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The regularity of points in multi-projective spaces

Huy Tài Hà[a], Adam Van Tuyl[b,*]

[a] Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
[b] Department of Mathematical Sciences, Lakehead University, Thunder Bay, Ont., Canada P7B 5E1

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Abstract

Let $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$ be the defining ideal of a scheme of fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with support in generic position. When all the m_i's are 1, we explicitly calculate the Castelnuovo–Mumford regularity of I. In general, if at least one $m_i \geqslant 2$, we give an upper bound for the regularity of I, which extends a result of Catalisano, Trung and Valla.

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0. Introduction

In this paper, we study the Castelnuovo–Mumford regularity of defining ideals of sets of points (reduced and non-reduced) in a multi-projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

If $I \subseteq \mathbf{k}[x_0, \cdots, x_n]$ is the defining ideal of a projective variety $X \subseteq \mathbb{P}^n$, then the Castelnuovo–Mumford regularity of I, denoted by $\text{reg}(I)$, is a very important invariant associated to X. It has been the objective of many authors to estimate $\text{reg}(I)$ since not only does it bound the degrees of a minimal set of defining equations for X, it also gives a uniform bound on the degrees of syzygies of I. The most fundamental situation is when X is a set of points. Examples of work on $\text{reg}(I)$ in this case can be seen in [5,7,8,15]. Recently, many authors (cf. [4,9–11,16]) have been interested in extending

* Corresponding author. Tel.: +1-807-343-8228; fax: +1-807-343-8023.
E-mail address: avantuyl@sleet.lakeheadu.ca (A. Van Tuyl).

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our understanding of points in $\mathbb{P}^n$ to sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We continue this trend by studying $\mathrm{reg}(I)$ when I defines a scheme of fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

In the context of $\mathbb{N}^2$-graded rings, Aramova et al. [1] have introduced a finer notion of regularity that places bounds on each coordinate of the degree of a multi-graded syzygy. Extending the definition of regularity to multi-graded rings is also considered recently in [12,13]. The usual notion of regularity could be treated as a bound on the total degree of the multi-graded syzygies. Our results, thus, naturally provide corresponding results for the new notion of regularity in multi-graded rings.

The $\mathbb{N}^k$-graded ring $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ where $\deg x_{i,j} = e_i$, the ith basis vector of $\mathbb{N}^k$, is the associated coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. The defining ideal of P_i is $\wp_i = (L_{1,1}, \ldots, L_{1,n_1}, \ldots, L_{k,1}, \ldots, L_{k,n_k})$ with $\deg L_{i,j} = e_i$. If $m_1, \ldots, m_s$ are positive integers, then we want to study the regularity of ideals of the form $I_Z = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$. Such an ideal I_Z defines a scheme of fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. The ideal I_Z is both $\mathbb{N}^k$-homogeneous, and homogeneous in the normal sense. Thus, when we refer to $\mathrm{reg}(I_Z)$, we shall mean its regularity as a homogeneous ideal in R, where R is viewed as a $\mathbb{N}^1$-graded ring.

A set of s points $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is said to be in generic position if it has maximal Hilbert function $H_{\mathbb{X}}(\underline{i}) = \min\{\dim_{\mathbf{k}} R_{\underline{i}}, s\}$ for all $\underline{i} \in \mathbb{N}^k$, where $R = \oplus_{\underline{i}} R_{\underline{i}}$ is the $\mathbb{N}^k$-homogeneous decomposition of R. It is shown in [17] that sets of s points in generic position form an open subset of the Hilbert scheme of all sets of s points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Our main results consist of explicitly calculating $\mathrm{reg}(I_Z)$ when Z is in generic position and reduced (i.e. there is no multiplicity at each point), and giving a bound on $\mathrm{reg}(I_Z)$ in general.

In the special case that each $m_i = 1$ and the set of points is in generic position, we show

$$\mathrm{reg}(I_Z) = \max\{d_1 + 1, \ldots, d_k + 1\},$$

where

$$d_i = \min\left\{ d \in \mathbb{N} \;\middle|\; \binom{d + n_i}{d} \geqslant s \right\} \quad \text{for each } i = 1, \ldots, k.$$

To prove this we use the fact that I_Z is both $\mathbb{N}^k$-homogeneous and $\mathbb{N}^1$-homogeneous to obtain information about $\mathrm{reg}(I_Z)$. We also use the Bayer–Stillman criterion for detecting m-regularity [2].

We then show that if $\mathbb{X}$ is generic position, and if $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$ with at least one $m_i \geqslant 2$, then

$$\mathrm{reg}(I_Z) \leqslant \max\left\{ m_1 + m_2 - 1, \left\lceil \frac{\sum_{i=1}^{s} m_i - 1}{n_1} \right\rceil, \ldots, \left\lceil \frac{\sum_{i=1}^{s} m_i - 1}{n_k} \right\rceil \right\} + k.$$

Our strategy is to investigate the regularity index $\mathrm{ri}(R/I_Z)$ of R/I_Z, considered as a $\mathbb{N}^1$-graded ring, by extending the results of [5] for fat point schemes in $\mathbb{P}^n$ to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and then use the fact that $\mathrm{reg}(I_Z) \leqslant \mathrm{ri}(R/I_Z) + k$.

We have organized this paper as follows. In Section 1 we introduce the relevant information about regularity, the regularity index, and points in multi-projective spaces.

In the Section 2 we compute the regularity of a defining ideal of a set of points in generic position. In the last section we bound the regularity for a set of fat points with generic support.

1. Preliminaries

Throughout this paper $\mathbf{k}$ denotes an algebraically closed field of characteristic zero. In this section, we recall the needed facts about the Castelnuovo–Mumford regularity, the regularity index, and points in multi-projective spaces. Let $S=\mathbf{k}[x_0,\ldots,x_n]$ be a polynomial ring.

Definition 1.1. A graded S-module M is m-regular if there exists a free resolution

$$0\to\bigoplus_j S(-e_{r,j})\to\cdots\to\bigoplus_j S(-e_{1,j})\to\bigoplus_j S(-e_{0,j})\to M\to 0$$

of M with $e_{i,j}-i\leqslant m$ for all i,j. The *Castelnuovo–Mumford regularity* (or simply, *regularity*) of M, denoted $\operatorname{reg}(M)$, is the least integer m for which M is m-regular.

If $I\subseteq S$, then $\operatorname{reg}(I)=\operatorname{reg}(S/I)+1$. The *saturation* $\bar{I}$ of the ideal $I\subseteq S$ is the ideal

$$\bar{I}:=\{F\in S \mid \text{ for } i=1,\ldots,n, \text{ there exists a } r \text{ such that } x_i^r\cdot F\in I\}.$$

I is said to be *saturated* if $I=\bar{I}$. The regularity of a saturated ideal does not change if we add a non-zero divisor. In fact,

Lemma 1.2 (Bayer and Stillman [2, Lemma 1.8]). *Let $I\subseteq S$ be a saturated ideal, and suppose h is a non-zero divisor of S/I. Then I is m-regular if and only if (I,h) is m-regular. Thus,* $\operatorname{reg}(I)=\operatorname{reg}((I,h))$.

The following theorem provides a means to determine if an ideal is m-regular.

Theorem 1.3. (Bayer and Stillman, [2, Theorem 1.10] criterion for m-regularity) *Let $I\subseteq S$ be an ideal generated in degrees $\leqslant m$. The following conditions are equivalent*:

(i) *I is m-regular.*
(ii) *There exists $h_1,\ldots,h_j\in S_1$ for some $j\geqslant 0$ so that*
  (a) $((I,h_1,\ldots,h_{i-1}):h_i)_m=(I,h_1,\ldots,h_{i-1})_m$ *for* $i=1,\ldots,j$, *and*
  (b) $(I,h_1,\ldots,h_j)_m=S_m$.

The *Hilbert function* $H_M:\mathbb{N}\to\mathbb{N}$ of a graded S-module M is defined $H_M(t):=\dim_{\mathbf{k}} M_t$. It is well known (cf. Bruns and Herzog [3, Theorem 4.1.3]) that there exists a unique polynomial $HP_M(t)$, called the *Hilbert polynomial* of M, such that $H_M(t)=HP_M(t)$ for $t\gg 0$.

Definition 1.4. The *regularity index* of a S-module M, denoted $\operatorname{ri}(M)$, is defined to be

$$\operatorname{ri}(M):=\min\{t \mid H_M(j)=HP_M(j) \text{ for all } j\geqslant t\}.$$

The regularity and regularity index of a S-module are then related as follows.

Lemma 1.5 (Migliore and Nagel [14, Lemma 5.8]). *If M is a graded S-module, then*

$$\mathrm{reg}(M) - \dim M + 1 \leqslant \mathrm{ri}(M) \leqslant \mathrm{reg}(M) - \text{depth } M + 1.$$

If $M = S/I$, then $\mathrm{ri}(S/I) \leqslant \mathrm{reg}(S/I) - \text{depth } S/I + 1 \leqslant \mathrm{reg}(I)$. Hence, we have

Corollary 1.6. *If $I \subseteq S$, then for all $t \geqslant \mathrm{reg}(I)$, $H_{S/I}(t) = HP_{S/I}(t)$.*

Our goal is to investigate $\mathrm{reg}(I)$ when I defines either a reduced or non-reduced set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ whose support is in generic position.

Let $R = \mathbf{k}[x_{1,0},\ldots,x_{1,n_1},\ldots,x_{k,0},\ldots,x_{k,n_k}]$, with $\deg x_{i,j} = e_i$ where e_i is the ith basis vector of $\mathbb{N}^k$, be the $\mathbb{N}^k$-graded coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $R_{e_i} = \mathbf{k}[x_{i,0},\ldots,x_{i,n_i}]$ be the graded coordinate ring of $\mathbb{P}^{n_i}$ for $i = 1,\ldots,k$. If $P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a point, then the ideal $\wp \subseteq R$ associated to P is the prime ideal $\wp = (L_{1,1},\ldots,L_{1,n_1},\ldots,L_{k,1},\ldots,L_{k,n_k})$ with $\deg L_{i,j} = e_i$. Suppose $\mathbb{X} = \{P_1,\ldots,P_s\}$ is a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and $m_1,\ldots,m_s$ are s positive integers. Let

$$I_Z = \wp_1^{m_1} \cap \wp_2^{m_2} \cap \cdots \cap \wp_s^{m_s},$$

where $\wp_i$ is the defining ideal of P_i, then I_Z defines a scheme of fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with support $\mathbb{X}$. When $m_i = 1$ for all i, $Z \equiv \mathbb{X}$ is reduced, and we usually use $I_{\mathbb{X}}$ instead of I_Z.

Since $\mathrm{ht}(\wp_i) = \sum_{j=1}^{k} n_j$ for each i, it follows that K-dim $R/I_Z = k$. Thus, by Lemma 1.5 we have

$$\mathrm{reg}(I_Z) \leqslant \mathrm{ri}(R/I_Z) + k.$$

Note that we have equality if $k = 1$ because then depth $R/I_Z = 1$.

We shall find it useful to consider R/I_Z as both an $\mathbb{N}^k$-graded ring and as a $\mathbb{N}^1$-graded ring. We shall, therefore, use $\mathscr{H}_Z(\underline{t})$ to denote the multi-graded Hilbert function $\mathscr{H}_Z(\underline{t}) := \dim_{\mathbf{k}}(R/I_Z)_{\underline{t}}$ with $\underline{t} = (t_1,\ldots,t_k) \in \mathbb{N}^k$, and $H_Z(t)$ to denote the $\mathbb{N}^1$-graded Hilbert function $H_Z := H_{R/I_Z}$. Because $(R/I_Z)_t = \oplus_{t_1+\ldots+t_k=t} (R/I_Z)_{t_1,\ldots,t_k}$, we have the identity:

$$H_Z(t) = \sum_{t_1+\ldots+t_k=t} \mathscr{H}_Z(t_1,\ldots,t_k) \text{ for all } t \in \mathbb{N}.$$

Definition 1.7. A set of s points $\mathbb{X} = \{P_1,\ldots,P_s\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is said to be in *generic position* if

$$\mathscr{H}_{\mathbb{X}}(\underline{t}) = \min\left\{\dim_{\mathbf{k}} R_{\underline{t}} = \binom{t_1+n_1}{n_1}\cdots\binom{t_k+n_k}{n_k}, s\right\} \text{ for all } \underline{t} \in \mathbb{N}^k.$$

Further results about points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ can be found in [16,17].

Remark 1.8. If $I \subseteq R$ is an $\mathbb{N}^k$-homogeneous ideal, then the $\mathbb{N}^k$-graded minimal free resolution of I is

$$0 \to \mathscr{F}_r \to \mathscr{F}_{r-1} \to \cdots \to \mathscr{F}_0 \to I \to 0,$$

where $\mathscr{F}_i = \oplus_j R(-d_{i,j,1}, -d_{i,j,2}, \ldots, -d_{ij,k})$. Since I is also homogeneous in the normal sense, the above resolution also gives a graded minimal free resolution of I:

$$0 \to \mathscr{F}'_r \to \mathscr{F}'_{r-1} \to \cdots \to \mathscr{F}'_0 \to I \to 0,$$

where $\mathscr{F}'_i = \oplus_j R(-d_{i,j,1} - d_{i,j,2} - \cdots - d_{i,j,k})$ where we view R as $\mathbb{N}^1$-graded. So if I is a $\mathbb{N}^k$-homogeneous ideal with $k \geqslant 2$, $\operatorname{reg}(I)$ can be interpreted as a crude invariant that bounds the total degree of the multi-graded syzygies.

The following lemma, which generalizes [16, Lemma 3.3], enables us to find non-zero divisors of specific multi-degrees.

Lemma 1.9. *Suppose $\mathbb{X} = \{P_1, \ldots, P_s\}$ is a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $\wp_1, \cdots, \wp_s$ are the defining ideals of $P_1, \ldots, P_s$, respectively, and $m_1, \ldots, m_s$ are positive integers. Set $I_Z = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$, and fix an $i \in \{1, \ldots, k\}$. Then there exists a form $L \in R_{e_i}$ such that $\bar{L}$ is a non-zero divisor in R/I_Z.*

2. The regularity of the defining ideal of points in generic position

Let $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a set of s reduced points in generic position. In this section we calculate the Castelnuovo–Mumford regularity of the defining ideal of $\mathbb{X}$.

For each $i = 1, \ldots, k$, set

$$d_i := \min \left\{ d \;\middle|\; \binom{d + n_i}{d} \geqslant s \right\},$$

and let $D := \max\{d_1 + 1, \ldots, d_k + 1\}$.

Note that if $n_i = \min\{n_1, \ldots, n_k\}$, then $D = d_i + 1$. Beginning with a combinatorial lemma, we use this notation to describe some of the properties of points in generic position.

Lemma 2.1. *Let $n \geqslant 1$. Then, for all $a, b \geqslant 1$,*

$$\binom{a+b+n}{a+b} \leqslant \binom{a+n}{a} \binom{b+n}{b}.$$

Proof. Because

$$\binom{a+b+n}{a+b} = \frac{(a+b+n)\cdots(a+1+n)}{(a+b)(a+b-1)\cdots(a+1)} \binom{a+n}{a},$$

it is enough to show that the inequality

$$\frac{(a+b+n)(a+b-1+n)\cdots(a+1+n)}{(a+b)\cdots(a+1)} \leqslant \binom{b+n}{b}$$

is true. This is equivalent to showing that

$$\frac{(a+b+n)(a+b-1+n)\cdots(a+1+n)}{(b+n)(b-1+n)\cdots(1+n)} \leqslant \frac{(a+b)(a+b-1)\cdots(a+1)}{b(b-1)\cdots 2\cdot 1}.$$

Rewriting the above expression, we see that we need to show that

$$\left[1+\frac{a}{b+n}\right]\left[1+\frac{a}{b-1+n}\right]\cdots\left[1+\frac{a}{1+n}\right] \leqslant \left[1+\frac{a}{b}\right]\left[1+\frac{a}{b-1}\right]\cdots\left[1+\frac{a}{1}\right].$$

But since

$$\left[1+\frac{a}{b+n-j}\right] \leqslant \left[1+\frac{a}{b-j}\right]$$

for $j=0,\ldots,b-1$ we are finished. □

Corollary 2.2. *Let $\mathbb{X} \subseteq \mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ be s points in generic position. If $(t_1,\ldots,t_k)\in\mathbb{N}^k$ is such that $t_1+\cdots+t_k = D-1$, then $\mathscr{H}_{\mathbb{X}}(t_1,\ldots,t_k)=s$.*

Proof. Suppose that $n_i=\min\{n_1,\ldots,n_k\}$, and hence, $D-1=d_i$. Lemma 2.1 then gives

$$\binom{t_1+n_1}{t_1}\binom{t_2+n_2}{t_2}\cdots\binom{t_k+n_k}{t_k} \geqslant \binom{t_1+n_i}{t_1}\binom{t_2+n_i}{t_2}\cdots\binom{t_k+n_i}{t_k}$$
$$\geqslant \binom{d_i+n_i}{d_i}.$$

Since

$$\binom{d_i+n_i}{d_i} \geqslant s,$$

we have $\mathscr{H}_{\mathbb{X}}(t_1,\ldots,t_k)=s$. □

Proposition 2.3. *Let $I_{\mathbb{X}}$ be the defining ideal of s points $\mathbb{X} \subseteq \mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ in generic position.*

(i) *As a $\mathbb{N}^1$-graded ideal, $I_{\mathbb{X}}$ is generated by forms of degree $\leqslant D$.*
(ii) *As a $\mathbb{N}^1$-graded ring, $R/I_{\mathbb{X}}$ has Hilbert polynomial $HP_{R/I_{\mathbb{X}}}(t)=s\binom{t+k-1}{k-1}$.*
(iii) *Fix an $i\in\{1,\ldots,k\}$ and let L be the non-zero divisor of Lemma 1.9 of degree e_i. If $\underline{t}=(t_1,\ldots,t_k)\in\mathbb{N}^k$ is such that $t_1+\cdots+t_k \geqslant D$ and $t_i>0$, then $(I_{\mathbb{X}},L)_{\underline{t}}=R_{\underline{t}}$.*

Proof. For (i) it suffices to show that for all $\underline{t}=(t_1,\ldots,t_k)\in\mathbb{N}^k$ with $t_1+\cdots+t_k\geqslant D+1$, $(I_{\mathbb{X}})_{\underline{t}}$ contains no new minimal generators. If $\underline{t}\in\mathbb{N}^k$ is such a tuple, then there exists $l,m\in\{1,\ldots,k\}$, not necessarily distinct, such that $\underline{t}-e_l-e_m\in\mathbb{N}^k$. By Corollary 2.2 it follows that $\mathscr{H}_{\mathbb{X}}(\underline{t}-e_l-e_m)=\mathscr{H}_{\mathbb{X}}(\underline{t}-e_l)=s$, since $t_1+\cdots+t_k-2\geqslant D-1$. Now apply the results of [17] to conclude that $(I_{\mathbb{X}})_{\underline{t}}$ contains no minimal generators.

Since $\mathbb{X}$ is in generic position, for $t\gg 0$ we have

$$H_{\mathbb{X}}(t)=\sum_{t_1+\cdots+t_k=t}\mathscr{H}_{\mathbb{X}}(t_1,\ldots,t_k)=\sum_{t_1+\cdots+t_k=t}s=s\binom{t+k-1}{k-1}.$$

Since $HP_{R/I_{\mathbb{X}}}$ is the unique polynomial that agrees with $H_{\mathbb{X}}$ for $t\gg 0$, (ii) now follows.

To prove (iii) we only need to consider the case $i=1$. Since $\bar{L}$ is a non-zero divisor, the exact sequence

$$0\to(R/I_{\mathbb{X}})(-e_1)\overset{\times\bar{L}}{\to}R/I_{\mathbb{X}}\to R/(I_{\mathbb{X}},L)\to 0$$

implies that

$$\mathscr{H}_{R/(I_{\mathbb{X}},L)}(t_1,\ldots,t_k)=\mathscr{H}_{\mathbb{X}}(t_1,\ldots,t_k)-\mathscr{H}_{\mathbb{X}}(t_1-1,t_2,\ldots,t_k)\text{ for all }\underline{t}\in\mathbb{N}^k,$$

where $\mathscr{H}_{\mathbb{X}}(t_1-1,t_2,\ldots,t_k)=0$ if $t_1-1<0$. Now suppose that $t_1+\cdots+t_k\geqslant D$ with $t_1>0$. Since $(t_1-1)+t_2+\cdots+t_k\geqslant D-1$, by Corollary 2.2 we have $\mathscr{H}_{\mathbb{X}}(t_1,\ldots,t_k)=\mathscr{H}_{\mathbb{X}}(t_1-1,t_2,\ldots,t_k)=s$. Thus $\mathscr{H}_{R/(I_{\mathbb{X}},L)}(t_1,\ldots,t_k)=0$, or eqivalently, $(I_{\mathbb{X}},L)_{(t_1,\ldots,t_k)}=R_{t_1,\ldots,t_k}$. □

Theorem 2.4. *Let $I_{\mathbb{X}}$ be the defining ideal of s points $\mathbb{X}\subseteq\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ in generic position. Then*

$$\operatorname{reg}(I_{\mathbb{X}})=\max\{d_1+1,\ldots,d_k+1\},$$

where $d_i:=\min\left\{d\;\middle|\;\binom{d+n_i}{d}\geqslant s\right\}$ *for* $i=1,\ldots,k$.

Proof. Without loss of generality, we assume that $n_1\geqslant n_2\geqslant\cdots\geqslant n_k\geqslant 1$. It thus suffices to show that $\operatorname{reg}(I_{\mathbb{X}})=d_k+1=\max\{d_1+1,\ldots,d_k+1\}$.

We first show that $\operatorname{reg}(I_{\mathbb{X}})>d_k$. By Lemma 1.9 there is a non-zero divisor $\bar{L}$ of $R/I_{\mathbb{X}}$ with $\deg L=e_k$. As a $\mathbb{N}$-homogeneous element of R, $\deg L=1$. Since $I_{\mathbb{X}}$ is saturated, by Lemma 1.2 it is enough to show $\operatorname{reg}(I_{\mathbb{X}},L)>d_k$.

From the short exact sequence

$$0\to(R/I_{\mathbb{X}})(-1)\overset{\times\bar{L}}{\to}R/I_{\mathbb{X}}\to R/(I_{\mathbb{X}},L)\to 0$$

of $\mathbb{N}^1$-graded rings, and from Proposition 2.3(ii) we deduce that

$$HP_{R/(I_{\mathbb{X}},L)}(t)=HP_{R/I_{\mathbb{X}}}(t)-HP_{R/I_{\mathbb{X}}}(t-1)=s\binom{t+(k-2)}{k-2}.$$

If we can show that $HP_{R/(I_{\mathbb{X}},L)}(d_k) \neq H_{R/(I_{\mathbb{X}},L)}(d_k)$, then by Corollary 1.6, we can conclude that $\operatorname{reg}(I_{\mathbb{X}},L) > d_k$. So, write $H_{R/(I_{\mathbb{X}},L)}(d_k) = A + B$, where

$$A := \sum_{t_1+\cdots+t_{k-1}=d_k} \mathscr{H}_{R/(I_{\mathbb{X}},L)}(t_1,\ldots,t_{k-1},0) \text{ and } B := \sum_{t_1+\cdots+t_k=d_k, t_k>0} \mathscr{H}_{R/(I_{\mathbb{X}},L)}(t_1,t_2,\ldots,t_k).$$

From the short exact sequence

$$0 \to (R/I_{\mathbb{X}})(-e_k) \xrightarrow{\times \bar{L}} R/I_{\mathbb{X}} \to R/(I_{\mathbb{X}},L) \to 0.$$

of $\mathbb{N}^k$-graded rings, we have

$$\mathscr{H}_{R/(I_{\mathbb{X}},L)}(t_1,\ldots,t_k) = \mathscr{H}_{R/I_{\mathbb{X}}}(t_1,\ldots,t_k) - \mathscr{H}_{R/I_{\mathbb{X}}}(t_1,\ldots,t_{k-1},t_k-1),$$

where $\mathscr{H}_{R/I_{\mathbb{X}}}(t_1,\ldots,t_{k-1},t_k-1) = 0$ if $t_k = 0$. Thus,

$$A = \sum_{t_1+\cdots+t_{k-1}=d_k} \mathscr{H}_{R/I_{\mathbb{X}}}(t_1,\ldots,t_{k-1},0).$$

Since $t_1 + \cdots + t_{k-1} = d_k$, by Corollary 2.2 we have $\mathscr{H}_{R/I_{\mathbb{X}}}(t_1,\ldots,t_{k-1},0) = s$. Hence,

$$A = \sum_{t_1+\cdots+t_{k-1}=d_k} s = s\binom{d_k+k-2}{k-2} = HP_{R/(I_{\mathbb{X}},L)}(d_k).$$

On the other hand, because $d_k = \min\left\{ d \,\middle|\, \binom{d+n_k}{d} \geqslant s \right\}$,

$$B \geqslant \mathscr{H}_{R/(I_{\mathbb{X}},L)}(0,\ldots,0,d_k) = \mathscr{H}_{\mathbb{X}}(0,\ldots,0,d_k) - \mathscr{H}_{\mathbb{X}}(0,\ldots,0,d_k-1)$$
$$= s - \binom{d_k-1+n_k}{d_k-1} > 0.$$

Thus, $H_{R/(I_{\mathbb{X}},L)}(d_k) = HP_{R/(I_{\mathbb{X}},L)}(d_k) + B > HP_{R/(I_{\mathbb{X}},L)}(d_k)$, as desired.

We now show that $\operatorname{reg}(I_{\mathbb{X}}) \leqslant d_k+1$ by demonstrating that $I_{\mathbb{X}}$ is (d_k+1)-regular. By Proposition 2.3(i), as a $\mathbb{N}^1$-graded ideal $I_{\mathbb{X}}$ is generated by elements of degree $\leqslant d_k+1$. For each $i \in \{1,\ldots,k\}$, by Lemma 1.9 there exists a non-zero divisor $\bar{L}_i \in R/I_{\mathbb{X}}$ with $\deg L_i = e_i$. After a change of variables in the $x_{1,j}$'s, a change of variables in the $x_{2,j}$'s, etc., we can assume that $L_i = x_{i,0}$ for $i = 1,\ldots,k$.

By the Bayer–Stillman criterion (Theorem 1.3), to show that $I_{\mathbb{X}}$ is (d_k+1)-regular, it is enough to prove:

(a) $((I_{\mathbb{X}},x_{1,0},\ldots,x_{j-1,0}) : x_{j,0})_{d_k+1} = (I_{\mathbb{X}},x_{1,0},\ldots,x_{j-1,0})_{d_k+1}$ for $j = 1,\ldots,k$,
(b) $(I_{\mathbb{X}},x_{1,0},\ldots,x_{k,0})_{d_k+1} = R_{d_k+1}$.

Proof. (a) We need to only show the non-trivial inclusion $[(I_{\mathbb{X}},x_{1,0},\ldots,x_{j-1,0}):x_{j,0}]_{d_k+1} \subseteq (I_{\mathbb{X}},x_{1,0},\ldots,x_{j-1,0})_{d_k+1}$ for each j. If $j = 1$, then the statement holds because $x_{1,0}$ is a non-zero divisor.

So, suppose $j > 1$. Set $J := [(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{j-1,0}) : x_{j,0}]$. Because J is also $\mathbb{N}^k$-homogeneous, if $F \in J_{d_k+1}$, then we can assume that $\deg F = \underline{t} = (t_1, \ldots, t_k)$ with $t_1 + \cdots + t_k = d_k + 1$. There are now two cases to consider.

In the first case, one of $t_1, \ldots, t_{j-1} > 0$. Suppose $t_l > 0$ with $1 \leqslant l \leqslant (j-1)$. Then by Proposition 2.3(iii) we have $F \in R_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{l,0})_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{j-1,0})_{\underline{t}}$. Since $(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{j-1,0})_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{j-1,0})_{d_k+1}$ (as vector spaces), we are finished.

In the second case, $t_1 = t_2 = \cdots = t_{j-1} = 0$. Then $F x_{j,0} \in (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{j-1,0})_{(0,\ldots,0,t_j+1,\ldots,t_k)}$. But since

$$(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{j-1,0})_{(0,\ldots,0,t_j+1,\ldots,t_k)} = (I_{\mathbb{X}})_{(0,\ldots,0,t_j+1,\ldots,t_k)},$$

we have $F x_{j,0} \in (I_{\mathbb{X}})_{(0,\ldots,0,t_j+1,\ldots,t_k)}$. But because $x_{j,0}$ is a non-zero divisor of $R/I_{\mathbb{X}}$,

$$F \in (I_{\mathbb{X}})_{(0,\ldots,0,t_j,\ldots,t_k)} \subseteq (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{j-1,0})_{(0,\ldots,0,t_j,\ldots,t_k)} \subseteq (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{j-1,0})_{d_k+1}.$$

(b) Since $R_{d_k+1} = \oplus_{t_1+\cdots+t_k=d_k+1} R_{t_1,\ldots,t_k}$ and because $(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})$ is also $\mathbb{N}^k$-homogeneous, it is enough to show that $R_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})_{\underline{t}}$ for all $\underline{t} = (t_1, \ldots, t_k) \in \mathbb{N}^k$ with $t_1 + \cdots + t_k = d_k + 1$. But for any $\underline{t} \in \mathbb{N}^k$ with $t_1 + \cdots + t_k = d_k + 1$, there exists at least one $t_l > 0$. Thus, by Proposition 2.3(iii) we have $R_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{l,0})_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})_{\underline{t}}$, thus completing the proof of (b).

Since we have just shown $d_k < \mathrm{reg}(I_{\mathbb{X}}) \leqslant d_k + 1$, the desired conclusion now follows. □

Remark 2.5. If $\mathbb{X}$ is a set of s points in generic position in $\mathbb{P}^n$, we recover the well known result that $\mathrm{reg}(I_{\mathbb{X}}) = d + 1$ where $d = \min \left\{ l \,\middle|\, \binom{l+n}{n} \geqslant s \right\}$.

3. Bounding the regularity of fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$

Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $m_1 \geqslant \cdots \geqslant m_s \in \mathbb{N}^+$. Suppose $\wp_i$, is the defining ideal of P_i for $i = 1, \ldots, s$ Let $I = I_Z = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$. In this section, we give an upper bound for $\mathrm{reg}(I)$ when $\mathbb{X}$ is in generic position. If we consider R/I as a $\mathbb{N}^1$-graded ring, then by Lemma 1.5

$$\mathrm{reg}(I) = \mathrm{reg}\,(R/I) + 1 \leqslant \mathrm{ri}\,(R/I) + \dim R/I = \mathrm{ri}\,(R/I) + k.$$

To bound $\mathrm{reg}(I)$, it is therefore enough to bound $\mathrm{ri}(R/I)$. For convenience, we assume that $n_1 \geqslant \cdots \geqslant n_k$. In the sequel, we shall also abuse notation by writing L for the form $L \in \mathbf{k}[x_{j,0}, \ldots, x_{j,n_j}]$, the hyperplane L in $\mathbb{P}^{n_j}$ defined by L, and the subvariety of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ defined by L.

Lemma 3.1. *If $\wp$ is the defining ideal of point $P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then*

$$\mathrm{ri}(R/\wp^a) = a - k \ \ \textit{for all}\ \ a \geqslant 1.$$

Proof. Since $\wp$ defines a complete intersection of height $\sum_{i=1}^k n_i$, Lemma 1.5 gives $\mathrm{ri}(R/\wp^a) = \mathrm{reg}(R/\wp^a) - k + 1$. The conclusion follows since $\mathrm{reg}(\wp^a) = a\,\mathrm{reg}(\wp) = a$ by Conca and Herzog [6, Theorem 3.1]. □

Lemma 3.2. *Suppose $P_1,\dots,P_r,P$ are points in generic position in $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$, and let $\wp_i$ be the defining ideal of P_i and let $\wp$ be the defining ideal of P. Let $m_1,\dots,m_r$ and a be positive integers, $J=\wp_1^{m_1}\cap\cdots\cap\wp_r^{m_r}$, and $I=J\cap\wp^a$. Then*

$$\mathrm{ri}(R/I)\leqslant\max\{a-k,\mathrm{ri}(R/J),\mathrm{ri}(R/(J+\wp^a))\}.$$

Furthermore, $R/(J+\wp^a)$ is artinian.

Proof. The short exact sequence of $\mathbb{N}^1$-graded rings

$$0\to R/I\to R/J\oplus R/\wp^a\to R/(J+\wp^a)\to 0$$

yields $H_{R/I}(t)=H_{R/J}(t)+H_{R/\wp^a}(t)-H_{R/(J+\wp^a)}(t)$. Combining this with Lemma 3.1 gives

$$\mathrm{ri}(R/I)\leqslant\max\{a-k,\mathrm{ri}(R/J),\mathrm{ri}(R/(J+\wp^a))\}.$$

To show that $R/(J+\wp^a)$ is artinian, we need to show that there exists b such that for all $\underline{t}=(t_1,\dots,t_k)\in\mathbb{N}^k$, if there is $t_j\geqslant b$, then $(R/(J+\wp^a))_{\underline{t}}=0$. So, it suffices to show that there exists such a b so that for all $\underline{t}=(t_1,\dots,t_k)$ with $t_j\geqslant b$ for some j, then all monomials of R of degree $\underline{t}$ are in $(J+\wp^a)$. Suppose M is a monomial in R of degree $\underline{t}$. Then $M=N_1N_2\cdots N_k$ where N_l are monomials in $\{x_{l,0},\dots,x_{l,n_l}\}$ and of degree t_l. It is enough to show $N_j\in(J+\wp^a)$.

Let $Q_1,\dots,Q_r,Q$ be the projections of $P_1,\dots,P_r,P$ in $\mathbb{P}^{n_j}$. Since the points are in generic position, the projections are distinct. Let $\mathcal{Q}_1,\dots,\mathcal{Q}_r$ and $\mathcal{Q}$ be the defining ideals of $Q_1,\dots,Q_r,Q$ in $A=\mathbf{k}[x_{j,0},\dots,x_{j,n_j}]$. Then it is easy to see that $A/(\mathcal{Q}_1^{m_1}\cap\cdots\cap\mathcal{Q}_r^{m_r}+\mathcal{Q}^a)$ is artinian. As well, $\mathcal{Q}_1^{m_1}\cap\cdots\cap\mathcal{Q}_r^{m_r}\subseteq J$ and $\mathcal{Q}^a\subseteq\wp^a$, and thus $\mathcal{Q}_1^{m_1}\cap\cdots\cap\mathcal{Q}_r^{m_r}+\mathcal{Q}^a\subseteq(J+\wp^a)$, and this is what needs to be shown. □

From Lemma 3.2, to estimate $\mathrm{ri}(R/I)$ we need to estimate $(\mathrm{ri}(R/(J+\wp^a))$, or equivalently, the least integer t such that $(R/I(J+\wp^a))_t=0$, when this ring is considered as $\mathbb{N}$-graded.

Lemma 3.3. *With the same hypotheses as in Lemma* 3.2, *and considering the $\mathbb{N}^1$-gradation, we have*

(i) $H_{R/(J+\wp^a)}(t)=\sum_{i=0}^{a-1}\dim_{\mathbf{k}}[(J+\wp^i)/(J+\wp^{i+1})]_t$ *for all* $t\geqslant 0$.
(ii) *If* $P=[1:0:\cdots:0]\times\cdots\times[1:0\cdots:0]$ *then* $[(J+\wp^i)/(J+\wp^{i+1})]_t=0$ *if and only if either* $i>t$, *or* $i<t$ *and* $GM\in(J+\wp^{i+1})$ *for every monomial* M *of degree* i *in* $\{x_{1,1},\dots,x_{1,n_1},\dots,x_{k,1},\dots,x_{k,n_km}\}$, *and every monomial* G *of degree* $t-i$ *in* $\{x_{1,0},x_{2,0},\dots,x_{k,0}\}$.

Proof. The first assertion follows from the short exact sequences:

$$0\to(J+\wp^i)/(J+\wp^{i+1})\to R/(J+\wp^{i+1})\to R/(J+\wp^i)\to 0,$$

where $i=0,\dots,a-1$.

To prove (ii), if $i > t$, then $(J + \wp^i)_t = (J + \wp^{i+1})_t = J_t$. So suppose $i < t$. We see that $\wp = (x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k})$. Thus $((J+\wp^i)/(J+\wp^{i+1}))_t = 0$ if and only if $(\wp^i)_t \subseteq (J + \wp^{i+1})_t$ if and only if $FM \in (J + \wp^{i+1})$ for every monomial M of degree i in $\{x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}\}$ and every form $F \in R_{t-i}$. But because $(J + \wp^{i+1})$ is $\mathbb{N}^k$-homogenous, we can take F to be $\mathbb{N}^k$-homogeneous, and so $F = G + H$ where G is a monomial of degree $t - i$ in $x_{1,0}, \ldots, x_{k,0}$ and $H \in \wp$. Since $HM \in \wp^{i+1}$, we have $((J + \wp^i)/(J + \wp^{i+1}))_t = 0$ if and only if $GM \in (J + \wp^{i+1})_t$, as desired. □

Lemma 3.4. *Let $P_1, \ldots, P_r$, P be points in generic position in $\mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_k}$ with $n_1 \geqslant \cdots \geqslant n_k$, and let $m_1 \geqslant \cdots \geqslant m_r$ be positive integers. Set $J = \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$. Suppose $\underline{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ is such that $n_k(\sum_{i=1}^k a_i) \geqslant \sum_{i=1}^r m_i$ and $\sum_{i=1}^k a_i \geqslant m_1$. Then we can find a_j hyperplanes $L_{j,1}, \ldots, L_{j,a_j}$ in $\mathbb{P}^{n_j}$, that is, $L_{j,l} \in \mathbf{k}[x_{j,0}, \ldots, x_{j,n_j}]$ for all $l = 1, \ldots, a_j$, such that*

$$L = \prod_{j=1}^{k} \left(\prod_{l=1}^{a_j} L_{j,l} \right) \in J$$

and L avoids P.

Proof. If $r \leqslant n_j$ for all j, then for each j we can find a linear form $L_j \in \mathbf{k}[x_{j,0}, \ldots, x_{j,n_j}]$ that passes through $P_1, \ldots, P_r$ and avoids P. If we take $L_{j,t} = L_j$ for all j, we have

$$L = \prod_{j=1}^{k} L_j^{a_j} \in \wp_1^{|\underline{a}|} \cap \cdots \cap \wp_r^{|\underline{a}|} \subseteq \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r} = J,$$

where $|\underline{a}| = \sum_{i=1}^k a_i$, since $|\underline{a}| \geqslant m_1 \geqslant \cdots \geqslant m_r$. Moreover, L avoids P.

Suppose now that $n_k \leqslant n_{k-1} \leqslant \cdots \leqslant n_{l+1} < r \leqslant n_l \leqslant \ldots \leqslant n_1$. We shall use induction on $\sum_{i=1}^r m_i$. Note that if $\sum_{i=1}^r m_i \leqslant n_k$ then the conclusion follows since in this case $r \leqslant n_k \leqslant n_j$ for all j. If $a_k = a_{k-1} = \cdots = a_{l+1} = 0$, then the conclusion follows as in the case $r \leqslant n_j$ for all j. Suppose there is $p \in \{l+1, \ldots, k\}$ such that $a_p \neq 0$. Choose a hyperplane L_1 in $\mathbb{P}^{n_p}$ ($L_1 \in \mathbf{k}[x_{p,0}, \ldots, x_{p,n_p}]$) that avoids P and passes through $P_1, \ldots, P_{n_p}$. Since $n_k(\sum_{i=1}^k a_i) \geqslant \sum_{i=1}^r m_i$, we have

$$n_k \left(\sum_{i=1}^{k} a_i \right) - n_k \geqslant \sum_{i=1}^{r} m_i - n_k \geqslant \sum_{i=1}^{r} m_i - n_p$$
$$= (m_1 - 1) + \cdots + (m_{n_p} - 1) + m_{n_p+1} + \cdots + m_r.$$

If we set $(b_1, \ldots, b_{p-1}, b_p, b_{p+1}, \ldots, b_k) = (a_1, \ldots, a_{p-1}, a_p - 1, a_{p+1}, \ldots + a_k)$, then we have

$$n_k \left(\sum_{i=1}^{k} b_i \right) = n_k \left(\sum_{i=1}^{k} a_i \right) - n_k \geqslant (m_1 - 1) + \ldots + (m_{n_p} - 1) + m_{n_p+1} + \cdots + m_r.$$

By induction there exists $L_{j,1},\ldots,L_{j,b_j}$ in $\mathbb{P}^{n_j}$ for all j that avoids P such that

$$L=\prod_{j=1}^{k}\left(\prod_{l=1}^{b_j}L_{j,l}\right)\in\wp_1^{m_1-1}\cap\cdots\cap\wp_{n_p}^{m_{n_p}-1}\cap\wp_{n_p+1}^{m_{n_p+1}}\cap\cdots\cap\wp_r^{m_r}.$$

If we take $L\cdot L_1$ we have the conclusion since $L_1\in\wp_1\cap\cdots\cap\wp_{n_p}$ (the a_p hyperplanes in $\mathbb{P}^{n_p}$ are $L_{p,1},\ldots,L_{p,b_p}$ and L_1). □

Proposition 3.5. *Let $P_1,\ldots,P_r$, P be points in generic position in $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ with $n_1\geqslant\cdots\geqslant n_k$. Suppose $m_1\geqslant\cdots\geqslant m_r\geqslant a$ are positive integers. Set $J=\wp_1^{m_1}\cap\cdots\cap\wp_r^{m_r}$. Let t be the least integer such that $n_kt\geqslant\sum_{i=1}^{r}m_i+a-1$. Then*

$$\mathrm{ri}(R/(J+\wp^a))\leqslant\max\{m_1+a-1,t\}.$$

Proof. Without loss of generality take $P=[1:0:\cdots:0]\times\cdots\times[1:0:\cdots:0]$. Then $\wp=(x_{1,1},\ldots,x_{1,n_1},\ldots,x_{k,1},\ldots,x_{k,n_k})$. If $r\leqslant n_j$ for all j, then we can find a hyperplane L_j in $\mathbb{P}^{n_j}$, i.e., $L_j\in\mathbf{k}[x_{j,0},\ldots,x_{j,n_j}]$, containing $P_1,\ldots,P_r$ and avoids P for each j. Then $L_j\in\wp_1\cap\cdots\cap\wp_r$ for all j.

Suppose $G=x_{1,0}^{a_1}\cdots x_{k,0}^{a_k}$ is a monomial of degree m_1 in $\{x_{1,0},\ldots,x_{k,0}\}$. Then $L:=L_1^{a_1}\cdots L_k^{a_k}\in\wp_1^{m_1}\cap\cdots\cap\wp_r^{m_1}\subseteq\wp_1^{m_1}\cap\cdots\cap\wp_r^{m_r}=J$. We can rewrite $L_j=x_{j,0}+H_j$ where $H_j\in(x_{j,1},\ldots,x_{j,n_j})\subseteq\wp$. Then $L\in J$ implies $G\in J+\wp$. Thus, for any monomial M of degree i in $\wp^i$ for some $0\leqslant i\leqslant a-1$, $GM\in J+\wp^{i+1}$. Since $a-1\geqslant i$, this implies that for any monomial $\bar{G}$ of degree $m_1+a-1-i$ in $\{x_{1,0},\ldots,x_{k,0}\}$, and any monomial M of degree i in $\wp^i$, $\bar{G}M\in(J+\wp^{i+1})$ because $\bar{G}$ is divisible by a monomial of degree m_1. By Lemma 3.3, this implies that $\mathrm{ri}(R/(J+\wp^a))\leqslant m_1+a-1$.

Suppose now that $r>n_k$. Since $n_1\geqslant\cdots\geqslant n_k$, by a change of coordinates we may assume that

$$\begin{aligned}P_1&=[0:1:0:\cdots:0]\times[0:1:0:\cdots:0]\times\cdots\times[0:1:0:\cdots:0]\\&\ \vdots\\P_{n_k}&=[\underbrace{0:\cdots:0}_{n_k}:1:0:\cdots:0]\times[\underbrace{0:\cdots:0}_{n_k}:1:0:\cdots:0]\times\cdots\\&\quad\times[\underbrace{0:\cdots:0:1}_{n_k}].\end{aligned}$$

So for $0\leqslant j\leqslant n_k$, $\wp_j=(\{x_{l,q}|l=1,\ldots,k,\ q\neq j\})$.

Let $h=\max\{m_1+a-1,t\}$ and $0\leqslant i\leqslant a-1$. Suppose now that $G=x_{1,0}^{a_1}\cdots x_{k,0}^{a_k}$ is a monomial of degree $h-i$ in $\{x_{1,0},\ldots,x_{k,0}\}$, and $M=\prod_{l=1}^{k}\prod_{q\neq0}x_{l,q}^{c_{l,q}}$ is a monomial of degree i in $\wp^i$. Because of Lemma 3.3 we need to show that $GM\in(J+\wp^{i+1})$.

It can be seen that

$$M\in\wp_1^{i-\sum_{l=1}^{k}c_{l,1}}\cap\wp_2^{i-\sum_{l=1}^{k}c_{l,2}}\cap\cdots\cap\wp_{n_k}^{i-\sum_{l=1}^{k}c_{l,n_k}}.$$

We also have, since $i \leqslant a-1$,

$$\sum_{i=1}^{k} a_i = h - i \geqslant m_1 \geqslant \max\left\{ m_1 - i + \sum_{i=1}^{k} c_{l,1}, \ldots, m_{n_k} - i + \sum_{i=1}^{k} c_{l,n_k} \right\}$$

and

$$\begin{aligned} n_k\left(\sum_{j=1}^{k} a_j\right) &= n_k(h-i) = n_k h - i n_k \\ &\geqslant \sum_{j=1}^{r} m_j + a - 1 - i n_k \geqslant \sum_{j=1}^{r} m_j + i - i n_k \\ &\geqslant \sum_{j=1}^{r} m_j + \sum_{l=1}^{k}\sum_{q=1}^{n_k} c_{l,q} - i n_k \\ &= \left(m_1 - i + \sum_{l=1}^{k} c_{l,1}\right) + \cdots + \left(m_{n_k} - i + \sum_{l=1}^{k} c_{l,n_k}\right) + m_{n_k+1} + \cdots + m_r. \end{aligned}$$

Using Lemma 3.4, there exists $L_{j,1},\ldots,L_{j,a_j} \in \mathbf{k}[x_{j,0},\ldots,x_{j,n_j}]$ for each $1 \leqslant j \leqslant k$ such that

$$L = \prod_{j=1}^{k}\left(\prod_{q=1}^{a_j} L_{j,q}\right) \in \wp_1^{m_1 - i + \sum c_{l,1}} \cap \cdots \cap \wp_{n_k}^{m_{n_k} - i + \sum c_{l,n_k}} \cap \wp_{n_k+1}^{m_{n_k+1}} \cap \cdots \cap \wp_r^{m_r}$$

and L avoids P. This implies that $LM \in J$.

Since $L_{j,q}$ avoids P. we can write $L_{j,q} = x_{j,0} + H_{j,q}$, where $H_{j,q} \in (x_{j,1},\ldots,x_{j,n_j}) \subseteq \wp$. Then $L = x_{1,0}^{a_1} \cdots x_{k,0}^{a_k} + N$, where $N \in \wp$. Thus, since $LM \in J$, then $GM \in (J + \wp^{i+1})$ which is what we need to prove. □

Theorem 3.6. *Suppose $P_1,\ldots,P_s$ are points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ ($s \geqslant 2$ and $n_1 \geqslant \cdots \geqslant n_k$), and $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$ are positive integers. Set $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$. Then*

$$\mathrm{ri}(R/I) \leqslant \max\left\{ m_1 + m_2 - 1, \left\lceil \frac{\sum_{i=1}^{s} m_i - 1}{n_k} \right\rceil \right\},$$

where $\lceil q \rceil$ denotes the smallest integer t such that $t \geqslant q$.

Proof. Note that $n_1 \geqslant \cdots \geqslant n_k$, so

$$\left\lceil \frac{\sum_{i=1}^{s} m_i - 1}{n_k} \right\rceil = \max\left\{ \left\lceil \frac{\sum_{i=1}^{s} m_i - 1}{n_j} \right\rceil \right\}_{j=1}^{k}$$

Also, $\min\{t \mid n_k t \geqslant q\} = \lceil q/n_k \rceil$. So, if we take $q = \sum_{i=1}^{r} m_i + m_{r+1} - 1$, and use Proposition 3.5 and Lemma 3.2, together with induction successively, we will have the conclusion. □

We obtain an immediate corollary which gives a bound on the regularity of the defining ideal of a scheme of fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Corollary 3.7. *With the hypotheses as in Theorem* 3.6 *we have*

$$\operatorname{reg}(I) \leqslant \max\left\{ m_1 + m_2 - 1, \left\lceil \frac{\sum_{i=1}^{s} m_i - 1}{n_k} \right\rceil \right\} + k.$$

Remark 3.8. When $k = 1$ we recover the result of [5] which was proved to be sharp. One may expect the bound in Corollary 3.7 to be sharp also.

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