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Embedding Rings in Completed Graded Rings
2. Algebras over a Field

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It is shown that any $\mathbb{Z}$-filtered associative algebra R over a field k can be embedded in the completion $H^{\oplus}$ of a $\mathbb{Z}$-graded k-algebra $\bigoplus H_i$, so that the filtration on R is induced by the grading of H. A normal form for the universal such H is found. A consequence is that if R is a k-algebra such that $R^n = \{0\}$, then R can be embedded in a graded k-algebra $\bigoplus H_i$ of which all homogeneous components except $H_1,..., H_{n-1}$ are zero, and so (by the results of part 1 of this series) can be embedded in $n \times n$ strictly upper triangular matrices over a commutative k-algebra. This last result has also been proved by I. V. L'vov.

Throughout this note, k will be an associative commutative ring with 1. In the sections where we prove the main results, we will essentially assume k is a field. Results obtainable without this assumption will be studied in [3, 4].

All *k-algebras* will be assumed associative, but not unital unless this is stated.

1. Filtered Algebras—Definitions

By a *$\mathbb{Z}$-filtered k-algebra* (or simply a *filtered algebra*) we shall mean a k-algebra R expressed as a union $R = \bigcup_{i\in\mathbb{Z}} R_{(i)}$, such that

$$\cdots \supseteq R_{(-2)} \supseteq R_{(-1)} \supseteq R_{(0)} \supseteq R_{(1)} \supseteq R_{(2)} \supseteq \cdots; \tag{1.1}$$

$$\text{each } R_{(i)} \text{ is a } k\text{-submodule of } R; \tag{1.2}$$

$$R_{(i)} R_{(j)} \subseteq R_{(i+j)} \qquad (i, j \in \mathbb{Z}); \tag{1.3}$$

$$\bigcap_{i\in\mathbb{Z}} R_{(i)} = \{0\}. \tag{1.4}$$

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Such a filtration on R is equivalent to a function $v: R \to \mathbb{Z} \cup \{+\infty\}$ satisfying

$$v(x+y) \geqslant \min(v(x), v(y)) \qquad (x, y \in R), \tag{1.5}$$

$$v(cx) \geqslant v(x) \qquad (c \in k, x \in R), \tag{1.6}$$

$$v(xy) \geqslant v(x) + v(y) \qquad (x, y \in R), \tag{1.7}$$

$$v(x) = +\infty \Leftrightarrow x = 0 \qquad (x \in R). \tag{1.8}$$

The relation between the system of submodules $R_{(i)}$ and the *filtration function* v is given by

$$v(x) = \sup\{i \mid x \in R_{(i)}\}, \qquad \text{equivalently} \qquad R_{(i)} = \{x \mid v(x) \geqslant i\}. \tag{1.9}$$

For $x \in R$, the value of the filtration function v on x will be called the *order* of x.

A *homomorphism of filtered algebras* $f: R \to S$ will mean a map f such that

$$f \text{ is a } k\text{-algebra homomorphism}, \tag{1.10}$$

and

$$v(f(x)) \geqslant v(x) \ \ (x \in R); \qquad \text{equivalently} \qquad f(R_{(i)}) \subseteq S_{(i)} \ \ (i \in \mathbb{Z}). \tag{1.11}$$

Such a homomorphism will be called an *embedding of filtered algebras* (or *v-embedding*) if in place of (1.11) it satisfies the stronger condition

$$v(f(x)) = v(x) \ \ (x \in R); \qquad \text{equivalently} \qquad R_{(i)} = f^{-1}(S_{(i)}) \ \ (i \in \mathbb{Z}). \tag{1.12}$$

By (1.8) or (1.4) this implies that f is one-to-one, but it is stronger; it says that R is isomorphic to a subalgebra of S *with filtration induced by that of* S. Thus (as in topology) the embeddings are a more restricted class than the one-to-one morphisms.

When we speak of *unital* algebras, we mean not merely that these are to have elements 1 satisfying $1x = x = x1$, but that we consider among them only homomorphisms taking 1 to 1. ("Nonunital" algebras may or may not have such elements; if they do we don't require homomorphisms to respect them.) *Filtered unital* k-algebras are required to satisfy

$$1 \in R_{(0)}, \qquad \text{equivalently} \qquad v(1) \geqslant 0. \tag{1.13}$$

If $R \neq \{0\}$, (1.13) is easily shown to imply

$$v(1) = 0. \tag{1.14}$$

The *completion* $\hat{R}$ of a filtered algebra R is defined in the familiar manner as the algebra of limits of Cauchy sequences. As a k-module it may be described by

$$\hat{R} = \text{inv. lim.}_i \, R/R_{(i)}. \tag{1.15}$$

2. Examples of Filtrations

(2.1) Let R be a k-algebra and $I \subseteq R$ a 2-sided ideal such that $\bigcap_i I^i = \{0\}$. (Note: even if R is nonunital we require ideals to be k-submodules.) Then we get the "I-adic filtration" on R by defining $R_{(i)} = I^i$ for $i > 0$, $R_{(i)} = R$ for $i \leqslant 0$. Under this filtration, all elements have nonnegative order. (Conversely, in any filtration under which all elements have nonnegative order, equivalently, $R = R_{(0)}$, each $R_{(i)}$ is an ideal of R, though not necessarily of the form I^i. So in such cases (1.15) is a limit *of algebras*.) Taking $I = \{0\}$, the I-adic filtration gives the *trivial* (=least nonnegative) filtration.

(2.2) If S is any k-algebra, the polynomial ring in a central indeterminate, $R = S[t]$, may be filtered by putting $v(f) = -\deg f$. Here all nonzero elements have *nonpositive* order. (Authors having such examples in mind often reverse the signs on their filtrations, relative to our definition. Cf. [2, p. 69] on the one hand, and [2, p. 96] on the other.)

(2.3) A filtration function v such that equality holds in (1.7) is called a $\mathbb{Z}$-valued *valuation*. Examples are the (p)-adic valuation on a commutative UFD R where $p \in R$ is a prime, which is nonnegative-valued, the filtration of (2.2) when S has no zero-divisors, which is nonpositive-valued, and the valuations induced by each of these on their fields of fractions, which assume both positive and negative values.

(2.4) One can get pathological examples by composing a given filtration function v with any function α from the range of v (excluding $+\infty$) into $\mathbb{Z}$ which satisfies $i \leqslant j \Rightarrow \alpha(i) \leqslant \alpha(j)$, $\alpha(i+j) \geqslant \alpha(i) + \alpha(j)$, and, in the unital case, also $\alpha(0) = 0$. Examples are $\alpha(n) = n^2$ for v nonnegative-valued, $\alpha(n) = \max(n, -10)$ for v nonpositive-valued, $\alpha(n) =$ greatest integer in $n/2$ for arbitrary v, and in the nonunital case $\alpha(n) = n + 1$ for arbitrary v.

(2.5) The infimum of two filtration functions on an algebra R (e.g., valuations) is again a filtration function.

3. Graded Algebras—Definitions

In dealing with graded objects, we prefer a terminology which distinguishes between homogeneous and not necessarily homogeneous elements, giving precedence to the former. Therefore, we define a *$\mathbb{Z}$-graded k-algebra* (or simply *graded algebra*) H to mean a collection of k-modules $(H_i)_{i\in\mathbb{Z}}$ given with k-bilinear maps $H_i \times H_j \to H_{i+j}$, written as multiplication, and satisfying the associative law

$$(xy)z = x(yz) \qquad (x \in H_h, y \in H_i, z \in H_j). \tag{3.1}$$

If H is a graded k-algebra, the k-module $\bigoplus_i H_i$ becomes an ordinary k-algebra in an obvious manner. We shall denote this algebra $H^{\oplus}$. It has a natural $\mathbb{Z}$-filtration given by

$$H^{\oplus}_{(i)} = \bigoplus_{j \geqslant i} H_j. \tag{3.2}$$

Its completion with respect to this filtration will be denoted $H^{\hat{\oplus}}$; we see that

$$H^{\hat{\oplus}}_{(i)} = \prod_{j \geqslant i} H_j. \tag{3.3}$$

We will identify the homogeneous components H_i with their images in $H^{\oplus}$ and $H^{\hat{\oplus}}$; thus a typical element of $H^{\hat{\oplus}}$ can be written as the sum of a Cauchy series

$$x = \sum_{v(x) \leqslant i < \infty} x_i \qquad (x_i \in H_i). \tag{3.4}$$

A homomorphism $f: G \to H$ of graded algebras means a family of k-module homomorphisms $f_i: G_i \to H_i$, respecting multiplication. *Unital* graded algebras will be understood to have identity elements $1 \in H_0$, and their homomorphisms are required to respect 1.

("Conventional" notation was used for graded algebras in [1] because they made such a brief appearance that the point wasn't worth dealing with. But the present notation, as well as the definitions concerning filtrations in Section 1, will be used consistently in the remaining papers of this series.)

4. Modules

Let a *filtered k-module* mean a k-module M expressed as the union of a chain of submodules $\cdots \supseteq M_{(-1)} \supseteq M_{(0)} \supseteq M_{(1)} \supseteq \cdots$ such that $\bigcap M_{(i)} = \{0\}$. The *order function* v on such a module is defined as on a filtered algebra (1.9).

For such an M, let us define for each $i \in \mathbb{Z}$

$$\mathrm{End}_{\mathrm{filt}}(M)_{(i)} = \{r \in \mathrm{End}_k(M) \mid rM_{(j)} \subseteq M_{(j+i)} \text{ for all } j \in \mathbb{Z}\}. \tag{4.1}$$

It is easy to see that the union of this chain,

$$\mathrm{End}_{\mathrm{filt}}(M) = \bigcup_i \mathrm{End}_{\mathrm{filt}}(M)_{(i)}, \tag{4.2}$$

is a filtered k-algebra. For any filtered k-algebra R, a *filtered R-module* will mean a filtered k-module M given with a homomorphism of filtered rings, $R \to \mathrm{End}_{\mathrm{filt}}(M)$; i.e., with a structure of R-module such that (1.7) holds with $x \in R$, $y \in M$. (This agrees with the above definition of "filtered k-module" if we give k the trivial filtration, see (2.1).) The filtered R-module M will be called *filtration faithful* if the map $R \to \mathrm{End}_{\mathrm{filt}}(M)$ is an embedding of filtered k-algebras; in other words, if for each $r \in R - \{0\}$ there exists $x \in M - \{0\}$ such that $v(rx) = v(r) + v(x)$. Every filtered algebra R has filtration faithful filtered modules; for example, R itself if it is unital, or if not, $k \oplus R$, made first into a unital over-algebra of R, and thence a filtered R-module.

Completions of filtered k-modules are defined as for algebras. It is not hard to see that for any filtered k-module M, $\mathrm{End}_{\mathrm{filt}}(M)$ embeds naturally in $\mathrm{End}_{\mathrm{filt}}(\hat{M})$, and that a filtered R-module structure on a filtered k-module M therefore extends to such a structure on $\hat{M}$.

A *graded* k-module N will mean a system of k-modules $(N_i)_{i \in \mathbb{Z}}$. If we define

$$\mathrm{End}_{\mathrm{gr}}(N)_i = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_k(N_j, N_{j+i}) \qquad (i \in \mathbb{Z}), \tag{4.3}$$

then these form a graded k-algebra $\mathrm{End}_{\mathrm{gr}}(N)$, and we can define a *graded module over a graded algebra* in obvious analogy to the definition for filtered objects.

As was the case for algebras, we have functors $(\ )^{\oplus}$ and $(\ )^{\hat{\oplus}}$ from graded modules to filtered modules. It is not hard to verify that for a graded k-module N,

$$\mathrm{End}_{\mathrm{filt}}(N^{\hat{\oplus}}) \cong (\mathrm{End}_{\mathrm{gr}}(N))^{\hat{\oplus}}. \tag{4.4}$$

(On either side of (4.4), the submodule $R_{(i)}$ can be pictured as consisting of infinite matrices in which the (h, j) position is occupied by elements of $\mathrm{Hom}_k(N_h, N_j)$ which are zero unless $j - h \geqslant i$.)

5. The Embedding Theorem

Before giving the general proof of our embedding result, let us show the idea in a case that is particularly easy to visualize (the case for which the second author first noted it.) This is the problem of embedding a nilpotent algebra R

over a field k in upper triangular matrices over an associative algebra. For concreteness let us consider an R such that $R^3 = \{0\}$.

We filter R by powers of itself; in this case, $R \supseteq R^2 \supseteq \{0\}$. Since k is a field, the k-subspace R^2 has a complement in R, so we get a k-vector-space decomposition $R = N_1 \oplus N_2$ with $N_2 = R^2$. If we think of $N_1 \oplus N_2$ as a right R-module, it is not faithful, since R^2 annihilates it, but if we adjoin another summand $N_0 = k$, on which R acts by $1(n_1 + n_2) = n_1 + n_2$ $(n_i \in N_i)$, we get a faithful module $N^{\oplus} = N_0 + N_1 + N_2$. Now R acts on $N^{\oplus}$ in a "triangular" fashion, giving an embedding

$$R \subseteq \begin{pmatrix} 0 & \mathrm{Hom}(N_0, N_1) & \mathrm{Hom}(N_0, N_2) \\ 0 & 0 & \mathrm{Hom}(N_1, N_2) \\ 0 & 0 & 0 \end{pmatrix} \subseteq T_3(\mathrm{End}_k(N^{\oplus})), \tag{5.1}$$

as desired.

It was noted in [1] that embeddability in upper triangular matrices is equivalent to embeddability in certain sorts of algebras $H^{\oplus}$. It is this version of our result that is most convenient to generalize to arbitrary filtered algebras.

THEOREM 1. *If k is a field and R a (unital or nonunital) filtered k-algebra, then R can be embedded as a (unital or nonunital) filtered k-algebra in a completed graded k-algebra $H^{\hat{\oplus}}$.*

More generally, this is true if the hypothesis that k is a field is weakened to say that for each $i \in \mathbb{Z}$, $R_{(i+1)}$ is a k-module direct summand in $R_{(i)}$, or still more generally, if R has a filtration faithful filtered module M such that for each $i \in \mathbb{Z}$, $M_{(i+1)}$ is a k-module direct summand in $M_{(i)}$.

Proof. The hypotheses are successively weaker; let us assume the last one, and write

$$M_{(i)} = N_i \oplus M_{(i+1)} \qquad (i \in \mathbb{Z}). \tag{5.2}$$

We think of the N_i as forming a graded k-module N. Then $N^{\oplus}$ embeds as a dense filtered submodule of M, so

$$\hat{M} \cong N^{\hat{\oplus}}, \tag{5.3}$$

We now have embeddings of filtered k-algebras

$$R \to \mathrm{End}_{\mathrm{filt}}(M) \to \mathrm{End}_{\mathrm{filt}}(\hat{M}) \cong \mathrm{End}_{\mathrm{filt}}(N^{\hat{\oplus}}) \cong \mathrm{End}_{\mathrm{gr}}(N)^{\hat{\oplus}}, \tag{5.4}$$

the last having the desired form $H^{\hat{\oplus}}$. ∎

Let us record separately the consequence for nilpotent R.

COROLLARY 2. *If k is a field and R an associative k-algebra satisfying $R^n = \{0\}$, then R can be embedded in $H^{\oplus}$ for a graded k-algebra H such that all components H_i are zero except for $i = 1,\ldots, n-1$. Hence by* [1, *Theorem* 1], *R can be embedded in strictly upper triangular $n \times n$ matrices over an associative, and even an associative commutative k-algebra.* (*Cf. Note below.*)

Remarks on proof. The existence of an upper triangular embedding over an associative k-algebra can be proved directly, as sketched above for $R^3 = \{0\}$. Alternatively, the first sentence, which is equivalent to either version of the last sentence by [1], can be obtained from the above Theorem: Note that in that Theorem, if $R = R_{(1)}$, then by replacing H by a graded subalgebra if necessary, one can assume H_i nonzero only in positive degrees; and if further $R_{(n)} = \{0\}$ we can then assume by going to a quotient of H if necessary that H is zero in all degrees $\geqslant n$. Now if $R^n = \{0\}$ and we filter R by $R_{(i)} = R^i$, we get the asserted result. ∎

(5.5) Note: The last (and formally strongest) statement of the above Corollary has also been proved by I. V. L'vov [8]. (The statement in [8] adds the conclusion that the associative commutative algebra can be taken to satisfy $C^n = \{0\}$. But it is not hard to see that if any C gives the desired embedding, then $C' = tC[t]/t^nC[t]$ yields a similar embedding and satisfies $C'^n = \{0\}$.) L'vov's result is presented as a refinement of a result of A. Z. Anan'in, who gives in [9] sufficient conditions for an algebra over a field k to be embeddable in *non-strictly* upper triangular matrices over a commutative associative k-algebra, which in particular are satisfied by nilpotent algebras.

In the summer of 1978, Bergman and Anan'in learned of each other's then still unpublished results in this area, but not their proofs. (Bergman's results at that time were those of [1], and Section 6 of this paper. They were later announced in [11]. Anan'in's result had been announced in [10, p. 4].) Subsequent contributions by L'vov and Vovsi, respectively, were also independent.

6. THE UNIVERSAL CONSTRUCTION

Let k again be an arbitrary commutative ring, and R any filtered k-algebra. Then there is a *universal* example of a graded k-algebra H with a homomorphism of filtered algebras $f: R \to H^{\hat{\oplus}}$. Indeed, consider the graded k-algebra $U(R)$ presented by generators

$$a_i \in U(R)_i \qquad (a \in R, i \geqslant v(a)), \tag{6.1}$$

and the relations needed to make the map

$$a \mapsto u(a) = \sum_{i \geqslant v(a)} a_i \in U(R)^{\widehat{\oplus}} \tag{6.2}$$

a homomorphism. (Beware our notation! Observe which elements are in R, and which in $U(R)$.) To express these relations without complicated restrictions on ranges of indices, we make the convention that $a_i = 0$ for $i < v(a)$ (values for which no such generators are introduced in (6.1)). Then these relations can be written

$$(a+b)_i = a_i + b_i \qquad (a, b \in R, i \in \mathbb{Z}), \tag{6.3}$$

$$(ca)_i = c(a_i) \qquad (a \in R, c \in k, i \in \mathbb{Z}), \tag{6.4}$$

$$(ab)_i = \sum_{v(a) \leqslant j \leqslant i - v(b)} a_j b_{i-j} \qquad (a, b \in R, i \in \mathbb{Z}). \tag{6.5}$$

In the unital case, one also has a set of relations saying that $u(1) = 1$. For simplicity we postpone introducing these till the end of this discussion.

(Brief digression: If we did not refer to *filtered* algebra homomorphisms, such a universal object would not exist. For instance, the k-algebra $R = k[x]$ admits ordinary k-algebra homomorphisms into algebras $H^{\widehat{\oplus}}$ which take x to elements of arbitrarily large negative order. Hence these cannot all factor through a single homomorphism of R into one algebra $U^{\widehat{\oplus}}$, since the image of x in $U^{\widehat{\oplus}}$ would have to have some finite order. Even using homomorphisms of filtered algebras, R does not in general have a universal such homomorphism into a *noncompleted* algebra $U^{\oplus}$. So filtered algebra homomorphisms into rings $H^{\widehat{\oplus}}$ give precisely the "right" conditions for this universal construction.)

We shall obtain below a normal form for elements of $U(R)$, assuming that k is a field, or more generallly, that for each $i \in \mathbb{Z}$, $R_{(i)}/R_{(i+1)}$ is free as a k-module. It will be evident from this normal form that the map $R \to U(R)^{\widehat{\oplus}}$ is an *embedding* of filtered algebras. This was, in fact, the first author's original proof of Theorem 1.

Since $R_{(i)}/R_{(i+1)}$ is a free k-module, $R_{(i+1)}$ is a k-module direct summand in $R_{(i)}$ with free complement:

$$R_{(i)} = F_i \oplus R_{(i+1)}. \tag{6.6}$$

Let us take a k-basis X_i for each F_i, and let $X = \bigcup X_i$. It is not hard to verify that every element $a \in R$ can be written uniquely as a sum converging in the filtration topology:

$$a = \sum_{i \geqslant v(a), x \in X_i} \alpha(x) x. \tag{6.7}$$

Here for each $i \geqslant v(a)$, $\alpha(x) \in k$ is nonzero for only finitely many $x \in X_i$, and the equation (6.7) means that if, for $j \in \mathbb{Z}$, we denote by $a^{(j)}$ the partial sum, over i ranging from $v(a)$ to j, then $v(a - a^{(j)}) > j$ for all j.

If we apply $u: R \to U(R)^{\oplus}$ to (6.7), and project onto the jth component $U(R)_j$, we get

$$a_j = \sum_{v(a) \leqslant i \leqslant j, x \in X_i} \alpha(x)\, x_j \qquad \text{in } U(R)_j \quad (a, \alpha \text{ as in (6.7)}). \tag{6.8}$$

This says that in $U(R)$, all of the generators a_j ($a \in R$, $j \geqslant v(a)$) can be expressed in terms of the x_j ($x \in X$). Hence we can eliminate all the generators (6.2) except for those with $a \in X$ from our presentation of $U(R)$, and take (6.7)–(6.8) as our *definition* of a_j for general $a \in R$. Relations (6.3) and (6.4) now become immediate consequences of this definition (6.8), and all cases of (6.5) become consequences of those cases with $a, b \in X$, because of the k-bilinearity of the multiplication of R. Hence $U(R)$ is presented by generators

$$x_j \in U(R)_j \qquad (x \in X_i, j \geqslant i), \tag{6.9}$$

and relations

$$(xy)_i = \sum_{v(x) \leqslant j \leqslant i - v(y)} x_j y_{i-j} \qquad (x, y \in X, i \geqslant v(x) + v(y)), \tag{6.10}$$

where the left-hand side of (6.10) is defined using (6.8) with $a = xy \in R$.

We now want to use the equations (6.10) as "reductions," in the sense of [3], to simplify expressions for elements of $U(R)$. For this purpose, we isolate one length-2 term from each of these equations, and use this relation to eliminate all occurrences of this term from expressions in $U(R)$. Let us make this the first term of the summation. Thus we rewrite (6.10) as

$$x_{v(x)}\, y_{i - v(x)} = (xy)_i - \sum_{v(x) < j \leqslant i - v(y)} x_j y_{i-j} \qquad (x, y \in X, i \geqslant v(x) + v(y)). \tag{6.11}$$

Let us call a generator x_j of $U(R)$ a "head generator" if $j = v(x)$. Then (6.11) allows us to eliminate or "push to the right" all head generators involved in monomials occurring in expressions for elements of $U(R)$. If we assign to each formal monomial p in the generators (6.9) a nonnegative integer "index," defined to be the sum over all head generators occurring in p of their positions in the monomial, counting from the right (e.g., the index of $x_i x'_{i'} x''_{i''}$, if x_i and $x'_{i'}$ are head generators, is $3 + 2 = 5$), and partially order these monomials by writing $p > q$ if either p is longer than q, or it has the same length but greater index, then we see that every application of (6.11) to a monomial sends it to a k-linear combination of monomials lower than itself under this partial ordering. This partial ordering has descending chain condition, hence successive applications of these reductions eventually carry every expression to a sum of *irreducible* monomials, that is, monomials involving no head generators except perhaps as their final factor.

The irreducible monomials thus span $U(R)$; we wish to verify that they form a free basis for $U(R)$ as a (graded) k-module. To do this, we must show that all "ambiguities" of our reduction system (6.11) in the sense of [5] are "resolvable" relative to the above partial ordering. These ambiguities involve terms

$$x_{v(x)}\, y_{v(y)}\, z_j \qquad (x, y, z \in X, j \geqslant v(z)), \tag{6.12}$$

which can be reduced either beginning with the first pair of terms or with the last pair. A brute force verification of resolvability is possible, but would be tedious (especially since the terms $(xy)_i$ in (6.11) will be general linear combinations of generators, some of which will be head generators and some not) and, ultimately, unenlightening. Let us instead use the trick introduced in [5, Section 4]. Let F denote the free graded k-algebra on the generators x_j $(x \in X, j \geqslant v(x))$, which is where our *formal* expressions live, and let us write down in $\hat{F^{\oplus}}$ the associative law of multiplication for the product $\sum_{h \geqslant v(x)} x_h$ $\sum_{i \geqslant v(y)} y_i$ $\sum_{j \geqslant v(z)} z_j$ for some particular $x, y, z \in X$ namely:

$$\Big(\sum_h x_h \sum_i y_i\Big) \sum_j z_j = \sum_h x_h \Big(\sum_i y_i \sum_j z_j\Big). \tag{6.13}$$

We now apply the reduction (6.11) to all terms $x_{v(x)} y_i$ in the parenthesized expression on the left, and to all terms $y_{v(y)} z_j$ in the parenthesized expression on the right. Since (6.11) is equivalent to (6.10), the results can be seen to be

$$\sum_i (xy)_i \sum_j z_j \qquad \text{and} \qquad \sum_i x_i \sum_j (yz)_j. \tag{6.14}$$

Now applying (6.11) to monomials of the form $w_{v(w)} z_j$ occurring on the left, and $x_{v(x)} w_j$ on the right, these products reduce, respectively, to

$$\sum_i ((xy)z)_i \qquad \text{and} \qquad \sum_i (x(yz))_i, \tag{6.15}$$

which are equal term-by-term by the associativity of R. Now if we look at the homogeneous component of a given degree, say $v(x) + v(y) + j$, we observe that our first round of reductions involved exactly one application of (6.11) to the first pair of factors of the monomial $x_{v(x)} y_{v(y)} z_j$, namely, in going from the left-hand side of (6.13) to that of (6.14), and exactly one application of (6.11) to the last pair of factors of that same product, namely, in going from the right-hand side of (6.13) to that of (6.14). Since we began with an equality, (6.13), and ended with equal terms, (6.15), these two reductions of $x_{v(x)} y_{v(y)} z_j$ must give results that are equal modulo the other reductions that we performed—and all of these affected terms lower under our partial ordering than $x_{v(x)} y_{v(y)} z_j$. This establishes that the ambiguities in question are "resolvable relative to the partial ordering," which as shown in [5] means that they are actually resolvable, and that our reduction system does give a normal form.

Let us now note the adjustments to be made in the case of unital graded and filtered algebras. First, to (6.3)–(6.5) we add

$$1_i = \delta_{i0} \qquad (i \in \mathbb{Z}). \tag{6.16}$$

Next, we must assume that we can choose our k-bases X_i such that $1 \in X_0$. Then we can use (6.16) to eliminate all occurrences of generators 1_i; hence we take for our generating set the elements (6.9) with x ranging only over $X - \{1\}$. However, we keep x ranging over all of X in the definition (6.8). In formulating our relations (6.10) and reductions (6.11) we again have x and y ranging only over $X - \{1\}$, but the case of (6.10) (and hence (6.11)) where x or y is 1 follows from (6.16). The resolution of ambiguities is achieved exactly as for the nonunital case.

In summary,

THEOREM 3. *Let k be a field and R a filtered (unital or nonunital) k-algebra; or more generally, let k be a commutative ring and R a filtered k-algebra which admits k-module splittings* (6.6). *For each i, let X_i be a k-basis for F_i, and in the unital case assume also that we can take $1 \in X_0$. Write $X = \bigcup X_i$.*

Let $U(R)$ be the graded algebra, unital or nonunital, respectively, with a universal homomorphism of filtered k-algebras $u: R \to U(R)^{\hat{\oplus}}$, and for $a \in R$, $j \in \mathbb{Z}$ define $a_j \in U(R)_j$ as the degree j component of $u(a) \in U(R)^{\hat{\oplus}}$.

Then in the nonunital case, $U(R)$ is defined by the generators x_j ($x \in X$, $j \geqslant v(x)$) and relations (6.11), *and has for basis the set of all monomials in these generators that do not involve generators $x_{v(x)}$ except possibly as the last factor.*

In the unital case the same is true with the generating set X replaced by $X - \{1\}$, but with "monomials" understood to include the empty monomial "1." ∎

Since all the head generators, and in the unital case 1_0 as well, become distinct basis elements of $U(R)$, we see that $u: R \to U(R)^{\hat{\oplus}}$ is an *embedding* of filtered k-algebras, which when k is a field yields the result of Theorem 1.

If we use the method of Section 6 of [5] instead of that of Section 1, we can get a slightly more general result by essentially the same arguments. Let us record this as

COROLLARY 4 (to proof of Theorem 3). *If in the context of Theorem* 3 *we do not assume a splitting with free complements* F_i *as in* (6.6), *but only a k-module splitting*

$$R_{(i)} = N_i \oplus R_{(i+1)} \tag{6.17}$$

(*but in the unital case, for* $i = 0$, *replace this by* $R_{(0)} = N_0 \oplus k \oplus R_{(1)}$), *and we write* $N_{i,j}$ *for the projection of* $u(N_i)$ *in* $U(R)_j$, *then we can conclude that* $U(R)$ *is generated as a graded k-algebra by the* $N_{i,j}$ ($i \leqslant j$), *that each* $U(R)_j$ *is the k-module direct sum of the products*

$$N_{i_1,j_1} \cdots N_{i_n,j_n} \qquad (j = j_1 + \cdots + j_n;\ i_m \leqslant j_m \text{ with strict inequality except for } m = n), \tag{6.18}$$

and that these products are naturally isomorphic to the tensor products $N_{i_1} \otimes_k \cdots \otimes_k N_{i_n}$. (*Here* $n \geqslant 1$ *in the nonunital case*; $n \geqslant 0$ *in the unital case, with the empty tensor product of k-modules understood to mean* $k \subseteq U(R)_0$.) ∎

7. $U(R)$ IS UBIQUITOUS

The following result is amusing.

PROPOSITION 5. *Let k be a field, R a filtered unital k-algebra, H a graded unital k-algebra, and* $f\colon R \to \hat{H}^{\oplus}$ *a v-embedding. Let E be the graded k-algebra obtained by adjoining to H an indeterminate* $z \in E_1$. *Define the conjugated embedding* $g\colon R \to \hat{E}^{\oplus}$ *by*

$$g(x) = (1-z)f(x)(1-z)^{-1} \qquad (x \in R), \tag{7.1}$$

and let V be the graded subalgebra of E generated by the homogeneous components of the elements of $g(R)$. *Then V is naturally isomorphic to* $U(R)$.

Sketch of proof. By the property characterizing the universal map u: $R \to \widehat{U(R)}^{\oplus}$, there exists a unique homomorphism

$$\gamma\colon U(R) \to E \tag{7.2}$$

such that the embedding of filtered algebras g (7.1) factors $R \to^{u} \widehat{U(R)}^{\oplus} \to^{\hat{\gamma}^{\oplus}} \hat{E}^{\oplus}$. The conclusion of our Proposition, stated precisely, is that

γ is one-to-one on each homogeneous component. To prove this result, we examine the structure of E.

Let us choose for each i a basis B_i for H_i as a k-vector-space, with $1 \in B_0$. Then for each j, a basis of E_j is given by the set of words (including, when $j=0$, the empty word 1) in the elements of $\{z\} \cup (\bigcup B_i) - \{1\}$ that have degree j (z being assigned degree 1, and elements of B_i degree i), such that no two elements of $\bigcup B_i$ occur in succession. (Cf. [5, Corollary 8.2].)

If such a word has z's in positions $p_1, p_2, \ldots, p_r$, let us call r the *weight* of the monomial, and $p_1 + p_2 + \cdots + p_r$ its *moment*. (For instance, $zbzzb'$ has weight 3, and moment $8 = 1 + 3 + 4$.) If $a \in E_j - \{0\}$, we may write a as a linear combination of monomials, collect together the summands of maximal weight, and among those the terms of maximum moment; we shall call the sum of these terms the *dominant part* of a.

For an element $a \in R$, let us write its images under $u: R \to U(R)^{\hat{\oplus}}$ and f: $R \to H^{\hat{\oplus}}$ using, respectively, (6.2):

$$u(a) = \sum_{i \geqslant v(a)} a_i \in U(R)^{\hat{\oplus}}$$

and

$$f(a) = \sum_{i \geqslant v(a)} \bar{a}_i \in H^{\hat{\oplus}}. \tag{7.3}$$

Hence

$$\begin{aligned} g(a) &= (1-z) f(a) (1-z)^{-1} \\ &= (1-z)\left(\sum_{i \geqslant v(a)} \bar{a}_i\right)(1 + z + z^2 + \cdots). \end{aligned}$$

We can write the degree j term of this equation as

$$\tilde{a}_j = \sum_{v(a) \leqslant i \leqslant j} \bar{a}_i z^{j-i} - \sum_{v(a) \leqslant i < j} z \bar{a}_i z^{j-i-1}. \tag{7.4}$$

We see that the part of (7.4) of highest weight will be given by the $i = v(a)$ terms *if these terms are nonzero*. These terms are

$$\begin{aligned} &[\bar{a}_{v(a)}, z]\, z^{j-v(a)-1} && \text{if } j > v(a), \\ &\bar{a}_{v(a)} && \text{if } j = v(a). \end{aligned} \tag{7.5}$$

A sufficient condition for (7.5) to be nonzero is

$j \geqslant v(a)$, and if $v(a) = 0$ the expression for $\bar{a}_0 \in H_0$ does *not* involve the basis element 1, (7.6) and in this case we see that the highest-moment part of (7.5), and thus the dominant part of (7.4), is

$$\bar{a}_{v(a)} z^{j-v(a)},$$

We now need to go back and put an additional condition on our choice of basis B for H. As in the proof of Theorem 3, let $R_{(i)} = F_i \oplus R_{(i+1)}$ and let X_i be a basis for F_i, with $1 \in X_0$, so that $U(R)$ has the basis described in Theorem 3. From the fact that $f\colon R \to \hat{H}^{\oplus}$ is a v-embedding, it follows that elements $f(x)$ $(x \in X_i)$ are linearly independent modulo $(\hat{H}^{\oplus})_{(i+1)}$, and hence that their images $\bar{x}_i \in H_i$ are linearly independent. Indeed, these are the leading terms of the $f(x)$, since for $x \in X_i$, $v(x) = i$. Hence we can assume B chosen so that

$$B_i \supseteq \{\bar{x}_i \mid x \in X_i\} \qquad (i \in \mathbb{Z}). \tag{7.7}$$

We now note that for $x \in X - \{1\}$ and $j \geqslant v(x)$, condition (7.6) holds, hence by our preceding observation the dominant part of $\tilde{x}_j$ is the single monomial

$$\bar{x}_{v(x)} z^{j-v(x)}. \tag{7.8}$$

Now consider a member of the basis for $U(R)_i$ found in Theorem 3:

$$w = x^{(1)}_{j(1)} \cdots x^{(n)}_{j(n)} \qquad (x^{(p)} \in X - \{1\}, \sum j(p) = i, j(p) \geqslant v(x^{(p)})$$

$$\text{with strict inequality for } p < n). \tag{7.9}$$

Its image in E_i will be

$$\gamma(w) = \tilde{x}^{(1)}_{j(1)} \cdots \tilde{x}^{(n)}_{j(n)}. \tag{7.10}$$

We claim that the dominant part of (7.10) is

$$\bar{x}^{(1)}_{v(x^{(1)})} z^{j(1)-v(x^{(1)})} \cdots \bar{x}_{v(x^{(n)})} z^{j(n)-v(x^{(n)})}. \tag{7.11}$$

It is not true in general that the dominant part of a product is the product of the dominant parts of the factors. But this is true if, first, in the maximum-weight part of each factor all monomials involve the same number of occurrences of elements of B, and second, none of the multiplications of monomials that must be performed among monomials of the dominant part brings together two elements of B—these conditions ensure that the moment-function will behave well. Both these conditions hold in this case in view of the form of (7.8), and the fact that $j(p) - v(x^{(p)}) > 0$ except perhaps for $p = n$.

It follows that the images under γ of distinct monomials (7.9) in our basis of $U(R)_j$ have distinct monomials in the basis of E as their dominant parts. Hence no nontrivial linear combination of these images can be 0, proving that $\gamma: U(R) \to E$ is one-to-one, as required. ∎

What about a nonunital version? The problem is to define (7.1). In a nonunital ring it is generally convenient to let an expression like $(1 + a)b$ or $b(1 + c)$ be used as an abbreviation for $b + ab$ or $b + bc$; and in our complete ring $H^{\hat{\oplus}}$ the factor "$(1 - z)^{-1}$" can be rewritten $1 + (z + z^2 + \cdots)$ and interpreted in this way. One finds that

$$\begin{aligned}(1 - z)\,c(1 - z)^{-1} &= c + [c, z](1 - z)^{-1}\\ &= c + [c, z] + [c, z](z + z^2 + \cdots)\end{aligned} \tag{7.12}$$

and one *can* verify directly that this defines an automorphism of $H^{\hat{\oplus}}$, though it is easiest to see this by adjoining a unit. In any case, we get

COROLLARY 6 TO PROOF. *The statement of Proposition* 5 *holds for nonunital algebras, if* (7.1) *is interpreted using* (7.12). ∎

8. REMARKS ON UNIVERSAL CONSTRUCTIONS

(a) We noted in Section 6 that given a filtered algebra R, there does not, in general, exist a graded algebra H with a universal homomorphism of filtered algebras from R to the non-completed graded algebra $H^{\oplus}$—essentially because every element $a \in R$ would have to go to a sum of homogeneous terms in *finitely many* degrees $\geqslant v(a)$, but no *specific* finite bound is given on this set of degrees.

However, we can set up a variant of the concept of filtered algebra for which such a universal map does exist. Note that for any graded ring H, we can define *two* filtration functions $H^{\oplus} \to \mathbb{Z} \cup \{+\infty\}$:

$$v\left(\sum a_i\right) = \inf\{i \mid a_i \neq 0\}, \qquad w\left(\sum a_i\right) = \inf\{-i \mid a_i \neq 0\}. \tag{8.1}$$

(For instance, in $k[x]$ these are, respectively, the "order in x" and the negative of the "degree in x" functions.) They are related by the condition

$$v(a) + w(a) \leqslant 0 \qquad (a \neq 0). \tag{8.2}$$

Let us call a k-algebra R with a pair of filtrations satisfying (8.2) "bifiltered." A bifiltration is equivalent to an expression of R as the union of a system of k-submodules $R_{(i,j)}$ (where $R_{(i,j)} = \{a \mid i \leqslant v(a) \leqslant -w(a) \leqslant j\}$), satisfying appropriate conditions. Given a bifiltered algebra R, there *will* exist a graded algebra H with a universal homomorphism of bifiltered algebras $R \to H^{\oplus}$, taking each $a \in R$ to a sum $a_{v(a)} + \cdots + a_{-w(a)}$. We have not investigated normal forms for such universal expressions, but we note that for these purposes, the analog of (6.6) should be splittings

$$R_{(i,j)} = F_{i,j} \oplus (R_{(i+1,j)} + R_{(i,j-1)}). \tag{8.3}$$

(b) Suppose G is an arbitrary group, and $H = (H_i)_{i \in G}$ a G-graded k-algebra, defined in the obvious way. For any formal infinite sum

$$a = \sum_{i \in G} a_i \in \prod_{i \in G} H_i, \tag{8.4}$$

let us define the *support* of a in G,

$$\operatorname{spt}(a) = \{i \in G \mid a_i \neq 0\} \subseteq G. \tag{8.5}$$

Suppose now that G is an *ordered group.* If we define

$$WO = \{I \subseteq G \mid I \text{ is well-ordered, under the ordering of } G\}, \tag{8.6}$$

then

$$H^{\oplus WO} = \left\{ a \in \prod H_i \;\middle|\; \operatorname{spt}(a) \in WO \right\} \tag{8.7}$$

can be made a k-algebra with good properties. (Cf. [7, Theorem VII.3.8, p. 276], and [6].)

We could again define a G-valued "filtration function," $v(a) = \inf \operatorname{spt}(a) \in G \cup \{+\infty\}$ on $H^{\oplus WO}$, making it a "G-filtered k-algebra." This function has some useful properties, but in general there will *not* exist a construction associating to every G-filtered algebra R a G-graded algebra H with a universal filtered algebra homomorphism $R \to H^{\oplus WO}$. This is because, unless $G \cong \mathbb{Z}$ or $\{0\}$, the sets $\{i \in G \mid i \geqslant v(a)\}$ are not well-ordered, and so have no maximal well-ordered subsets.

But again we can get a universal construction by varying our concept of filtration. Note that the support function on a ring $H^{\oplus WO}$ satisfies the following conditions, analogous to (1.5)–(1.8):

$$\operatorname{spt}(x+y) \subseteq \operatorname{spt}(x) \cup \operatorname{spt}(y), \tag{8.8}$$

$$\operatorname{spt}(cx) \subseteq \operatorname{spt}(x) \qquad (c \in k), \tag{8.9}$$

$$\operatorname{spt}(xy) \subseteq \operatorname{spt}(x)\operatorname{spt}(y) \qquad (G \text{ written multiplicatively}), \tag{8.10}$$

$$\operatorname{spt}(x) = \varnothing \Leftrightarrow x = 0. \tag{8.11}$$

Let us call a k-algebra R with a function $V: R \to WO$ satisfying the conditions listed above for the function spt a "G-well-filtered k-algebra." A *homomorphism* of G-well-filtered k-algebras means an algebra homomorphism f such that for all a, $V(f(a)) \subseteq V(a)$. Now for every G-well-filtered algebra R one can find, by the exact analog of the construction of Section 5, a G-graded algebra $U(R)$ with a universal homomorphism of well-filtered algebras, $R \to U(R)^{\oplus wo}$.

The concept of well-filtered algebra subsumes those of $\mathbb{Z}$-filtered and $\mathbb{Z}$-bifiltered algebras: We can identify the former with $\mathbb{Z}$-well-filtered algebras such that for all $a \neq 0$, $V(a)$ has the form $\{i \mid i \geqslant i_0\}$ (where, of course, $i_0 = v(a)$), and the latter with $\mathbb{Z}$-well-filtered algebras such that for all $a \neq 0$, $V(a)$ has the form $\{i \mid i_0 \leqslant i \leqslant i_1\}$. Of course, starting with a $\mathbb{Z}$-filtered or bifiltered algebra $H^{\hat{\oplus}}$ or $H^{\oplus}$ arising from a graded algebra H, these constructions do not give the same well-filtrations as the support function does (by (8.8)–(8.11)), but rather certain "coarsenings" thereof.

Classes of algebras of formal sums (8.4) defined by conditions on the supports of elements other than well-ordering are discussed in [4, Sections 4, 10].

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