{j_m})$$

of $\underline{V}$, called the *components* of f; here $A_x = A_y$ whenever x and y are mates under ξ, and for each such pair of mates there is a *naturality condition* to be satisfied by these components. (In practice one suppresses, in writing the components of f, one of each pair of equal arguments in $f(A_{i_1}, ..., A_{i_n}, A_{j_1}, ..., A_{j_m})$, and one often writes the remaining arguments as subscripts. Thus one writes $e_{AB}: [A, B] \otimes A \to B$ or $e(A, B)$, and not $e(A, B, A, B)$.) If $g: |S| \to |R|$ is another natural transformation, of graph $\eta: S \to R$, and if η and ξ are *compatible*, we can define as in [1] a composite natural transformation $gf: |T| \to |R|$ of graph $\eta\xi: T \to R$; the component

$$(gf)(A_{i_1}, ..., A_{i_n}, A_{k_1}, ..., A_{k_l}): |T|(A_{i_1}, ..., A_{i_n}) \to |R|(A_{k_1}, ..., A_{k_l})$$

of gf is the composite of the components

$$f(A_{i_1}, ..., A_{i_n}, A_{j_1}, ..., A_{j_m})$$

and

$$g(A_{j_1}, ..., A_{j_m}, A_{k_1}, ..., A_{k_l}),$$

where $A_x = A_y$ if x and y are either mates under ξ or mates under η. In fact we can in the present circumstances define the composite gf, of graph $\eta\xi$, even when η and ξ are not compatible, for we have here a recourse not available in the more general situation of [1]: we define the components of gf just as above, setting $A_{j_r} = I$ for any $j_r \in v(S)$ which occurs in one of the closed loops. That the composite so formed is still natural is clear, as we have merely modified f and g by specializing some of the arguments before composing them as in [1]. This law of composition is associative, and there is an evident identity natural transformation $1: |T| \to |T|$ of graph $1: T \to T$.

We can define, therefore, a new category $\underline{N}(\underline{V})$ depending upon $\underline{V}$. The objects of $\underline{N}(\underline{V})$, like those of $\underline{G}$, are to be all the shapes; a morphism $f: T \to S$ in $\underline{N}(\underline{V})$ is to be a natural transformation $f: |T| \to |S|$, which we shall often call "a natural transformation $f: T \to S$"; and composition in $\underline{N}(\underline{V})$ is to be the above composition of natural transformations. We can call $\underline{N}(\underline{V})$ "the category of shapes and natural transformations for $\underline{V}$"; and we shall often abbreviate $\underline{N}(\underline{V})$ to $\underline{N}$ when $\underline{V}$ is clear from the context. There is an evident functor $\Gamma: \underline{N} \to \underline{G}$ which is the identity on objects and which takes each natural transformation f to its graph Γf.

From natural transformations $f: T \to T'$ and $g: S \to S'$ of graphs ξ and η we get a natural transformation $f \otimes g: T \otimes S \to T' \otimes S'$ of graph $\xi \otimes \eta$ by taking the components of $f \otimes g$ to be the $\otimes$-products of the components of f and those of g. Similarly we get a natural transformation $[f, g]: [T', S] \to [T, S']$ of graph $[\xi, \eta]$. It is easy to verify that $\otimes$ and $[\ ,\ ]$ are thereby made into functors $\underline{N} \times \underline{N} \to \underline{N}$ and $\underline{N}^{\mathrm{op}} \times \underline{N} \to \underline{N}$; clearly $\Gamma: \underline{N} \to \underline{G}$ commutes with $\otimes$ and $[\ ,\ ]$.

For any shapes T, S, R we get a natural transformation $a_{TSR}: (T \otimes S) \otimes R \to T \otimes (S \otimes R)$ of graph $\alpha_{TSR}: (T \otimes S) \otimes R \to T \otimes (S \otimes R)$ by taking the component

$$a_{TSR}(A_{i_1}, \ldots, A_{i_n}, A_{j_1}, \ldots, A_{j_m}, A_{k_1}, \ldots, A_{k_l})$$

of a_{TSR} to be the component

$$a(|T|(A_{i_1}, \ldots, A_{i_n}), |S|(A_{j_1}, \ldots, A_{j_m}), |R|(A_{k_1}, \ldots, A_{k_l}))$$

of a. Then it follows easily that the morphism a_{TSR} of $\underline{N}$ is the (T, S, R)-component of a natural isomorphism between the functors $(-\otimes-)\otimes-$ and $-\otimes(-\otimes-)$ of $\underline{N} \times \underline{N} \times \underline{N}$ into $\underline{N}$. This natural isomorphism we again call a, and we often write a. $(T \otimes S) \otimes R \to T \otimes (S \otimes R)$, abbreviating as usual a_{TSR} to a. In the same way we define natural isomorphisms $b: T \otimes I \to T$, $c: T \otimes S \to S \otimes T$ and natural transformations $d: T \to [S, T \otimes S]$, $e: [T, S] \otimes T \to S$, of respective graphs $\beta, \gamma, \delta, \epsilon$; and we verify that a, b, c, d, e give to $\underline{N}$ the structure of a closed category.

We now have closed categories $\underline{N} = \underline{N}(\underline{V})$ and $\underline{G}$, and a functor $\Gamma: \underline{N} \to \underline{G}$ which is the identity on objects, which commutes with $\otimes$ and $[\ ,\ ]$, and which sends a, b, c, d, e to $\alpha, \beta, \gamma, \delta, \epsilon$. In order to make statements that will embrace at once the closed categories $\underline{N}$ and $\underline{G}$, we shall suppose, throughout this paper, that $\underline{H}$ is some closed category with the same objects as $\underline{G}$, and that $\Gamma: \underline{H} \to \underline{G}$ is a functor which is the identity on objects, which commutes with $\otimes$ and $[\ ,\ ]$, and which sends a, b, c, d, e to $\alpha, \beta, \gamma, \delta, \epsilon$. The cases of interest are that where $\underline{H} = \underline{N}(\underline{V})$ and Γ is as above, and that where $\underline{H} = \underline{G}$ and $\Gamma = 1$.

Given any such $\underline{H}$, we define a subcategory of $\underline{H}$, whose objects are all shapes, and whose morphisms shall be called the *allowable* morphisms of $\underline{H}$. These are to be the smallest class of morphisms of $\underline{H}$ satisfying the following five conditions (in which $T, S, R, \ldots$ denote arbitrary shapes):

AM1. For any T, S, R each of the following morphisms is in the class:

$$1: T \to T$$
$$a: (T \otimes S) \otimes R \to T \otimes (S \otimes R),$$
$$a^{-1}: T \otimes (S \otimes R) \to (T \otimes S) \otimes R,$$
$$b: T \otimes I \to T,$$
$$b^{-1}: T \to T \otimes I,$$
$$c: T \otimes S \to S \otimes T.$$

AM2. For any T, S each of the following morphisms is in the class:

d: $T \to [S, T \otimes S]$,

e: $[T, S] \otimes T \to S$.

AM3. If f: $T \to T'$ and g: $S \to S'$ are in the class so is $f \otimes g$: $T \otimes S \to T' \otimes S'$.

AM4. If f: $T \to T'$ and g: $S \to S'$ are in the class so is $[f, g]$: $[T', S] \to [T, S']$.

AM5. If f: $T \to S$ and g: $S \to R$ are in the class so is gf: $T \to R$.

The allowable morphisms of $\underline{G}$ are called the *allowable graphs*, and those of $\underline{N}(\underline{V})$ are called the *allowable natural transformations* (for $\underline{V}$). It is evident that the functor Γ: $\underline{N} \to \underline{G}$ takes allowable natural transformations to allowable graphs, since those natural transformations $f \in \underline{N}$ for which Γf is allowable clearly satisfy AM1–AM5.

The first two of our principal results deal with the case $\underline{H} = \underline{G}$, and are:

Theorem 2.1. *There is an algorithm for deciding whether a graph ξ: $T \to S$ is allowable.*

Theorem 2.2. *If the graphs ξ: $T \to S$, η: $S \to R$ are allowable, they are compatible.*

The proofs will be given in §7 and in §6 respectively.

Since we shall be interested only in *allowable* natural transformations, we see from Theorem 2.2 that there was no real need to introduce the composition of incompatible ones; it was merely a convenience so that $\underline{N}$ could be described as a category. Our third principal result is:

Theorem 2.3. *Let $\underline{V}$ be any closed category. If ξ: $T \to S$ is an allowable graph, there is in $\underline{N}(\underline{V})$ at least one allowable natural transformation f: $T \to S$ of graph ξ.*

Proof. Those allowable graphs ξ which are images under Γ of allowable natural transformations satisfy AM1–AM5, and therefore constitute the totality of allowable graphs.

For our final main result we pick out a subset of the shapes called the *proper* shapes. Call a shape T *constant* if its set of variables $v(T)$ is empty. Then the proper shapes are defined inductively by:

PS1 $\quad I$ is a proper shape.

PS2 $\quad 1$ is a proper shape.

PS3 $\quad$ If T and S are proper shapes so is $T \otimes S$.

PS4 $\quad$ If T and S are proper shapes so is $[T, S]$, *unless* S is constant and T is not constant.

Our final principal result then is:

Theorem 2.4. *Let $\underline{V}$ be any closed category, and let $f, f' : T \to S$ be two allowable natural transformations in $\underline{N}(\underline{V})$ with the same graph $\xi = \Gamma f = \Gamma f'$. Then, provided that the shapes T and S are proper, we have $f = f'$.*

The proof will be given in § 7.

§3. The monoidal case

We are going to build on the known coherence theorem for the monoidal case, proved in [9] and simplified a little in [5]. The purpose of this section is to re-state this result in terms entirely analogous to those used in §2 above, so that it is easily available for our use.

We have seen that we get the description of a monoidal category from that of a closed category by omitting the data [,], d, e and the axioms C5 and C6. All the concepts introduced in §2 have analogues in the monoidal case, as follows.

The shapes we need here are those defined by the inductive rules S1, S2, S3 of §2, omitting S4; we call these the *integral shapes* (for it is reasonable to think of $\otimes$ as a kind of multiplication, and of [,] as a kind of division). For integral T the rules V1, V2, V3 suffice to describe the set of variables $v(T)$; clearly each element of $v(T)$ is covariant. Because of this, a pair of mates under a graph ξ: $T \to S$, where T and S are integral, consists of an element of $v(T)$ and an element of $v(S)$; thus we may identify the graph ξ with the corresponding bijection of $v(T)$ onto $v(S)$. It is especially for integral T (where there are no complications of variance) that it is convenient to identify $v(T)$, when it has n elements, with the ordered set $\{1, 2, ..., n\}$; and we shall do so freely. The integral shapes and the graphs connecting them form a full subcategory $\underline{G}_0$ of $\underline{G}$; we can look upon $\otimes$ as a functor $\underline{G}_0 \otimes \underline{G}_0 \to \underline{G}_0$, and the graphs $\alpha: (T \otimes S) \otimes R \to T \otimes (S \otimes R)$, $\beta: T \otimes I \to T$, $\gamma: T \otimes S \to S \otimes T$ turn $\underline{G}_0$ into a monoidal category.

If $\underline{V}$ is any monoidal category, each integral shape T determines a functor $|T|: \underline{V} \times ... \times \underline{V} \to \underline{V}$ by the rules F1, F2, F3 of §2. Since we can again speak of a natural transformation $f: |T| \to |S|$ of graph $\xi: T \to S$, we have a category $\underline{N}_0(\underline{V})$, whose objects are the integral shapes and whose morphisms $f: T \to S$ are the natural transformations $f: |T| \to |S|$ of arbitrary graph. Like the category $\underline{N}(\underline{V})$ of §2, $\underline{N}_0(\underline{V})$ becomes a monoidal category with the obvious definitions of $f \otimes g$ and of a, b, c; and there is a functor $\Gamma: \underline{N}_0(\underline{V}) \to \underline{G}_0$ which is the identity on objects and which sends each natural transformation to its graph. The functor Γ commutes with $\otimes$ and sends a, b, c, to α, β, γ.

In this monoidal case we shall need to compare the $\underline{N}_0(\underline{V})$'s for different monoidal categories $\underline{V}$. If $\underline{V}$ and $\underline{V}'$ are monoidal categories, a *strict monoidal functor* $\Delta: \underline{V} \to \underline{V}'$ shall mean a functor that commutes with $\otimes$ and for which $\Delta a = a'$, $\Delta b = b'$, and $\Delta c = c'$ (where, for example, this last assertion means that $\Delta c_{A,B} = c'_{\Delta A, \Delta B}$). In particular, $\Gamma: \underline{N}_0(\underline{V}) \to \underline{G}_0$ is a strict monoidal functor. It is easily seen that a strict monoidal functor Δ: $\underline{V} \to \underline{V}'$ induces a strict monoidal functor $\underline{N}_0(\Delta): \underline{N}_0(\underline{V}) \to \underline{N}_0(\underline{V}')$, which is the identity on objects and which sends the natural transformation $f: T \to S$ to the natural transformation whose components are the images under Δ of those of f. It is further clear that the composite of $\Gamma': \underline{N}_0(\underline{V}') \to \underline{G}_0$ with $\underline{N}_0(\Delta)$ is $\Gamma: \underline{N}_0(\underline{V}) \to \underline{G}_0$.

For any monoidal category $\underline{V}$ we define the *central* morphisms of $\underline{V}$ to be the smallest class of morphisms of $\underline{V}$ satisfying the conditions AM1, AM3, and AM5 of

§2, where $T, S, R, \ldots$ now denote arbitrary objects of $\underline{V}$; since the isomorphisms of $\underline{V}$ satisfy AM1, AM3 and AM5, every central morphism is an isomorphism. These central morphisms constitute a subcategory Cent $\underline{V}$ of $\underline{V}$ with the same objects as $\underline{V}$; clearly Cent $\underline{V}$ is itself a monoidal category, and the inclusion Cent $\underline{V} \to \underline{V}$ is a strict monoidal functor. It is clear that any strict monoidal functor $\Delta : \underline{V} \to \underline{V}'$ carries central morphisms of $\underline{V}$ into central morphisms of $\underline{V}'$.

The analogue of Theorem 2.2 for the monoidal case is trivially true, for any graphs $\xi : T \to S$ and $\eta : S \to R$ are clearly compatible when T, S and R are integral. The analogues of Theorems 2.1, 2.3 and 2.4 are contained in the following result, which expresses essentially what was proved in [9]:

Theorem 3.1. *Let $\underline{V}$ be any monoidal category. If T and S are integral shapes, then any graph $\xi : T \to S$ is central in $\underline{G}_0$, and there is in* Cent $\underline{N}_0(\underline{V})$ *one and only one natural transformation $f : T \to S$ of graph ξ.*

In other words we have Cent $\underline{N}_0(\underline{V}) \cong$ Cent $\underline{G}_0 = \underline{G}_0$. We shall write $|\xi|_{\underline{V}} : T \to S$ for the unique morphism of Cent $\underline{N}_0(\underline{V})$ with $\Gamma|\xi|_{\underline{V}} = \xi$. It is immediate that $|\eta\xi|_{\underline{V}} = |\eta|_{\underline{V}}|\xi|_{\underline{V}}$, and that $|\xi \otimes \eta|_{\underline{V}} = |\xi|_{\underline{V}} \otimes |\eta|_{\underline{V}}$; and further that $|\alpha|_{\underline{V}} = a$, $|\beta|_{\underline{V}} = b$, $|\gamma|_{\underline{V}} = c$. Moreover if $\Delta : \underline{V} \to \underline{V}'$ is a strict monoidal functor, it is clear that $\underline{N}_0(\Delta)$ carries $|\xi|_{\underline{V}}$ to $|\xi|_{\underline{V}'}$.

§4. Central morphisms in $\underline{N}(\underline{V})$ and in $\underline{G}$

This section will use Theorem 3.1 to handle, for a closed category $\underline{V}$, that part of the coherence problem involving only a, b and c. In other words, we shall deal here with the *central* morphisms of $\underline{N}(\underline{V})$. These, like the composite

$$(T\otimes [S,R])\otimes I \xrightarrow[a]{} T\otimes([S,R]\otimes I) \xrightarrow[1\otimes b]{} T\otimes [S,R] \ ,$$

involve in general non-integral shapes. We bring them within the ambit of Theorem 3.1 by showing that the central morphisms of $\underline{N}(\underline{V})$ and of $\underline{G}$ admit an alternative description: they arise from the morphisms of $\underline{G}_0$ by the substitution of "$\otimes$-irreducible" or "prime" shapes for the variables.

We suppose then that $\underline{V}$ is a closed category, and as in §2 we use $\underline{H}$ to denote either $\underline{N}(\underline{V})$ or $\underline{G}$, with $\Gamma: \underline{H} \to \underline{G}$ sending f to its graph in the first case and being the identity in the second case. Since $\underline{H}$, being a closed category, is a monoidal category, we can speak as in §3 of the central morphisms of $\underline{H}$; it is immediate from the definition of these that they are a subset of the allowable morphisms of $\underline{H}$. Since $\Gamma: \underline{N}(\underline{V}) \to \underline{G}$ is a strict monoidal functor, it takes a central morphism of $\underline{N}(\underline{V})$ (which we shall call a *central natural transformation*) to a central morphism of $\underline{G}$ (which we shall call a *central graph*). As $\underline{V}$ will be fixed, we shall abbreviate $\underline{N}(\underline{V})$ to $\underline{N}$.

If P is any integral shape we have as in §3, since $\underline{H}$ is a monoidal category, a functor $|P|: \underline{H} \times \ldots \times \underline{H} \to \underline{H}$. Thus for arbitrary shapes $X_1, \ldots, X_n$ (where n is the number of elements of $v(P)$) we get a shape $|P|(X_1, \ldots, X_n)$, and for arbitrary morphisms $f_i: X_i \to X_i'$ in $\underline{H}$ we get a morphism $|P|(f_1, \ldots, f_n): |P|(X_1, \ldots, X_n) \to |P|(X_1', \ldots, X_n')$ in $\underline{H}$. It is evident that $|P|(X_1, \ldots, X_n)$ is the same shape whether we take $\underline{H}$ to be $\underline{N}$ or $\underline{G}$, and that $\Gamma(|P|(f_1, \ldots, f_n)) = |P|(\Gamma f_1, \ldots, \Gamma f_n)$.

Now let P, Q be integral shapes. A graph $\xi: P \to Q$ may be identified with a bijection of $v(P)$ onto $v(Q)$ and hence, if $v(P)$ and $v(Q)$ have n elements, with a permutation ξ of $\{1, \ldots, n\}$. As in §3 we have a unique $|\xi|_{\underline{H}}: P \to Q$ in Cent $\underline{N}_0(\underline{H})$ of graph ξ. We can write its typical component as

$$(4.1) \qquad |\xi|_{\underline{H}}(X_1, \ldots, X_n): |P|(X_{\xi 1}, \ldots, X_{\xi n}) \to |Q|(X_1, \ldots, X_n) \ ;$$

it is a morphism of $\underline{H}$.

Proposition 4.1. *For any graph $\xi: P \to Q$ between integral shapes P, Q and for any shapes $X_1, \ldots, X_n$ the morphism* (4.1) *of $\underline{H}$ is central.*

Proof. Consider the family of all those graphs ξ in $\underline{G}_0$ for which (4.1) is indeed central in $\underline{H}$ for all $X_1, \ldots, X_n$; it suffices to show that this family satisfies AM1, AM3 and AM5, for then it contains Cent $\underline{G}_0$ which, by Theorem 3.1, is all of $\underline{G}_0$. Now this family satisfies AM1 because $|\alpha|_{\underline{H}} = a$, etc.; it satisfies AM3 because the components of $|\xi \otimes \eta|_{\underline{H}} = |\xi|_{\underline{H}} \otimes |\eta|_{\underline{H}}$ are the tensor products of the components of $|\xi|_{\underline{H}}$ and those of $|\eta|_{\underline{H}}$; and it satisfies AM5 because the components of $|\eta\xi|_{\underline{H}} = |\eta|_{\underline{H}}|\xi|_{\underline{H}}$ are the composites of certain components of $|\eta|_{\underline{H}}$ and of $|\xi|_{\underline{H}}$.

Since, as we saw in §3, $\underline{N}_0(\Gamma)$ takes $|\xi|_{\underline{N}}$ to $|\xi|_{\underline{G}}$, it follows from the definition in §3 of $\underline{N}_0(\Gamma)$ that

$$\Gamma(|\xi|_{\underline{N}}(X_1, \ldots, X_n)) = |\xi|_{\underline{G}}(X_1, \ldots, X_n)\,. \tag{4.2}$$

It is easy to calculate $|\xi|_{\underline{G}}(X_1, \ldots, X_n)$. First, it is clear by induction that the variable-set $v(|P|(X_1, \ldots, X_n))$ is $v(X_1) \hat{+} \ldots \hat{+} v(X_n)$.

Proposition 4.2. *The graph*

$$|\xi|_{\underline{G}}(X_1, \ldots, X_n)\colon |P|(X_{\xi 1}, \ldots, X_{\xi n}) \to |Q|(X_1, \ldots, X_n)$$

is the involution on the set

$$v(X_{\xi 1}) + \ldots + v(X_{\xi n}) + v(X_1) + \ldots + v(X_n)$$

corresponding to the evident bijection induced by ξ *of*

$$v(X_{\xi 1}) + \ldots + v(X_{\xi n}) \quad \text{with} \quad v(X_1) + \ldots + v(X_n)\,.$$

Proof. Again it suffices to show that the family of those ξ in $\underline{G}_0$ for which this is true satisfies AM1, AM3, and AM5; the verifications are immediate.

We now proceed to show that all the central morphisms of $\underline{H}$ are obtainable in the form (4.1). Define the *prime* shapes to be the shape 1 and all shapes of the form $[T, S]$. It follows easily from the inductive definition of shapes that any shape T can be expressed *uniquely* in the form $T = |P|(X_1, \ldots, X_n)$ where P is an integral shape and $X_1, \ldots, X_n$ are prime shapes. We call this the *prime factorization* of T, and call $X_1, \ldots, X_n$ the list of *prime factors* of T. Note that n may be 0, so that this list may be empty; namely when T is a constant integral shape. In general, if T is an integral shape, its prime factorization is $T = |T|(1, 1, \ldots, 1)$. Observe that if the prime factorizations of T and S are $T = |P|(X_1, \ldots, X_n)$ and $S = |Q|(Y_1, \ldots, Y_m)$, then that of $T \otimes S$ is $|P \otimes Q|(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$.

Proposition 4.3. *Let* $f\colon T \to S$ *be a central morphism of* $\underline{H}$, *and let the prime factorizations of* T *and of* S *be* $T = |P|(X_1, \ldots, X_n)$ *and* $S = |Q|(Y_1, \ldots, Y_m)$. *Then* $m = n$, *and there is a permutation* ξ *of* $\{1, \ldots, n\}$ *such that* $X_i = Y_{\xi i}$ *for each* i *and such that* $f = |\xi|_{\underline{H}}(Y_1, \ldots, Y_n)\colon |P|(Y_{\xi 1}, \ldots, Y_{\xi n}) \to |Q|(Y_1, \ldots, Y_n)$.

Proof. Consider the family of all those morphisms of $\underline{H}$ that *are* of the form

$$|\eta|_{\underline{H}}(Z_1, \ldots, Z_r)\colon |J|(Z_{\eta 1}, \ldots, Z_{\eta r}) \to |K|(Z_1, \ldots, Z_r)$$

for integral shapes J, K and prime shapes Z_i. It suffices to show that this family satisfies AM1, AM3 and AM5, and therefore contains all central morphisms of $\underline{H}$. When we advert to the relation between the prime factorization of $T \otimes S$ and those of T and of S, the verifications are immediate from the facts that $|\alpha|_{\underline{H}} = a$, etc., $|\eta \otimes \zeta|_{\underline{H}} = |\eta|_{\underline{H}} \otimes |\zeta|_{\underline{H}}$, and $|\zeta\eta|_{\underline{H}} = |\zeta|_{\underline{H}} |\eta|_{\underline{H}}$.

Since a shape T is integral exactly when its prime factors are all 1, and is constant exactly when its prime factors are all constant, we have

Corollary 4.4. *If $f: T \to S$ is a central morphism in $\underline{H}$ and if either one of T, S is integral (resp. constant), so is the other.*

In view of Proposition 4.2 we also have

Corollary 4.5. *If $\phi: T \to S$ is a central graph, each pair of mates under ϕ consists of an element of $v(T)$ and an element of $v(S)$.*

Returning to Proposition 4.3, we may observe that the permutation ξ therein is not in general uniquely determined by f. For instance if T and S are both $[I, I] \otimes [I, I]$, so that $P = Q = 1 \otimes 1$ and $X_1 = Y_1 = X_2 = Y_2 = [I, I]$, then it will follow from Proposition 4.8 below that $|\xi|_{\underline{H}}(Y_1, Y_2)$ is $1: T \to S$ for both permutations ξ of $\{1, 2\}$. However:

Proposition 4.6. *If in Proposition 4.3 permutations ξ and ξ' both satisfy the stated conditions, we have $\xi' = \lambda\xi$ where λ is a permutation of $\{1, ..., n\}$ for which $\lambda i \neq i$ implies that Y_i and $Y_{\lambda i}$ are equal constant shapes.*

Proof. Since $X_{\xi^{-1} i} = Y_i$ and also $X_{\xi^{-1} i} = Y_{\xi'\xi^{-1} i} = Y_{\lambda i}$ we have $Y_i = Y_{\lambda i}$. Since $|\xi|_{\underline{H}}(Y_1, ..., Y_n) = |\xi'|_{\underline{H}}(Y_1, ..., Y_n)$ we have by (4.2) that $|\xi|_{\underline{G}}(Y_1, ..., Y_n) = |\xi'|_{\underline{G}}(Y_1, ..., Y_n)$, and we conclude from Proposition 4.2 that $\xi' j = \xi j$ unless $v(Y_{\xi j})$ is empty; that is, $\lambda i = i$ unless Y_i is constant.

For the desired main result of this section, we need to show that the permutations λ of the type described in Proposition 4.6 are *exactly* those for which $|\lambda|_{\underline{H}}(Y_1, ..., Y_n) = 1$. First we prove:

Lemma 4.7. *If T is a constant shape there is an isomorphism $k_T: T \to I$ in $\underline{H}$ which, together with its inverse, is allowable.*

Proof. From the natural isomorphism

$$\underline{H}(A, I) \xrightarrow[\underline{H}(b, 1)]{} \underline{H}(A \otimes I, I) \xrightarrow[\pi]{} \underline{H}(A, [I, I])$$

we deduce, by the Yoneda Lemma, the existence of an isomorphism $h: [I, I] \to I$.

Using (1.1) and (1.2) we find that h and h^{-1} are the respective composites

$$[I, I] \xrightarrow[b^{-1}]{} [I, I] \otimes I \xrightarrow[e]{} I, \qquad I \xrightarrow[d]{} [I, I \otimes I] \xrightarrow[[1, b]]{} [I, I],$$

so that both are allowable. We now define k_T inductively for constant shapes T by setting $k_I = 1$, by taking $k_{T \otimes S}$ to be the composite

$$T \otimes S \xrightarrow[k_T \otimes k_S]{} I \otimes I \xrightarrow[b]{} I,$$

and by taking $k_{[T, S]}$ to be the composite

$$[T, S] \xrightarrow[[k_T^{-1}, k_S]]{} [I, I] \xrightarrow[h]{} I.$$

Proposition 4.8. *Let Q be an integral shape, Let $Y_1, \ldots, Y_n$ be prime shapes, and let λ be a permutation of $\{1, \ldots, n\}$ for which $\lambda i \neq i$ implies that Y_i and $Y_{\lambda i}$ are equal constant shapes. Then $|\lambda|_{\underline{H}}(Y_1, \ldots, Y_n) = 1 \colon |Q|(Y_{\lambda 1}, \ldots, Y_{\lambda n}) \to |Q|(Y_1, \ldots, Y_n)$.*

Proof. We can express λ as a product of transpositions; since $|\mu\nu|_{\underline{H}} = |\mu|_{\underline{H}} |\nu|_{\underline{H}}$ we may suppose that λ is such a transposition. Replacing λ by a suitable conjugate $\mu\lambda\mu^{-1}$, we may suppose that λ is the transposition interchanging 1 and 2 and leaving fixed $3, \ldots, n$, while $Y_1 = Y_2$ are equal constant shapes. In $\underline{G}_0$, Q is isomorphic to $(1 \otimes 1) \otimes R$ for some integral R, and since $|\mu \otimes 1|_{\underline{H}} = |\mu|_{\underline{H}} \otimes 1$ we may suppose that Q is in fact the shape $1 \otimes 1$. But then $|\lambda|_{\underline{H}} = c$, and it remains to prove that $c_{YY} = 1$: $Y \otimes Y \to Y \otimes Y$ if Y is a constant shape. Using the isomorphism k_Y of Lemma 4.7 we have by the naturality of c a commutative diagram

$$\begin{array}{ccc} Y \otimes Y & \xrightarrow{c_{YY}} & Y \otimes Y \\ {\scriptstyle k_Y \otimes k_Y} \downarrow & & \downarrow {\scriptstyle k_Y \otimes k_Y} \\ I \otimes I & \xrightarrow[c_{II}]{} & I \otimes I \end{array} \quad ;$$

since $c_{II} = 1$ by Theorem 3.1, it follows that $c_{YY} = 1$.

Theorem 4.9. *Let $\underline{V}$ be a closed category. Then each central graph $\phi \colon T \to S$ in $\underline{G}$ is Γf for a unique central natural transformation $f \colon T \to S$ in $\underline{N}(\underline{V})$.*

Proof. Let $T = |P|(X_1, \ldots, X_n)$, $S = |Q|(Y_1, \ldots, Y_m)$ be the prime factorizations. Applying Proposition 4.3 with $\underline{H} = \underline{G}$, we conclude that $m = n$, and that for some permutation ξ we have $X_i = Y_{\xi i}$ and $\phi = |\xi|_{\underline{G}}(Y_1, \ldots, Y_n)$. Setting $f = |\xi|_{\underline{N}}(Y_1, \ldots, Y_n)$, which is central by Proposition 4.1, we see by (4.2) that $\Gamma f = \phi$, thus proving the existence of f.

To prove the uniqueness of f, let $f': T \to S$ be another central natural transformation with $\Gamma f' = \phi$. Applying Proposition 4.3 with $\underline{H} = \underline{N}$, we conclude that $f' = |\xi'|_{\underline{N}}(Y_1, ..., Y_n)$ for some permutation ξ' with $X_i = Y_{\xi' i}$. Now (4.2) gives $\phi = \Gamma f' = |\xi'|_{\underline{G}}(Y_1, ..., Y_n)$; and Proposition 4.6 with $\underline{H} = \underline{G}$ shows that $\lambda = \xi'\xi^{-1}$ has the properties described therein. We conclude from Proposition 4.8 with $\underline{H} = \underline{N}$ that $|\lambda|_{\underline{N}}(Y_1, ..., Y_n) = 1$. Thus, since $|\lambda\xi|_{\underline{N}} = |\lambda|_{\underline{N}}\,|\xi|_{\underline{N}}$, we have $|\xi'|_{\underline{N}}(Y_1, ..., Y_n) = |\xi|_{\underline{N}}(Y_1, ..., Y_n)$, or $f' = f$.

We conclude this section with two useful propositions that could in fact have been proved immediately after Proposition 4.3. In the situation of that proposition, we may call ξ the *association of the prime factors of T and of S*, and then call $Y_{\xi i}$ *the prime factor of S associated, via f, with the prime factor X_i of T*. This language is a little imprecise, because of the non-uniqueness of ξ; we sometimes have a *choice of association*. The statements of the results below allow for this choice.

Proposition 4.10. *Let $f: A \otimes B \to C \otimes D$ be a central morphism of $\underline{H}$. For some choice of association, let each prime factor of A, considered as a prime factor of $A \otimes B$, be associated via f with a prime factor of C. Then there are a shape E and central morphisms $g: A \otimes E \to C$ and $h: B \to E \otimes D$ such that f is the composite*

$$A \otimes B \xrightarrow[1 \otimes h]{} A \otimes (E \otimes D) \xrightarrow[a^{-1}]{} (A \otimes E) \xrightarrow[g \otimes 1]{} C \otimes D.$$

Proof. Let the prime factorizations be $A = |P|(X_1, ..., X_n)$, $B = |Q|(Y_1, ..., Y_m)$, $C = |R|(Z_1, ..., Z_l)$, $D = |S|(V_1, ..., V_k)$. We must have $n + m = l + k$, and the hypothesis of the proposition means that $f = |\xi|_{\underline{H}}(Z_1, ..., Z_l, V_1, ..., V_k)$ for some permutation ξ of $\{1, ..., n{+}m\}$ that maps the subset $\{1, ..., n\}$ into the subset $\{1, ..., l\}$. Let $j_1, ..., j_{l-n}$ be those elements of $\{1, ..., l\}$, in ascending order, that are not in the image under ξ of $\{1, ..., n\}$. Set $E = (Z_{j_1} \otimes Z_{j_2}) \otimes ... \otimes Z_{j_{l-n}}$; any way of inserting parentheses will do. Let ρ be the permutation of $\{1, ..., l\}$ given by $\rho i = \xi i$ for $i \leq n$, $\rho(n + i) = j_i$ for $i \leq l - n$. Let $\bar{\rho}$ be the permutation of $\{1, ..., n + m\}$ which is equal to ρ on $\{1, ..., l\}$ and which is the identity on $\{l + 1, ..., n + m\}$. Let $\bar{\sigma}$ be the permutation $\bar{\rho}^{-1}\xi$ of $\{1, ..., n + m\}$; clearly $\bar{\sigma}$ is the identity on $\{1, ..., n\}$. Let σ be the permutation of $\{1, ..., m\}$ given by $\sigma i = \bar{\sigma}(n + i) - n$. Define g as $|\rho|_{\underline{H}}(Z_1, ..., Z_l)$ and h as $|\sigma|_{\underline{H}}(Z_{j_1}, ..., Z_{j_{l-n}}, V_1, ..., V_k)$. Then $1 \otimes h = |\bar{\sigma}|_{\underline{H}}(X_1, ..., X_n, Z_{j_1}, ..., Z_{j_{l-n}}, V_1, ..., V_k)$; $a^{-1} = |\alpha^{-1}|_{\underline{H}}(X_1, ..., X_n, Z_{j_1}, ..., Z_{j_{l-n}}, V_1, ..., V_k) = |1|_{\underline{H}}(X_1, ..., V_k)$; and $g \otimes 1 = |\bar{\rho}|_{\underline{H}}(Z_1, ..., Z_l, V_1, ..., V_k)$. It follows that $(g \otimes 1)\, a^{-1}(1 \otimes h) = |\bar{\rho}\bar{\sigma}|_{\underline{H}}(Z_1, ..., Z_l, V_1, ..., V_k) = |\xi|_{\underline{H}}(Z_1, ..., V_k) = f$, as required.

Proposition 4.11. *Let $f: [P, Q] \otimes B \to [R, S] \otimes D$ be a central morphism of $\underline{H}$, and for some choice of association let the prime factor $[P, Q]$ of $[P, Q] \otimes B$ be asso-*

ciated via f with the prime factor $[R, S]$ of $[R, S] \otimes D$. Then $P = R$, $Q = S$, and there is a central morphism $k: B \to D$ such that $f = 1 \otimes k: [P, Q] \otimes B \to [P, Q] \otimes D$.

Proof. Since associated prime factors must be equal, we have $P = R$ and $Q = S$. We apply Proposition 4.10 with $A = [P, Q]$ and $C = [R, S]$; in this case $E = I$, and g is clearly $b: [P, Q] \otimes I \to [P, Q]$. Writing k for the composite

$$B \xrightarrow[h]{} I \otimes D \xrightarrow[c]{} D \otimes I \xrightarrow[b]{} D,$$

it follows from Theorem 3.1 that $f = 1 \otimes k$.

§5. Processes of construction

In the next section we shall show that the allowable natural transformations and the allowable graphs can be classified by a numerical *rank*, and that those of higher rank can be built up from those of lower rank, modulo central ones, by the use of three simple processes now to be described.

We consider a closed category $\underline{H}$, which will in our applications be either $\underline{G}$ or $\underline{N}(\underline{V})$. The first process of construction is the formation of the tensor product $f \otimes g : A \otimes B \to C \otimes D$ of two given morphisms $f: A \to C$ and $g: B \to D$. Observe that

$$(5.1) \qquad hf \otimes kg = (h \otimes k)(f \otimes g)$$

whenever hf and gk are defined. The second process of construction is the formation of the morphism $\pi(f): A \to [B, C]$ as in §1 from a given morphism $f: A \otimes B \to C$. Since π is natural we have commutativity in

$$(5.2) \qquad \begin{array}{ccc} A' & & \\ {\scriptstyle g}\downarrow & \searrow^{\pi(f(g \otimes 1))} & \\ A & \xrightarrow[\pi(f)]{} & [B, C] \, , \end{array}$$

where $f(g \otimes 1)$ is the obvious composite $A' \otimes B \to A \otimes B \to C$. The third process of construction begins with morphisms $f: A \to B$ and $g: C \otimes D \to E$ and produces the composite

$$(5.3) \qquad ([B, C] \otimes A) \otimes D \xrightarrow[(1 \otimes f) \otimes 1]{} ([B, C] \otimes B) \otimes D \xrightarrow[e \otimes 1]{} C \otimes D \xrightarrow[g]{} E \, .$$

Rather than introduce a special symbol for the composite (5.3), we find it convenient to denote the composite

$$(5.4) \qquad [B, C] \otimes A \xrightarrow[1 \otimes f]{} [B, C] \otimes B \xrightarrow[e]{} C$$

by $\langle f \rangle : [B, C] \otimes A \to C$, so that (5.3) may be written as $g(\langle f \rangle \otimes 1)$. The symbol $\langle f \rangle$ is of course ambiguous, inasmuch as the value of C must be understood from the context. It is clear that $\langle \; \rangle$ is natural, in the sense that, for

$$A' \xrightarrow[u]{} A \xrightarrow[f]{} B \xrightarrow[v]{} B' \, , \quad C' \xrightarrow[w]{} C$$

we have commutativity in

(5.5)
$$\begin{array}{ccc} [B,C]\otimes A & \xrightarrow{\langle f\rangle} & C \\ {\scriptstyle [v,w]\otimes u}\uparrow & & \uparrow{\scriptstyle w} \\ [B',C']\otimes A' & \xrightarrow[\langle vfu\rangle]{} & C' \end{array}$$

We need the following two lemmas giving connections between these processes.

Lemma 5.1. *Let $f: A\otimes B \to C$ and $g: D \to I$. Then the image under π of the composite*

$$(A\otimes D)\otimes B \xrightarrow[u]{} (A\otimes B)\otimes D \xrightarrow[f\otimes g]{} C\otimes I \xrightarrow[b]{} C\ ,$$

where u is the evident central morphism $a^{-1}(1\otimes c)a$, is the composite

$$A\otimes D \xrightarrow[\pi(f)\otimes g]{} [B,C]\otimes I \xrightarrow[b]{} [B,C]\ .$$

Proof. Set $\pi(f) = h$, so that $f = e(h\otimes 1)$ by the definition of π. Then

$$b(f\otimes g)u = b(e(h\otimes 1)\otimes g)u = b(e\otimes 1)((h\times 1)\otimes g)u\ ,$$

which is $ebu((h\otimes g)\otimes 1)$ by the naturality of b and of u. The bu in this last expression is, by Theorem 3.1, the unique central morphism $b\otimes 1: ([B,C]\otimes I)\otimes B \to [B,C]\otimes B$. Thus, by the definition of π again,

$$b(f\otimes g)u = e(b\otimes 1)((h\otimes g)\otimes 1) = \pi^{-1}(b(h\otimes g))\ .$$

Lemma 5.2. *For $f: A\to B$ and $g: C\otimes B \to D$, we have*

$$g(1\otimes f) = \langle f\rangle(\pi(g)\otimes 1): C\otimes A \to D\ .$$

Proof. By (5.1) and the definition of π,

$$\langle f\rangle(\pi(g)\otimes 1) = e(1\otimes f)(\pi(g)\otimes 1)) = e(\pi(g)\otimes 1)(1\otimes f) = g(1\otimes f)\ .$$

The remainder of this section concerns compatibility of graphs, for the closed category $\underline{G}$. We mostly omit the proofs, which are entirely evident but tedious to put into words.

Lemma 5.3. *Let $\xi : Q \to R$, $\eta : R \to S$, $\zeta : S \to T$ be graphs in $\underline{G}$; then the following assertions are equivalent:*

(i) *ζ is compatible with η and $\zeta\eta$ is compatible with ξ;*

(ii) *η is compatible with ξ and ζ is compatible with $\eta\xi$.*

When the assertions of Lemma 5.3 are true, we say that the three graphs ξ, η, ζ are compatible. The concept clearly extends to any number of graphs $\xi_i : T_{i-1} \to T_i$.

Lemma 5.4. (a) *Graphs $\xi : R \to S$ and $\eta : S \to T$ are compatible if either is central.*

(b) *If two graphs $\xi : R \to S$ and $\eta : S \to T$ are compatible while the graphs $\rho : R' \to R$, $\sigma : S \to S'$, $\tau : T \to T'$ are central, then the graphs $\sigma\xi\rho : R' \to S'$ and $\tau\eta\sigma^{-1} : S' \to T'$ are compatible.*

Proof. Immediate from Corollary 4.5.

Lemma 5.5. *In the situation of (5.1) above, if $\underline{H} = \underline{G}$, $h \otimes k$ is compatible with $f \otimes g$ if and only if h is compatible with f and k compatible with g.*

Lemma 5.6. *In the situation of (5.2) above, if $\underline{H} = \underline{G}$, then $\pi(f)$ is compatible with g if and only if f is compatible with $g \otimes 1$.*

Lemma 5.7. *Let $f : A \to B$, $g : C \otimes D \to E$, $k : E \otimes F \to G$ in $\underline{G}$. If k is compatible with $g \otimes 1 : (C \otimes D) \otimes F \to E \otimes F$, it is compatible with $(g \otimes 1)((\langle f\rangle \otimes 1) \otimes 1)$: $(([B, C] \otimes A) \otimes D) \otimes F \to E \otimes F$.*

Lemma 5.8. *Let $f : A \to B$, $g : C \otimes D \to E$, $u : A' \to A$, $v : D' \to D$ in $\underline{G}$. If f is compatible with u and if g is compatible with $1 \otimes v$ then $g(\langle f\rangle \otimes 1) : ([B, C] \otimes A) \otimes D \to E$ is compatible with $(1 \otimes u) \otimes v : ([B, C] \otimes A') \otimes D' \to ([B, C] \otimes A) \otimes D$.*

Lemma 5.9. *Let $\underline{H} = \underline{G}$, let f and g be as in Lemma 5.2, and let $h : D \otimes E \to F$. Then if $h : D \otimes E \to F$. Then if $h : D \otimes E \to F$, $g \otimes 1 : (C \otimes B) \otimes E \to D \otimes E$, and $(1 \otimes f) \otimes 1 : (C \otimes A) \otimes E \to (C \otimes B) \otimes E$ are compatible, so are $h(\langle f\rangle \otimes 1) : ([B, D] \otimes A) \otimes E \to F$ and $(\pi(g) \otimes 1) \otimes 1 : (C \otimes A) \otimes E \to ([B, D] \otimes A) \otimes E$.*

§6. Constructibility of allowable morphisms

We place ourselves once again in the general situation $\Gamma: \underline{H} \to \underline{G}$ envisaged in §2 and §4; we recall that the cases of interest are $\underline{H} = \underline{N}(\underline{V})$ and $\underline{H} = \underline{G}$. The object of this section is to show that the allowable morphisms may be built up, modulo central morphisms, by the three processes described in §5. It is convenient to introduce the temporary name of *constructible* morphisms for those allowable morphisms that can be so built up; our aim is then to show that all allowable morphisms are constructible. We also give in this section the proof of Theorem 2.2.

We therefore define the *constructible* morphisms of $\underline{H}$ to be the smallest class of morphisms of $\underline{H}$ satisfying the following five conditions:

CM1 Every central morphism is in the class.

CM2 If $f: T \to S$ is in the class and if $u: T' \to T$ and $v: S \to S'$ are central then $vfu: T' \to S'$ is in the class.

CM3 If $f: A \to C$ and $g: B \to D$ are in the class so is $f \otimes g: A \otimes B \to C \otimes D$.

CM4 If $f: A \otimes B \to C$ is in the class so is $\pi(f): A \to [B, C]$.

CM5 If $f: A \to B$ and $g: C \otimes D \to E$ are in the class so is $g(\langle f \rangle \otimes 1)$: $([B, C] \otimes A) \otimes D \to E$.

In view of the definitions (1.1) of π and (5.4) of $\langle\ \rangle$, it is evident that the allowable morphisms satisfy CM1–CM5, so that the constructible morphisms are a subclass of the allowable ones.

We call an allowable morphism $f: T \to S$ in $\underline{H}$ *trivial* if both T and S are constant integral shapes.

Lemma 6.1. *A trivial constructible morphism in $\underline{H}$ is central.*

Proof. Consider the subclass of the constructible morphisms consisting of the following morphisms $f: T \to S$: if T and S are both constant integral shapes, f is to be central; otherwise, f is to be constructible. This subclass clearly satisfies CM1, CM4 and CM5. It satisfies CM2 because, by Corollary 4.4, if T' and S' are constant integral shapes so are T and S; and then vfu is central if f is. It satisfies CM3 because if $A \otimes B$ and $C \otimes D$ are constant integral shapes so are A, B, C and D; and then $f \otimes g$ is central if f and g are. Hence this subclass contains all constructible morphisms.

Proposition 6.2. *For each constructible $h: T \to S$ in $\underline{H}$, at least one of the following is true:*

(i) *h is central*

(ii) *h is of the form*

$$T \xrightarrow{x} A \otimes B \xrightarrow{f \otimes g} C \otimes D \xrightarrow{y} S$$

where f and g are constructible and non-trivial, and x and y are central.

(iii) *h is of the form*

$$T \xrightarrow{\pi(f)} [B, C] \xrightarrow{y} S$$

where f is constructible and y is central.

(iv) *h is of the form*

$$T \xrightarrow{x} ([B, C] \otimes A) \otimes D \xrightarrow{\langle f \rangle \otimes 1} C \otimes D \xrightarrow{g} S$$

where g and f are constructible and x is central.

Proof. Consider those constructible morphisms that *are* of one of the above forms (i)–(iv); we show that this class satisfies CM1–CM5 and therefore consists of all constructible morphisms. That it satisfies CM1, CM4 and CM5 is clear.

To see that CM2 is satisfied, let $u: T' \to T$ and $v: S \to S'$ be central. Then if h is central, so is vhu. If h is as in (ii) above, vhu is $(vy)(f \otimes g)(xu)$, which is of the same form. If h is as in (iii) above, vhu is $(vy)\,\pi(f(u \otimes 1))$, which is of the same form, $f(u \otimes 1)$ being constructible by CM2 since $u \otimes 1$ is central. If h is as in (iv) above, vhu is $(vg)(\langle f \rangle \otimes 1)(xu)$, which is of the same form, vg being constructible by CM2.

That CM3 is satisfied is clear unless f or g is trivial. If g is trivial it is central by Lemma 6.1. In this case B, I, D are constant integral shapes, and the empty graphs $B \to I$ and $I \to D$ give, by Proposition 4.1, central morphisms $u: B \to I$ and $v: I \to D$ in $\underline{H}$. Then by Theorem 4.9 we have $g = vu$. It follows at once from the naturality of b that $f \otimes g$ is then the composite

$$A \otimes B \xrightarrow{1 \otimes u} A \otimes I \xrightarrow{b} A \xrightarrow{f} C \xrightarrow{b^{-1}} C \otimes I \xrightarrow{1 \otimes v} C \otimes D;$$

since $(1 \otimes v)b^{-1}$ and $b(1 \otimes u)$ are central, and since CM2 is satisfied, this lies in the class because f does. Finally if f is trivial then, by the naturality of c, $f \otimes g$ is the composite

$$A \otimes B \xrightarrow{c} B \otimes A \xrightarrow{g \otimes f} D \otimes C \xrightarrow{c} C \otimes D,$$

which is in the class since CM2 is satisfied and since, by what we have just proved, $g \otimes f$ is in the class.

Remark. For brevity, morphisms h of the forms (ii), (iii), (iv) of Proposition 6.2 will be said to be respectively *of type* $\otimes$, *of type* π, and *of type* $\langle\ \rangle$.

For the purposes of our inductive proofs we introduce for each shape T a non-negative integer $r(T)$ called its *rank*, defined by the following inductive rules:

R1 $\quad r(I) = 0$.

R2 $r(1) = 1$.

R3 $r(T \otimes S) = r(T) + r(S)$.

R4 $r([T, S]) = r(T) + r(S) + 1$.

Note that $r(T) = 0$ if and only if T is a constant integral shape.

Lemma 6.3. *If f: $T \to S$ is central then $r(T) = r(S)$.*

Proof. Those central morphisms for which this is true clearly satisfy AM1, AM3 and AM5, and therefore constitute the totality of central morphisms.

The non-trivial step in the proof that all the allowable morphisms are constructible is the proof that the constructible morphisms are closed under composition. In fact, because of the exigencies of the inductive argument, we prove the variant of closure-under-composition given in Proposition 6.4 below. Moreover, because the same inductive argument applies, we prove at the same time the corresponding fact about compatibility, which will lead to a proof of Theorem 2.2.

Proposition 6.4. *If the morphisms h: $T \to S$ and k: $S \otimes U \to V$ of $\underline{H}$ are constructible, so is the composite morphism*

$$T \otimes U \xrightarrow[h \otimes 1]{} S \otimes U \xrightarrow[k]{} V.$$

Moreover, if $\underline{H} = \underline{G}$, the graphs k and $h \otimes 1$ are compatible.

Proof. The proof is by a double induction; we suppose the results to be true for all pairs of constructible morphisms h': $T' \to S'$ and k': $S' \otimes U' \to V'$ for which $r(T') + r(S') + r(U') + r(V') < r(T) + r(S) + r(U) + r(V)$; we also suppose them to be true for any pair h', k' for which $r(T') + r(S') + r(U') + r(V') = r(T) + r(S) + r(U) + r(V)$, provided that $r(T') + r(S') < r(T) + r(S)$.

By Proposition 6.2, each of h and k is central, or of type $\otimes$, or of type π, or of type $\langle\ \rangle$; we distinguish cases accordingly. We shall use Lemma 5.4, the Axiom CM2, and Lemma 6.3 freely without further explicit mention to "ignore" or to "absorb" central morphisms wherever convenient.

Case 1: either h or k is central. If h is central, so is $h \otimes 1$; the results follow from CM2 and from Lemma 5.4.

Case 2: h is of type $\langle\ \rangle$. Let h be $g(\langle f \rangle \otimes 1)x$ as in Proposition 6.2 (iv). Then the desired composite is

$$k(h \otimes 1) = k(g \otimes 1)((\langle f \rangle \otimes 1) \otimes 1)(x \otimes 1)$$

$$= (k(g \otimes 1)a^{-1})(\langle f \rangle \otimes 1)(a(x \otimes 1)),$$

which is again of type $\langle\ \rangle$, provided only that $k(g\otimes 1)a^{-1}$ is constructible. Since a^{-1} is central, we need the constructibility of the composite

$$(C\otimes D)\otimes U \xrightarrow[g\otimes 1]{} S\otimes U \xrightarrow[k]{} V\,;$$

this follows by the inductive hypothesis since T, whose rank is equal by Lemma 6.3 to that of $([B,C]\otimes A)\otimes D$, has been replaced by $C\otimes D$, clearly of lower rank.

The same induction shows that k and $g\otimes 1$ are compatible; so by Lemma 5.7 and Lemma 5.4, k is compatible with $(g\otimes 1)((\langle f\rangle\otimes 1)\otimes 1)(x\otimes 1) = h\otimes 1$.

Case 3: h is of type $\otimes$. Let h be $y(f\otimes g)x$ as in Proposition 6.2 (ii). We are to consider the composite $k(h\otimes 1)$; without loss of generality we may suppose that $x = 1$ and absorb $y\otimes 1$ into k. Then we have

$$k(h\otimes 1) = k((f\otimes g)\otimes 1) = ka^{-1}(f\otimes(g\otimes 1))a$$

$$= ka^{-1}(f\otimes 1)(1\otimes(g\otimes 1))a\,,$$

so that finally $k(h\otimes 1)$ is the composite

$$(6.1)\qquad (A\otimes B)\otimes U \xrightarrow[wa]{} B\otimes(A\otimes U) \xrightarrow[g\otimes 1]{} D\otimes(A\otimes U) \xrightarrow[ka^{-1}(f\otimes 1)w^{-1}]{} V\,,$$

where the w's stand for two instances of the central morphism aca^{-1}. Now the composite

$$A\otimes(D\otimes U) \xrightarrow[f\otimes 1]{} C\otimes(D\otimes U) \xrightarrow[ka^{-1}]{} V$$

is constructible by the induction hypothesis, because U has been replaced by $D\otimes U$ (*this* is the reason for formulating Proposition 6.4 for $k(h\otimes 1)$ instead of just kh) and we have

$$r(T) + r(S) + r(U) + r(V) = r(A\otimes B) + r(C\otimes D) + r(U) + r(V)$$

$$> r(A) + r(C) + r(D\otimes U) + r(V)$$

unless $r(B) = 0$; in the latter case we get equality, but then $r(D) > 0$ since g is nontrivial, and

$$r(T) + r(S) = r(A \otimes B) + r(C \otimes D) > r(A) + r(C)\,,$$

so that the second half of the induction hypothesis applies. Since w^{-1} is central, $k' = ka^{-1}(f \otimes 1)w^{-1}$ is constructible; and since wa is central, (6.1) will be constructible if the composite $k'(g \otimes 1)$ is. But now the induction hypothesis shows that $k'(g \otimes 1)$ is indeed constructible, by essentially the same calculation with ranks as above, with g replacing f.

The same inductions show that ka^{-1} is compatible with $f \otimes 1$, so that k is compatible with $a^{-1}(f \otimes 1)w^{-1}$; and that k' is compatible with $g \otimes 1$, and hence with $(g \otimes 1)wa$. Therefore, by Lemma 5.3, k is compatible with $a^{-1}(f \otimes 1)w^{-1}(g \otimes 1)wa = h \otimes 1$.

Case 4: k is of type π. Thus $k = y\pi(f)$ for central y and constructible f. We can take $y = 1$, so that $k = \pi(f)\colon S \otimes U \to V = [B, C]$ for some constructible $f\colon (S \otimes U) \otimes B \to C$. By (5.2) we have

$$k(h \otimes 1) = \pi(f)(h \otimes 1) = \pi(f((h \otimes 1) \otimes 1))\,;$$

this is constructible if $f((h \otimes 1) \otimes 1)$ is, and hence if $f((h \otimes 1) \otimes 1)a^{-1} = fa^{-1}(h \otimes 1)$ is. This last is the composite

$$T \otimes (U \otimes B) \xrightarrow[h \otimes 1]{} S \otimes (U \otimes B) \xrightarrow[fa^{-1}]{} C\,,$$

which is constructible by induction, since $r(U \otimes B) + r(C) < r(U) + r([B, C])$.

The same induction proves fa^{-1} compatible with $h \otimes 1$, hence f with $a^{-1}(h \otimes 1)a = (h \otimes 1) \otimes 1$; and thence, by Lemma 5.6, $\pi(f)$ with $h \otimes 1$.

Case 5: h is of type π and k of type $\otimes$. There are central morphisms x, y and z such that $h = z\pi(m)$ for some constructible $m\colon T \otimes P \to Q$ and $k = y(f \otimes g)x$ for some constructible and non-trivial f and g. We may take $y = 1$ and absorb $z \otimes 1$ into x, so that the composite $k(h \otimes 1)$ to be considered has the form

$$T \otimes U \xrightarrow[h \otimes 1]{} [P, Q] \otimes U \xrightarrow[x]{} A \otimes B \xrightarrow[f \otimes g]{} C \otimes D = V\,.$$

Interchanging A and B if necessary, we can assume that the central morphism x associates $[P, Q]$ with a prime factor of A. Then Proposition 4.10 gives a shape R such that x has the form of a composite

$$[P, Q] \otimes U \xrightarrow[1 \otimes s]{} [P, Q] \otimes (R \otimes B) \xrightarrow[a^{-1}]{} ([P, Q] \otimes R) \otimes B \xrightarrow[t \otimes 1]{} A \otimes B\,,$$

for suitable central s and t. Since $(1 \otimes s)(h \otimes 1) = (h \otimes 1)(1 \otimes s)$ we can drop s and write $U = R \otimes B$, while t can be absorbed into f. The composite $k(h \otimes 1)$ to be considered now has the form

$$T \otimes (R \otimes B) \xrightarrow[h \otimes 1]{} [P,Q] \otimes (R \otimes B) \xrightarrow[a^{-1}]{} ([P,Q] \otimes R) \otimes B \xrightarrow[f \otimes g]{} C \otimes D .$$

This may be rewritten as

$$(f \otimes g)a^{-1}(h \otimes 1) = (f \otimes g)((h \otimes 1) \otimes 1)a^{-1} = (f(h \otimes 1) \otimes g)a^{-1} ,$$

which will be constructible by CM2 and CM3 if the composite

$$T \otimes R \xrightarrow[h \otimes 1]{} [P,Q] \otimes R \xrightarrow[f]{} C$$

is. That this is indeed so follows by induction, since g is non-trivial and therefore $r(U) + r(V) = r(R \otimes B) + r(C \otimes D) > r(R) + r(C)$.

The induction argument also shows that f is compatible with $h \otimes 1$, whence, by Lemmas 5.5 and 5.4, $f \otimes g$ is compatible with $((h \otimes 1) \otimes 1)a^{-1} = a^{-1}(h \otimes 1)$; finally, by Lemma 5.4 again, $k = (f \otimes g)a^{-1}$ is compatible with $h \otimes 1$.

Case 6: h is of type π and k is of type ⟨ ⟩. Thus there are central morphisms z and x such that $h = z\pi(m)$ and $k = g(\langle f \rangle \otimes 1)x$ for constructible morphisms

$$m\colon T \otimes P \to Q , \quad f\colon A \to B , \quad g\colon C \otimes D \to V .$$

By absorbing $z \otimes 1$ in x, we may suppose that $S = [P,Q]$ and that $z = 1$. Then the composite to be considered has the form

$$T \otimes U \xrightarrow[\pi(m) \otimes 1]{} [P,Q] \otimes U \xrightarrow[x]{} ([B,C] \otimes A) \otimes D \xrightarrow[g(\langle f \rangle \otimes 1)]{} V .$$

We distinguish three subcases, according as $[P,Q]$ is associated via the central morphism x with $[B,C]$, with a prime factor of A, or with a prime factor of D. (We recall that these possibilities need not be mutually exclusive, if $[P,Q]$ is constant.)

Subcase 1: $[P,Q]$ is associated with $[B,C]$. By Proposition 4.11, $P = B$, $Q = C$, and x is the composite

$$[B,C] \otimes U \xrightarrow[1 \otimes s]{} [B,C] \otimes (A \otimes D) \xrightarrow[a^{-1}]{} ([B,C] \otimes A) \otimes D$$

for a suitable central s. Since $(1 \otimes s)(h \otimes 1) = (h \otimes 1)(1 \otimes s)$ we may, arguing as in Case 5, suppose that $U = A \otimes D$ and $s = 1$. The composite to be considered then has the form

$$T \otimes (A \otimes D) \xrightarrow[\pi(m)\otimes 1]{} [B,C] \otimes (A \otimes D) \xrightarrow[a^{-1}]{} ([B,C] \otimes A) \otimes D \xrightarrow[g(\langle f\rangle \otimes 1)]{} V.$$

Now $a^{-1}(\pi(m) \otimes 1) = ((\pi(m) \otimes 1) \otimes 1)\, a^{-1}$ by naturality, while by Lemma 5.2 we have $\langle f\rangle\, (\pi(m) \otimes 1) = m(1 \otimes f)$. The composite thus becomes

$$k(h \otimes 1) = g(m(1 \otimes f) \otimes 1)a^{-1} = g(mc(f \otimes 1)c \otimes 1)a^{-1}\ .$$

This formula involves two successive composites, first a composite h',

$$A \otimes T \xrightarrow[f\otimes 1]{} B \otimes T \xrightarrow[mc]{} C = Q$$

and second the composite

$$(T \otimes A) \otimes D \xrightarrow[h'c\otimes 1]{} C \otimes D \xrightarrow[g]{} V\ .$$

Both are of the form considered in our induction. The induction assumption does apply to both because the original rank, with $U = A \otimes D$ and $S = [P, Q] = [B, C]$, is

$$rT + rS + rU + rV = rT + rV + rB + rC + rA + rD + 1\ ,$$

and this clearly exceeds either of the ranks $rA + rB + rT + rC$ or $rT + rA + rD + rC + rV$ involved in the two composites above.

The same induction shows that g is compatible with $h'c \otimes 1$ and mc is compatible with $f \otimes 1$; so that by Lemmas 5.3, 5.4 and 5.5 $g(m \otimes 1)$ is compatible with $(1 \otimes f) \otimes 1$. It follows from Lemma 5.9 and Lemma 5.4 that $g(\langle f\rangle \otimes 1)$ is compatible with $((\pi(m) \otimes 1) \otimes 1)\, a^{-1} = a^{-1}(\pi(m) \otimes 1)$, so that $k = g(\langle f\rangle \otimes 1)a^{-1}$ is compatible with $h \otimes 1 = \pi(m) \otimes 1$, as required.

Subcase 2: $[P, Q] = S$ is associated with a prime factor of A. By Proposition 4.10, there is a shape R such that $a(c \otimes 1)x$ is the composite

$$S \otimes U \xrightarrow[1\otimes s]{} S \otimes (R \otimes ([B,C] \otimes D))$$

$$\xrightarrow[a^{-1}]{} (S \otimes R) \otimes ([B,C] \otimes D) \xrightarrow[t\otimes 1]{} A \otimes ([B,C] \otimes D)$$

for suitable central s and t. By the naturality of a and c, therefore, x is the composite

$$S \otimes U \xrightarrow[1\otimes s]{} S \otimes (R \otimes ([B,C] \otimes D))$$

$$\xrightarrow[w]{} ([B,C] \otimes (S \otimes R)) \otimes D \xrightarrow[(1\otimes t)\otimes 1]{} ([B,C] \otimes A) \otimes D,$$

where w is the central natural transformation $w = (c \otimes 1)a^{-1}a^{-1}$. Once again, since $(1 \otimes s)(h \otimes 1) = (h \otimes 1)(1 \otimes s)$, we may suppose that $s = 1$ and $U = R \otimes ([B, C] \otimes D)$. Moreover, since $\langle f \rangle (1 \otimes t) = \langle ft \rangle$ by (5.5) (the naturality of $\langle\ \rangle$), we may absorb t in f and hence suppose that $A = S \otimes R$ and $t = 1$. The desired composite $k(h \otimes 1)$ thus has the form

$$g(\langle f \rangle \otimes 1)\, w(h \otimes 1) = g(\langle f \rangle \otimes 1)((1 \otimes (h \otimes 1)) \otimes 1)\, w$$

$$= g(\langle f(h \otimes 1) \rangle \otimes 1) w,$$

by the naturality of w and of $\langle\ \rangle$. It thus suffices by CM5 to prove the composite

$$T \otimes R \xrightarrow[h \otimes 1]{} S \otimes R \xrightarrow[f]{} B$$

constructible. But this is of the form considered in the induction, and since $r(U) > r(R) + r(B)$ the inductive hypothesis applies.

By the same inductive argument f is compatible with $h \otimes 1$; by Lemmas 5.8 and 5.4, therefore, $g(\langle f \rangle \otimes 1)$ is compatible with $((1 \otimes (h \otimes 1)) \otimes 1)w = w(h \otimes 1)$; finally, by Lemma 5.4 again, $k = g(\langle f \rangle \otimes 1)\, w$ is compatible with $h \otimes 1$.

Subcase 3: $[P, Q] = S$ is associated with a prime factor of D. By Proposition 4.10, there is a shape R such that cx is the composite

$$S \otimes U \xrightarrow[1 \otimes s]{} S \otimes (R \otimes ([B, C] \otimes A))$$

$$\xrightarrow[a^{-1}]{} (S \otimes R) \otimes ([B, C] \otimes A) \xrightarrow[t \otimes 1]{} D \otimes ([B, C] \otimes A)$$

for suitable central s and t. By the naturality of c, therefore, x is the composite

$$S \otimes U \xrightarrow[1 \otimes s]{} S \otimes (R \otimes ([B, C] \otimes A))$$

$$\xrightarrow[u]{} ([B, C] \otimes A) \otimes (S \otimes R) \xrightarrow[1 \otimes t]{} ([B, C] \otimes A) \otimes D,$$

where u is the central natural transformation $u = ca^{-1}$. Since $(1 \otimes s)(h \otimes 1) = (h \otimes 1)(1 \otimes s)$ we may again suppose that $s = 1$, so that $U = R \otimes ([B, C] \otimes A)$. Since $(\langle f \rangle \otimes 1)(1 \otimes t) = (1 \otimes t)(\langle f \rangle \otimes 1)$, we may absorb $1 \otimes t$ in g and hence suppose that $t = 1$ and $D = S \otimes R$. The desired composite $k(h \otimes 1)$ is then

$$g(\langle f \rangle \otimes 1)\, u(h \otimes 1) = g(\langle f \rangle \otimes 1)(1 \otimes (h \otimes 1))\, u$$

$$= g(1 \otimes (h \otimes 1))(\langle f \rangle \otimes 1)\, u$$

$$= gu(h \otimes 1)u^{-1}(\langle f \rangle \otimes 1)\, u,$$

using (5.1) and the naturality of u. It thus suffices by CM5 to prove the constructibility of $gu(h \otimes 1)u^{-1}$, and therefore of $gu(h \otimes 1)$. This is the composite

$$T \otimes (R \otimes C) \xrightarrow[h \otimes 1]{} S \otimes (R \otimes C) \xrightarrow[gu]{} V,$$

which is constructible by the inductive hypothesis since $r(R \otimes C) < r(U)$.

By the same inductive argument gu is compatible with $h \otimes 1$, so that g is compatible with $u(h \otimes 1)u^{-1} = 1 \otimes (h \otimes 1)$. By Lemmas 5.8 and 5.4, therefore, $g(\langle f \rangle \otimes 1)$ is compatible with $(1 \otimes (h \otimes 1))u = u(h \otimes 1)$; so that finally $k = g(\langle f \rangle \otimes 1)u$ is compatible with $h \otimes 1$.

This concludes the proof of Proposition 6.4.

Theorem 6.5. *The constructible morphisms of $\underline{H}$ are exactly the allowable ones.*

Proof. We have already observed that the allowable morphisms clearly satisfy CM1–CM5, so that every constructible morphism is allowable. It remains to show that the constructible morphisms satisfy AM1–AM5.

They satisfy AM1 because 1, a, b, c, a^{-1}, b^{-1} are central. As for AM2, d: $T \to [S, T \otimes S]$ is $\pi(1)$ where 1: $T \otimes S \to T \otimes S$, so that d is constructible by CM4; and e: $[T, S] \otimes T \to S$ is $\langle 1 \rangle$, which by the naturality of b is the composite

$$[T, S] \otimes T \xrightarrow[b^{-1}]{} ([T, S] \otimes T) \otimes I \xrightarrow[\langle 1 \rangle \otimes 1]{} S \otimes I \xrightarrow[b]{} S,$$

so that e is constructible by CM2 and CM5. AM3 is trivially satisfied, as it coincides with CM3. In AM4, let f: $T \to T'$ and $g: S \to S'$ be constructible. Then the composite

$$[T', S] \otimes T \xrightarrow[b^{-1}]{} ([T', S] \otimes T) \otimes I \xrightarrow[\langle f \rangle \otimes 1]{} S \otimes I \xrightarrow[gb]{} S'$$

is constructible by CM2 and CM5; but this composite is $g\langle f \rangle$ by the naturality of b. It follows from CM4 that $\pi(g\langle f \rangle)$ is constructible; but $\pi(g\langle f \rangle) = \pi(ge(1 \otimes f))$ is equal by the naturality of π to $[g, f]\, \pi(e) = [g, f]\, 1 = [g, f]$. Thus AM4 is satisfied.

There remains AM5. Let f: $T \to S$ and $g: S \to R$ be constructible. Then the composite

(6.2) $$S \otimes I \xrightarrow[b]{} S \xrightarrow[g]{} R$$

is constructible by CM2, whence the composite

(6.3) $$T \otimes I \xrightarrow[f \otimes 1]{} S \otimes I \xrightarrow[gb]{} R$$

is constructible by Proposition 6.4; by CM2 again, the composite of (6.3) with $b^{-1}: T \to T \otimes I$ is also constructible, and by the naturality of b this composite is gf.

Proof of Theorem 2.2. Let $f: T \to S$, $g: S \to R$ be allowable graphs in $\underline{G}$. By Theorem 6.5, they are constructible. The composite (6.2) is then also constructible, so that in (6.3) gb is compatible with $f \otimes 1$ by Proposition 6.4. We conclude from Lemma 5.4 that g is compatible with $b(f \otimes 1)b^{-1} = f$.

§7. Proofs of Theorem 2.1 and Theorem 2.4

We still use $\underline{H}$ to denote $\underline{N}(\underline{V})$ or $\underline{G}$, with $\Gamma: \underline{H} \to \underline{G}$ as before. For the purposes of this section we need a slight refinement of Proposition 6.2. Let us call an integral shape P *reduced* if it is either the shape I or else is constructed by the rules S2 and S3 alone; that is, it contains no I's unless it reduces to I alone. By *an iterated tensor product* of shapes $X_1, ..., X_n$ we mean $|P|(X_1, ..., X_n)$ for any reduced integral shape P with $v(P) = \{1, ..., n\}$; if $n = 0$ it is just I. Let us call an arbitrary shape T *reduced* if, in its prime factorization $T = |P|(X_1, ..., X_n)$, the integral shape P is reduced. (This is a consistent use of language since the prime factorization of the integral shape P is $|P|(1, 1, ..., 1)$.)

Lemma 7.1. *Given any shape T we can find a reduced shape T' and a central isomorphism $z: T \to T'$ in $\underline{H}$.*

Proof. Let the prime factorization of T be $|P|(X_1, ..., X_n)$. Let P' be a reduced integral shape with $v(P') = v(P) = \{1, ..., n\}$, and let $\xi: P \to P'$ be the graph corresponding to the identity permutation of $\{1, ..., n\}$. Set $T' = |P'|(X_1, ..., X_n)$ and set $z = |\xi|_{\underline{H}}(X_1, ..., X_n)$.

Lemma 7.2. *In Proposition 6.2 we can suppose that the shapes A, B, C, D in (ii) and the shapes A, D in (iv) are reduced.*

Proof. In case (ii), replace A, B, C, D by reduced isomorphs as in Lemma 7.1, absorbing the central isomorphims thereby introduced into x and y; similarly for case (iv).

We define the *rank* $r(h)$ of a morphism $h: T \to S$ in $\underline{H}$ to be the sum $r(T) + r(S)$ of the ranks of T and of S. If h is allowable, which by Theorem 6.5 is the same thing as constructible, Proposition 6.2 asserts that h has one of the four following forms:

$$(7.1) \qquad h = x\,, \quad h = y(f \otimes g)x\,, \quad h = y\pi(f)\,, \quad h = g(\langle f \rangle \otimes 1)x\,,$$

where x and y are central, f and g are allowable, and moreover in the $y(f \otimes g)x$ case neither f nor g is trivial. The basis of our inductive arguments is the obvious fact that in each case we have $r(f) < r(h)$ and (where applicable) $r(g) < r(h)$. In using the forms (7.1) we shall always suppose that the reductions of Lemma 7.2 have been carried out.

Proof of Theorem 2.1. We are to construct an algorithm for deciding whether a graph $h: T \to S$ in $\underline{G}$ is allowable. We suppose inductively that we possess such an algorithm for all smaller values, if any, of $r(h)$. Since finding the prime factorizations of T and of S is algorithmic, Propositions 4.3 and 4.2 enable us to decide whether h is central. It remains to test whether h is of one of the remaining types in (7.1), which we again refer to as type $\otimes$, type π, and type $\langle\ \rangle$.

To test whether h is of type $\otimes$, with the notation as in Proposition 6.2 and with A, B, C, D reduced, first observe that, by Proposition 4.3, the prime factors of A and of B must together make up those of T; so that there are only a finite number of possibilities for A and for B to be tried (because A and B are reduced!). Similarly there are only a finite number of possibilities for C and for D; and for a given choice of A, B, C, D there are only a finite number of possibilities for x and for y. When these choices are all made, the graph $y^{-1}hx^{-1}$ is either not of the form $f \otimes g$, or else is of this form for a unique f and g. Since $r(f) < r(h)$ and $r(g) < r(h)$, we can now test f and g for allowability.

Entirely similar procedures allow us to test whether h is of type π or of type $\langle\ \rangle$, so that we have the desired algorithm.

Remark. There are non-allowable graphs in $\underline{G}$; the unique graph $[1, 1] \to I$ is one such.

Before proving Theorem 2.4 we establish some facts about proper shapes, as defined in §2. Observe that every constant shape is proper; that if $[T, S]$ is proper then T and S are proper; and that $T \otimes S$ is proper if and only if T and S are proper – whence T is proper if and only if each of its prime factors is proper.

Lemma 7.3. *If $h: T \to S$ is a central morphism in $\underline{H}$ and if either T or S is proper so is the other.*

Proof. By Proposition 4.3, T and S have the same prime factors.

Proposition 7.4. *Let $h: T \to S$ be allowable in $\underline{H}$, with the shape S constant and the shape T proper. Then the shape T is constant.*

Proof. Suppose inductively that it is so for all smaller values, if any, of $r(h)$, and consider h of one of the four possible types in (7.1). If h is central the result is immediate by Corollary 4.4. By this same Corollary 4.4, together with Lemma 7.3 and Lemma 6.3, we may ignore central factors x and y in the other types in (7.1). If $h = f \otimes g: A \otimes B \to C \otimes D$ then C and D are constant because S is and A and B are proper because T is, so that by induction A and B are constant, whence T is constant. If $h = \pi(f): T \to [B, C]$ then B and C are constant because S is, so that $T \otimes B$ is proper because T is, and then T is constant by the inductive hypothesis applied to $f: T \otimes B \to C$. Finally, if $h = g(\langle f \rangle \otimes 1): ([B, C] \otimes A) \otimes D \to S$, then the inductive hypothesis applied to $g: C \otimes D \to S$ shows that C and D are constant. Since T is proper so is $[B, C]$, whence B is constant. Finally the inductive hypothesis applied to $f: A \to B$ shows that A is constant, so that T is constant.

We next show how to eliminate constant prime factors from a shape T.

Lemma 7.5. *Given a shape T we can find a shape S with $r(S) \leq r(T)$ and an allowable isomorphism $f: T \to S$ in $\underline{H}$ with allowable inverse such that*

(a) *S is reduced;*

(b) *S has no constant prime factors, its prime factors being precisely the non-constant ones of T;*

(c) *if T is proper, so is S;*

(d) *there is a constant shape R and a central isomorphism $T \to S \otimes R$ with the same graph as f.*

Proof. Let S be any iterated tensor product of the non-constant prime factors of T, and R any iterated tensor product of the constant prime factors of T. There is an evident central isomorphism $T \to S \otimes R$, and f is the composite of this with

$$S \otimes R \xrightarrow[1 \otimes k_R]{} S \otimes I \xrightarrow[b]{} S,$$

where k_R is the isomorphism of Lemma 4.7.

Proposition 7.6. *Let $h: P \otimes Q \to M \otimes N$ be an allowable morphism in $\underline{H}$, where P, Q, M, N are proper shapes. Suppose that the graph Γh is of the form $\xi \otimes \eta$ for graphs $\xi: P \to M$ and $\eta: Q \to N$. Then there are allowable morphisms $p: P \to M$, $q: Q \to N$ such that $h = p \otimes q$, $\Gamma p = \xi$, and $\Gamma q = \eta$.*

Proof. Suppose inductively that it is so for all smaller values, if any, of $r(h)$. By Lemma 7.5 we may without loss of generality suppose each of P, Q, M, N to be reduced and to have only non-constant prime factors.

If h is central, so is $\Gamma h = \xi \otimes \eta$. From Propositions 4.3 and 4.2, it is clear that ξ and η are then central. By Theorem 4.9 there are central $p: P \to M$, $q: Q \to N$ with $\Gamma p = \xi$ and $\Gamma q = \eta$; then $\Gamma h = \Gamma(p \otimes q)$, so that by Theorem 4.9 again we have $h = p \otimes q$.

If h is of type $\otimes$, say h is the composite

$$P \otimes Q \xrightarrow[x]{} A \otimes B \xrightarrow[f \otimes g]{} C \otimes D \xrightarrow[y]{} M \otimes N;$$

let an iterated tensor product, in the order in which they occur in P, of those prime factors of P that are associated via x with a prime factor of A [resp. B] be X [resp. Y]; similarly let an iterated $\otimes$-product of those prime factors of Q associated via x with a prime factor of A [resp. B] be U [resp. V]. In the same way let X', Y', U', V' be iterated $\otimes$-products of the prime factors "common" to M and C, M and D, N and C, N and D respectively. Define a graph $\rho: X \to X'$ as the restriction of Γh to $v(X) + v(X')$; this is indeed a graph because Γh is of the form $\xi \otimes \eta$. Define similarly graphs $\sigma: Y \to Y'$, $\tau: U \to U'$, $\kappa: V \to V'$. The graphs of the allowable morphisms

(7.2) $$X\otimes U \longrightarrow A \xrightarrow[f]{} C \longrightarrow X'\otimes U' ,$$

(7.3) $$Y\otimes V \longrightarrow B \xrightarrow[g]{} D \longrightarrow Y'\otimes V' ,$$

where the unnamed arrows denote the obvious central morphisms, are respectively $\rho\otimes\tau$ and $\sigma\otimes\kappa$. By the inductive hypothesis we conclude that (7.2) and (7.3) are respectively $r\otimes t$ and $s\otimes k$ for allowable morphisms r, t, s, k with the respective graphs $\rho, \tau, \sigma, \kappa$. Define p and q to be the composites

$$P \longrightarrow X\otimes Y \xrightarrow[r\otimes s]{} X'\otimes Y' \longrightarrow M ,$$

$$Q \longrightarrow U\otimes V \xrightarrow[t\otimes k]{} U'\otimes V' \longrightarrow N ,$$

where once again the unnamed arrows denote the obvious central morphisms. That $h = p\otimes q$ is then immediate from Theorem 4.9, while evidently $\Gamma p = \xi$ and $\Gamma q = \eta$.

If h is of type π, say h is the composite

$$P\otimes Q \xrightarrow[\pi(f)]{} [B, C] \xrightarrow[y]{} M\otimes N ,$$

then by Proposition 4.3 either $M = [B, C]$ and $N = I$ or else $N = [B, C]$ and $M = I$; by replacing h by chc if necessary we may suppose the former to be the case. Then h is the composite

$$P\otimes Q \xrightarrow[\pi(f)]{} [B, C] \xrightarrow[b^{-1}]{} [B, C]\otimes I .$$

Since $\Gamma h = \xi\otimes\eta$ it follows from Lemma 5.1 that the graph of the composite

$$(P\otimes B)\otimes Q \xrightarrow[u]{} (P\otimes Q)\otimes B \xrightarrow[f]{} C \xrightarrow[b^{-1}]{} C\otimes I ,$$

where u is the evident central morphism, is $\pi^{-1}(\xi)\otimes\eta$. By induction, therefore, $b^{-1}fu$ is $r\otimes q$ for allowable $r: P\otimes B \to C$ with graph $\pi^{-1}(\xi)$ and $q: Q \to I$ with graph η. Set $p = \pi(r)$; then $\Gamma p = \xi$ and $h = p\otimes q$ by another application of Lemma 5.1.

If h is of type $\langle\ \rangle$, with the notation of Proposition 6.2, we may (replacing h by chc if necessary) suppose that $[B, C]$ is associated via x with a prime factor of P. Let an iterated $\otimes$-product of those prime factors of A associated via x with a prime factor of P [resp. Q] be X [resp. Y]. The mate under Γh of an element of $v(Y)$ is in $v(A) + v(B)$ by the form $g(\langle f\rangle\otimes 1)x$ of h, but is in $v(Q) + v(N)$ by the hypothesis that $\Gamma h = \xi\otimes\eta$; it must therefore be in $v(Y)$. Thus the graph of the composite

(7.4) $$X\otimes Y \longrightarrow A \xrightarrow[f]{} B \xrightarrow[b^{-1}]{} B\otimes I ,$$

where the unnamed arrow is the obvious central morphism, is of the form $\rho \otimes \sigma$ for graphs $\rho : X \to B$ and $\sigma : Y \to I$. By the inductive hypothesis, (7.4) is $r \otimes s$ for allowable $r : X \to B$, $s : Y \to I$. It then follows from Proposition 7.4 that Y is constant; since none of the prime factors of Q is constant, this means that $Y = I$ and that all the prime factors of A are therefore associated via x with prime factors of P.

It follows then from Proposition 4.10 that there are a shape R and central morphisms y and z such that x is the composite

$$P \otimes Q \xrightarrow[y \otimes 1]{} (([B,C] \otimes A) \otimes R) \otimes Q \xrightarrow[a]{} ([B,C] \otimes A) \otimes (R \otimes Q)$$
$$\xrightarrow[1 \otimes z]{} ([B,C] \otimes A) \otimes D.$$

It is clear that the graph of the composite

$$(C \otimes R) \otimes Q \xrightarrow[a]{} C \otimes (R \otimes Q) \xrightarrow[1 \otimes z]{} C \otimes D \xrightarrow[g]{} M \otimes N$$

is $\zeta \otimes \eta$, where $\zeta : C \otimes R \to M$ is the restriction of ξ to $v(C) + v(R) + v(M)$. By induction, $g(1 \otimes z)a$ is $r \otimes q$ for allowable $r : C \otimes R \to M$ and $q : Q \to N$ with the appropriate graphs. Setting p equal to the composite

$$P \xrightarrow[y]{} ([B,C] \otimes A) \otimes R \xrightarrow[r(\langle f \rangle \otimes 1)]{} M,$$

we have $p \otimes q = (r \otimes q)((\langle f \rangle \otimes 1) \otimes 1)(y \otimes 1) = g(1 \otimes z)a((\langle f \rangle \otimes 1) \otimes 1)(y \otimes 1)$; by the naturality of a this is $g(1 \otimes z)(\langle f \rangle \otimes 1)a(y \otimes 1) = g(\langle f \rangle \otimes 1)(1 \otimes z)a(y \otimes 1) = g(\langle f \rangle \otimes 1)x = h$.

This completes the proof of Proposition 7.6.

Proposition 7.7. *Let $f : A \otimes B \to C$ be an allowable morphism in $\underline{H}$, where A, B, C are proper shapes. Suppose that the mate under Γf of each element of $v(B)$ is again in $v(B)$. Then B is constant.*

Proof. The composite

$$A \otimes B \xrightarrow[f]{} C \xrightarrow[b^{-1}]{} C \otimes I$$

has a graph of the form $\xi \otimes \eta$; therefore by Proposition 7.6 there is an allowable morphism $q : B \to I$. It follows from Proposition 7.4 that B is constant.

Proposition 7.8. *Let $h : ([Q,M] \otimes P) \otimes N \to S$ be an allowable morphism between proper shapes in $\underline{H}$, with $[Q,M]$ not constant. Suppose that the graph Γh is of the form $\eta(\langle \xi \rangle \otimes 1)$ for graphs $\xi : P \to Q$, $\eta : M \otimes N \to S$. Suppose finally that ξ cannot be written in the form*

$$P \xrightarrow[\omega]{} ([F,G] \otimes E) \otimes H \xrightarrow[\rho(\langle \omega \rangle \otimes 1)]{} Q \tag{7.5}$$

for any graphs ω, ρ, σ with ω central. Then there are allowable morphisms $p: P \to Q$, $q: M \otimes N \to S$ such that $h = q(\langle p \rangle \otimes 1)$, $\Gamma p = \xi$ and $\Gamma q = \eta$.

Proof. Suppose inductively that it is so for all smaller values, if any, of $r(h)$. Use Lemma 7.5 to replace P, N, S by reduced shapes which have no constant prime factors; we must show that in doing so we lose no generality. It follows from (5.5) that doing so makes no difference to the expressibility Γh in the form $\eta(\langle \xi \rangle \otimes 1)$ or the expressibility of h in the form $q(\langle p \rangle \otimes 1)$ for allowable p and q. We must show that it makes no difference to the expressibility of ξ in the form (7.5); but this is a very easy deduction from Lemma 7.5 (d).

Note that once we have $h = q(\langle p \rangle \otimes 1)$, it is automatic that $\Gamma p = \xi$ and $\Gamma q = \eta$.

Suppose that h is central. By Corollary 4.5, the mate under Γh of an element of $\upsilon(P)$ or of $\upsilon(Q)$ is then an element of $\upsilon(S)$. On the other hand, by the form $\eta(\langle \xi \rangle \otimes 1)$ of Γh the mate of an element of $\upsilon(P)$ is an element of $\upsilon(Q)$, and conversely. It follows that P and Q are both constant, so that by the reduction above we must have $P = I$. Lemma 4.7 now gives an allowable $p = k_Q^{-1}: I \to Q$. By the naturality of e,

$$\langle p \rangle = e(1 \otimes p) = e([p, 1] \otimes 1) : [Q, M] \otimes I \longrightarrow [I, M] \otimes I \longrightarrow M\ .$$

On the other hand, $[A, [I, M]] \cong [A \otimes I, M] \cong [A, M]$ in any closed category, so the Yoneda Lemma provides an isomorphism $[1, b]d: M \cong [I, M]$ with inverse eb^{-1}. Therefore $[1, b]de = b: [I, M] \otimes I \cong [I, M]$, so that from the display above

$$[1, b]d\langle p \rangle = b([p, 1] \otimes 1): [Q, M] \otimes I \longrightarrow [I, M]\ .$$

Since the right-hand morphism is an isomorphism we have constructed the following factorization of the identity

$$1 = t\langle p \rangle: [Q, M] \otimes I \longrightarrow [Q, M] \otimes I,$$

with an allowable $t = ([p^{-1}, 1] \otimes 1) b^{-1} [1, b] d$. The originally given allowable morphism h can now be factored as

$$h = h1 = h(t \otimes 1)(\langle p \rangle \otimes 1) = q(\langle p \rangle \otimes 1)$$

with q allowable, as required.

If h is of the form $y(f \otimes g)x$ for $f: A \to C, g: B \to D$, we may without loss of generality suppose that $[Q, M]$ is associated via x with a prime factor of A. Let an iterated $\otimes$-product of those prime factors of P associated via x with a prime factor of A [resp. B] be X [resp. Y]. The mate under Γh of an element of $\upsilon(Y)$ is in $\upsilon(B) + \upsilon(D)$ by the form $y(f \otimes g)x$ of h, but is in $\upsilon(Q) + \upsilon(P)$ by the hypothesis that $\Gamma h = \eta(\langle \xi \rangle \otimes 1)$; it must therefore be in $\upsilon(Y)$. It now follows from Proposition

7.7 that Y is constant; since none of the prime factors of P is constant, this means that $Y = I$ and that all the prime factors of P are associated via x with prime factors of A.

It follows then from Proposition 4.10 that there are a shape R and central morphisms s and t such that x is the composite

$$([Q,M] \otimes P) \otimes N \xrightarrow[1 \otimes t]{} ([Q,M] \otimes P) \otimes (R \otimes B) \xrightarrow[a^{-1}]{} (([Q,M] \otimes P) \otimes R) \otimes B \xrightarrow[s \otimes 1]{} A \otimes B .$$

It is clear that the graph of the composite

$$([Q,M] \otimes P) \otimes R \xrightarrow[s]{} A \xrightarrow[f]{} C$$

is $\zeta(\langle\zeta\rangle \otimes 1)$, where $\zeta : M \otimes R \to C$ is the restriction of η to $v(M) + v(R) + v(C)$. It follows by induction that fs is $r(\langle p\rangle \otimes 1)$ for allowable $p: P \to Q$ and $r: M \otimes R \to C$. Then

$$h = y(f \otimes g)x = y(f \otimes g)(s \otimes 1)a^{-1}(1 \otimes t)$$

$$= y(fs \otimes g)a^{-1}(1 \otimes t) = y(r \otimes g)((\langle p\rangle \otimes 1) \otimes 1)a^{-1}(1 \otimes t)$$

$$= y(r \otimes g)a^{-1}(\langle p\rangle \otimes 1)(1 \otimes t) = y(r \otimes g)a^{-1}(1 \otimes t)(\langle p\rangle \otimes 1)$$

is of the required form, with $q = y(r \otimes g)a^{-1}(1 \otimes t)$.

If h is of the form $y\pi(f)$ for some $y : [B,C] \to S$, we must have $[B,C] = S$ and $y = 1$. Then $\Gamma h = \eta(\langle\xi\rangle \otimes 1)$, $\pi^{-1}h = f$ and the naturality of π^{-1} shows that $\Gamma f = \pi^{-1}\eta((\langle\xi\rangle \otimes 1) \otimes 1)$. If we rewrite this as

$$\Gamma(fa^{-1}) = (\Gamma f)\alpha^{-1} = (\pi^{-1}\eta)\alpha^{-1}(\langle\xi\rangle \otimes 1) : ([Q,M] \otimes P) \otimes (N \otimes B) \to C ,$$

we can apply the induction assumption to fa^{-1} to get $fa^{-1} = r(\langle p\rangle \otimes 1)$ for allowable $p: P \to Q$ and $r: M \otimes (N \otimes B) \to C$. Then $f = ra\ \ ((\langle p\rangle \otimes 1) \otimes 1)$, so by the naturality of π we get

$$h = \pi f = \pi(ra^{-1})(\langle p\rangle \otimes 1) ,$$

which is in the desired form.

In the final case where h is of the form

$$([Q,M] \otimes P) \otimes N \xrightarrow[x]{} ([B,C] \otimes A) \otimes D \xrightarrow[\langle f\rangle \otimes 1]{} C \otimes D \xrightarrow[g]{} S ,$$

we distinguish cases according as $[Q,M]$ is associated via x with (i) $[B,C]$; (ii) a prime factor of A; or (iii) a prime factor of D.

Case (i). By Proposition 4.11, $B = Q$, $C = M$, and x is the composite

$$([Q,M] \otimes P) \otimes N \xrightarrow[a]{} [Q,M] \otimes (P \otimes N)$$

$$\xrightarrow[1 \otimes y]{} [Q,M] \otimes (A \otimes D) \xrightarrow[a^{-1}]{} ([Q,M] \otimes A) \otimes D$$

for some central y. Let X be an iterated $\otimes$-product of those prime factors of P associated via y with a prime factor of D. The mate under Γh of an element of $v(X)$ is in $v(M) + v(D) + v(S)$ by the form $g(\langle f \rangle \otimes 1)x$ of h, but is in $v(P) + v(Q)$ by the hypothesis that $\Gamma h = \eta(\langle \xi \rangle \otimes 1)$; it must therefore be in $v(X)$. Then X is constant by Proposition 7.7, and since P has no constant prime factors, this means that all the prime factors of P are associated via y with prime factors of A. A similar argument shows that all the prime factors of N are associated via y with prime factors of D. We may therefore, absorbing central morphisms into g and f where necessary, suppose without loss of generality that $A = P$, $D = Q$, and $x = 1$. Then $h = q(\langle p \rangle \otimes 1)$ with $q = g$ and $p = f$.

Case (ii). $[Q,M]$ is associated via x with a prime factor of A. Suppose if possible that $[B,C]$ were associated via x with a prime factor of P. Let X be an iterated $\otimes$-product of all those prime factors of $([Q,M] \otimes P) \otimes N$ that either are prime factors of P or else are associated via x with prime factors of A. The mate under Γh of an element of $v(A)$ is in $v(A) + v(B)$ by the form $g(\langle f \rangle \otimes 1)x$ of h, while the mate under Γh of an element of $v(P)$ is in $v(P) + v(Q)$ by the hypothesis that $\Gamma h = \eta(\langle \xi \rangle \otimes 1)$; thus the mate under Γh of an element of $v(X)$ is again in $v(X)$. It follows from Proposition 7.7 that X is constant, which contradicts the hypothesis that $[Q,M]$ is not constant.

Thus no prime factor of P is associated via x with $[B,C]$. Let Y be an iterated $\otimes$-product of those prime factors of P associated via x with prime factors of D. The mate under Γh of an element of $v(Y)$ is in $v(C) + v(D) + v(S)$ by the form $g(\langle f \rangle \otimes 1)x$ of h, but is in $v(P) + v(Q)$ by the hypothesis that $\Gamma h = \eta(\langle \xi \rangle \otimes 1)$; it must therefore be in $v(Y)$. Then Y is constant by Proposition 7.7, and since P has no constant prime factors this means that every prime factor of P is associated via x with a prime factor of A.

Then by Proposition 4.10 there are a shape R and central morphisms t and s such that $a(c \otimes 1)x$ is the composite

$$([Q,M] \otimes P) \otimes N \xrightarrow[1 \otimes s]{} ([Q,M] \otimes P) \otimes (R \otimes ([B,C] \otimes D))$$

$$\xrightarrow[a^{-1}]{} (([Q,M] \otimes P) \otimes R) \otimes ([B,C] \otimes D)$$

$$\xrightarrow[t \otimes 1]{} A \otimes ([B,C] \otimes D);$$

thus by naturality x is the composite

$$([Q,M]\otimes P)\otimes N \xrightarrow[1\otimes n]{} ([Q,M]\otimes P)\otimes(([B,C]\otimes R)\otimes D)$$
$$\xrightarrow[v]{} ([B,C]\otimes(([Q,M]\otimes P)\otimes R))\otimes D$$
$$\xrightarrow[(1\otimes t)\otimes 1]{} ([B,C]\otimes A)\otimes D\,,$$

where n is the central morphism $(c\otimes 1)a^{-1}s$ and v is the evident central morphism. It is clear that the graph of the composite

$$([Q,M]\otimes P)\otimes R \xrightarrow[t]{} A \xrightarrow[f]{} B$$

is $\zeta(\langle\xi\rangle\otimes 1)$, where $\zeta\colon M\otimes R\to B$ is the restriction of η to $v(M)+v(R)+v(B)$. So by induction ft is $r(\langle p\rangle\otimes 1)$ for allowable $p\colon P\to Q$ and $r\colon M\otimes R\to B$. Setting q equal to the composite

$$M\otimes N \xrightarrow[1\otimes n]{} M\otimes(([B,C]\otimes R)\otimes D)$$
$$\xrightarrow[v]{} ([B,C]\otimes(M\otimes R))\otimes D \xrightarrow[g(\langle r\rangle\otimes 1)]{} S\,,$$

we have $q(\langle p\rangle\otimes 1) = g(\langle r\rangle\otimes 1)v(1\otimes n)(\langle p\rangle\otimes 1) = g(\langle r\rangle\otimes 1)\,v(\langle p\rangle\otimes 1)(1\otimes n)$; by the naturality of v this is $g(\langle r\rangle\otimes 1)((1\otimes(\langle p\rangle\otimes 1))\otimes 1)\,v(1\otimes n)$. Using (5.5), $\langle r\rangle(1\otimes(\langle p\rangle\otimes 1)) = \langle r(\langle p\rangle\otimes 1)\rangle$; which is $\langle ft\rangle$, or $\langle f\rangle(1\otimes t)$ by (5.5) again. Thus finally,

$$q(\langle p\rangle\otimes 1) = g(\langle f\rangle\otimes 1)((1\otimes t)\otimes 1)\,v(1\otimes n) = g(\langle f\rangle\otimes 1)x = h\,.$$

Case (iii). $[Q,M]$ is associated via x with a prime factor of D. Suppose if possible that $[B,C]$ were associated via x with a prime factor of P. Let X be an iterated $\otimes$-product of those prime factors of A that are associated via x with prime factors of N. The mate under Γh of an element of $v(X)$ is in $v(A)+v(B)$ by the form $g(\langle f\rangle\otimes 1)x$ of h, but is in $v(M)+v(N)+v(S)$ by the hypothesis that $\Gamma h = \eta(\langle\xi\rangle\otimes 1)$; it must therefore be in $v(X)$. Then X is constant by Proposition 7.7, and since N has no constant prime factors we conclude that every prime factor of A is associated via x with a prime factor of P. This implies that ξ is of the form

$$P \xrightarrow[\omega]{} ([B,C]\otimes A)\otimes H \xrightarrow[\rho(\langle\sigma\rangle\otimes 1)]{} Q$$

for some integral ω, which is excluded by hypothesis.

Thus no prime factor of P is associated via x with $[B,C]$. Let Y be an iterated $\otimes$-product of those prime factors of P associated via x with prime factors of A. The mate under Γh of an element of $v(Y)$ is in $v(A)+v(B)$ by the form $g(\langle f\rangle\otimes 1)x$ of h, but is in $v(P) + v(Q)$ by the hypothesis that $\Gamma h = \eta(\langle\xi\rangle \otimes 1)$; it must therefore be in $v(Y)$. It follows from Proposition 7.7 that Y is constant, and since P has no constant prime factors this means that all the prime factors of P are associated via x with prime factors of D.

For brevity, let us write

$$M' = [Q, M] \otimes P, \qquad C' = [B, C] \otimes A$$

so that $\langle f\rangle : C' \to C$. In the central morphism $x : M' \otimes N \to C' \otimes D$, we now know that all the prime factors of M' are associated with prime factors of D. Apply Proposition 4.10 to this situation; it gives a shape R, a central morphism $N \to C' \otimes R$ (which without loss we can take to be the identity) and a central morphism $t : M' \otimes R \to D$, so that $x = (1 \otimes t)w$, as in the first row of the following diagram, in which w is the evident central morphism:

$$\begin{array}{ccccccc}
M' \otimes (C' \otimes R) & \xrightarrow{w} & C' \otimes (M' \otimes R) & \xrightarrow{1 \otimes t} & C' \otimes D & & \\
\downarrow{\scriptstyle 1 \otimes (\langle f\rangle \otimes 1)} & & \downarrow{\scriptstyle \langle f\rangle \otimes 1} & & \downarrow{\scriptstyle \langle f\rangle \otimes 1} & & \\
M' \otimes (C \otimes R) & \xrightarrow{w} & C \otimes (M' \otimes R) & \xrightarrow{1 \otimes t} & C \otimes D & \xrightarrow{g} & S\ .
\end{array}$$

The diagram evidently commutes. Now the hypothesis $\Gamma h = \eta(\langle\xi\rangle \otimes 1)$ for $\eta : M \otimes N \to S$ clearly means that the graph of the composite $g(1 \otimes t)w$ is $\zeta(\langle\xi\rangle \otimes 1)$, where $\zeta : M \otimes (C \otimes R) \to S$ is the restriction of η to $vM + vC + vR + vS$. By induction, $g(1 \otimes t)w$ is $r(\langle p\rangle \otimes 1)$ for allowable $p : P \to Q$ and $r : M \otimes (C \otimes R) \to S$. Therefore, since $\langle p\rangle : M' \to M$,

$$h = g(1 \otimes t)w(1 \otimes (\langle f\rangle \otimes 1)) = r(\langle p\rangle \otimes 1)(1 \otimes (\langle f\rangle \otimes 1)) \quad .$$

$$= r(1 \otimes (\langle f\rangle \otimes 1))(\langle p\rangle \otimes 1)$$

has the requisite form $q(\langle p\rangle \otimes 1)$ for $q = r(1 \otimes (\langle f\rangle \otimes 1))$.

This concludes the proof of Proposition 7.8.

Proof of Theorem 2.4. Let T, S be proper shapes and let $h, h' : T \to S$ be allowable natural transformations in $\underline{N}(\underline{V})$ with $\Gamma h = \Gamma h'$; we are to prove that $h = h'$. Suppose inductively that it is so for all smaller values (if any) of $r(h)$; note that $r(h) = r(T) + r(S) = r(h')$. By Lemma 7.5, we may suppose that none of the prime factors of T or of S is constant.

If both h and h' are central we have $h = h'$ by Theorem 4.9. So we may suppose that h is of one of the other forms (7.1).

If h is of the form $y\pi(f)$, we have $y^{-1}h = \pi(f)$ where f is allowable. But then $y^{-1}h' = \pi(f')$, where $f' = \pi^{-1}(y^{-1}h')$ is also allowable by (1.2). Since h and h' have the same graph, so do f and f'; hence $f' = f$ by the inductive hypothesis, whence $h' = h$.

If h is of the form $y(f \otimes g)x$, we have $y^{-1}hx^{-1} = f \otimes g$: $A \otimes B \to C \otimes D$. Then $\Gamma(y^{-1}h'x^{-1}) = \Gamma(y^{-1}hx^{-1}) = \Gamma f \otimes \Gamma g$; so that by Proposition 7.6 $y^{-1}h'x^{-1} = f' \otimes g'$ for allowable $f'\colon A \to C$ and $g'\colon B \to D$ with $\Gamma f = \Gamma f'$ and $\Gamma g = \Gamma g'$, whence $f' = f$ and $g' = g$ by the inductive hypothesis. Hence $h' = h$.

There remains the case where h is of the form $g(\langle f \rangle \otimes 1)x$. Then it may be the case that the graph Γf of f is of the form

$$A \xrightarrow[\omega]{} ([F, G] \otimes E) \otimes H \xrightarrow[\rho(\langle\sigma\rangle \otimes 1)]{} B$$

for some central ω and some ρ, σ; in this case we have $\Gamma h = \Gamma g(\langle \Gamma f \rangle \otimes 1)\Gamma x$. Here, since $\langle\ \rangle$ is natural, as in (5.5),

$$\langle \Gamma f \rangle = \langle \rho(\langle \sigma \rangle \otimes 1)\omega \rangle = \langle \rho \rangle (1 \otimes (\langle \sigma \rangle \otimes 1))(1 \otimes \omega)\,.$$

It now follows easily that $\Gamma h = \tau(\langle \sigma \rangle \otimes 1)\psi$ for some τ and for some central ψ. Perhaps σ is of the form

$$E \xrightarrow[\phi]{} ([X, Y] \otimes Z) \otimes W \xrightarrow[\kappa(\langle\lambda\rangle \otimes 1)]{} F$$

for some central ϕ and some κ, λ; but E has strictly fewer prime factors than A, since $[F, G]$ is a prime factor of A but not of E; Z has strictly fewer prime factors than E; and so on. Thus this process terminates, and ultimately we have an expression for Γh of the form

$$T \xrightarrow[\mu]{} ([Q, M] \otimes P) \otimes N \xrightarrow[\eta(\langle\xi\rangle \otimes 1)]{} S$$

where μ is central and ξ is not of the form (7.5). Moreover $[Q, M]$ is not constant since T has no constant prime factors. By Theorem 4.9 there is a central natural transformation $y\colon T \to ([Q, M] \otimes P) \otimes N$ with $\Gamma y = \mu$. From Proposition 7.8 applied to hy^{-1} and $h'y^{-1}$ we conclude that $hy^{-1} = q(\langle p \rangle \otimes 1)$ and $h'y^{-1} = q'(\langle p' \rangle \otimes 1)$ for allowable $p, p'\colon P \to Q$ and allowable $q, q'\colon M \otimes N \to S$ with $\Gamma p = \Gamma p'$ and $\Gamma q = \Gamma q'$. It follows from the inductive hypothesis that $p = p'$ and $q = q'$, so that $h = h'$.

This completes the proof of Theorem 2.4.

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