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Oscillation of solutions of second order mixed nonlinear differential equations under impulsive perturbations

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ABSTRACT

New oscillation criteria are obtained for second order forced mixed nonlinear impulsive differential equations of the form

$$(r(t)\Phi_\alpha(x'))' + q(t)\Phi_\alpha(x) + \sum_{k=1}^{n} q_k(t)\Phi_{\beta_k}(x) = e(t), \quad t \neq \theta_i$$
$$x(\theta_i^+) = a_i x(\theta_i), \quad x'(\theta_i^+) = b_i x'(\theta_i)$$

where $\Phi_\gamma(s) := |s|^{\gamma-1}s$ and $\beta_1 > \beta_2 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$.

If $\alpha = 1$ and the impulses are dropped, then the results obtained by Sun and Wong [Y.G. Sun, J.S.W. Wong, Oscillation criteria for second order forced ordinary differential equations with mixed nonlinearities, J. Math. Anal. Appl. 334 (2007) 549–560] are recovered. Examples are given to illustrate the results.

1. Introduction

Impulsive differential equations providing a natural description of the motion of several real world processes subject to short time perturbations are more richer than the differential equations without impulse effect. Such models are often encountered in various fields of science and technology such as physics, population dynamics, ecology, biological systems, optimal control, etc.; see [1–6] and the references cited therein.

Compared to equations without impulses there is a little work done regarding the oscillation problem for impulsive differential equations due to difficulties caused by impulsive perturbations [7–15]. In particular, there are only a few papers concerning the interval oscillation of impulsive differential equations. In [16], Liu and Xu have given interval oscillation criteria for equations of the form

$$(r(t)x'(t))' + q(t)\Phi_\gamma(x(t)) = e(t), \qquad t \neq \theta_i$$
$$x(\theta_i^+) = a_i x(\theta_i), \qquad x'(\theta_i^+) = b_i x'(\theta_i)$$

where $\gamma > 1$ is a constant and $b_i \geq a_i > 0$. The present authors [13,14], making use of a Picone type identity, have studied super-half-linear impulsive equations of the form

$$(r(t)\Phi_\alpha(x'))' + p(t)\Phi_\alpha(x') + q(t)\Phi_\beta(x) = e(t), \quad t \neq \theta_i$$
$$\Delta(r(t)\Phi_\alpha(x')) + q_i\Phi_\beta(x) = e_i, \quad t = \theta_i$$

where $\beta > \alpha$. No sign restriction was imposed on the sequence $\{q_i\}$ and the forcing sequence $\{e_i\}$.

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In this paper, motivated by the works in [16,17], we consider the oscillation problem for mixed nonlinear impulsive differential equations of the form

$$\begin{aligned} &(r(t)\Phi_\alpha(x'))' + q(t)\Phi_\alpha(x) + \sum_{k=1}^{n} q_k(t)\Phi_{\beta_k}(x) = e(t), \quad t \neq \theta_i \\ &x(\theta_i^+) = a_i x(\theta_i), \qquad x'(\theta_i^+) = b_i x'(\theta_i) \end{aligned} \tag{1.1}$$

where $\Phi_\gamma(s) := |s|^{\gamma-1}s$ and $z(t^\pm) = \lim_{\tau\to t^\pm} z(\tau)$;

(i) the nonlinearities satisfy

$$\beta_1 > \beta_2 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0; \tag{1.2}$$

(ii) $\{\theta_i\}$ is a strictly increasing unbounded sequence of real numbers; $\{a_i\}$ and $\{b_i\}$ are real sequences such that

$$b_i/a_i \geq 1, \quad i \in \mathbb{N}; \tag{1.3}$$

(iii) $r, q, q_k, e \in \mathrm{PLC}[t_0, \infty) := \big\{h : [t_0, \infty) \to \mathbb{R}$ is continuous on each interval (θ_i, θ_{i+1}), $h(\theta_i^\pm)$ exist, $h(\theta_i) = h(\theta_i^-)$ for $i \in \mathbb{N}\big\}$, $k = 1, 2, \dots, n$; $r(t) > 0$ is a nondecreasing function.

By a solution of Eq. (1.1), we mean a function $x \in \mathrm{PLC}[t_0, \infty)$ such that $x' \in \mathrm{PLC}[t_0, \infty)$ and $x(t)$ satisfies Eq. (1.1) for $t \geq t_0$. As usual, a solution of Eq. (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. The equation is called oscillatory if and only if every solution is oscillatory.

Note that when the impulses are absent, Eq. (1.1) takes the form

$$(r(t)\Phi_\alpha(x'))' + q(t)\Phi_\alpha(x) + \sum_{k=1}^{n} q_k(t)\Phi_{\beta_k}(x) = e(t). \tag{1.4}$$

Eq. (1.4) was studied by Sun and Wong when $\alpha = 1$ in [17], where the authors give some interval oscillation criteria. For some works related to interval oscillation criteria, we refer in particular to [18–21].

The following lemmas are needed.

Lemma 1.1. *Let $\{\beta_j\}$, $j = 1, \dots, n$, be the n-tuple satisfying (1.2). Then there exists an n-tuple $(\eta_1, \eta_2, \dots, \eta_n)$ satisfying*

(a) $\sum_{j=1}^{n} \beta_j\eta_j = \alpha$, *and*
(b) $\sum_{j=1}^{n} \eta_j < 1,\ 0 < \eta_j < 1$.

Lemma 1.2. *Let $\{\beta_j\}$, $j = 1, \dots, n$, be the n-tuple satisfying (1.2). Then there exists an n-tuple $(\eta_1, \eta_2, \dots, \eta_n)$ satisfying*

(a) $\sum_{j=1}^{n} \beta_j\eta_j = \alpha$, *and*
(b) $\sum_{j=1}^{n} \eta_j = 1,\ 0 < \eta_j < 1$.

The proofs of Lemmas 1.1 and 1.2 can be obtained easily from that of [17, Lemma 1] by taking $\alpha_i = \beta_i/\alpha$.

Note that if $n = 2$, we have $\beta_1 > \alpha > \beta_2 > 0$. Then, in the case of Lemma 1.1 we may take

$$\eta_1 = \frac{\alpha - \beta_2(1-\epsilon)}{\beta_1 - \beta_2}, \qquad \eta_2 = \frac{\beta_1(1-\epsilon) - \alpha}{\beta_1 - \beta_2}$$

where ϵ is any number satisfying $0 < \epsilon < 1 - \alpha/\beta_1$. In the second case, solving the system of equations in (a) and (b) of Lemma 1.2, one easily gets

$$\eta_1 = \frac{\alpha - \beta_2}{\beta_1 - \beta_2}, \qquad \eta_2 = \frac{\beta_1 - \alpha}{\beta_1 - \beta_2}.$$

We also need the Young inequality

$$\frac{{x_1}^p}{p} + \frac{{x_2}^q}{q} - x_1x_2 \geq 0, \quad x_1, x_2 \geq 0 \tag{1.5}$$

where p and q are positive real numbers such that $1/p + 1/q = 1$.

Let z_1, z_2 be nonnegative real numbers and $\gamma > 0$ a constant. Applying (1.5) with $p = \gamma + 1$, $q = 1 + 1/\gamma$, $x_1 = z_1$ and $x_2 = z_2^\gamma$, we obtain a simple inequality

$$z_1^{\gamma+1} + \gamma z_2^{\gamma+1} - (\gamma+1)z_1z_2^\gamma \geq 0. \tag{1.6}$$

The following lemma can be proved by using (1.6); also see [22].

Lemma 1.3. *Let z, A, B, C and D be nonnegative real numbers. Then*

(i) $Az^\lambda + B \geq \lambda(\lambda-1)^{1/\lambda-1}A^{1/\lambda}B^{1-1/\lambda}z$ *for* $\lambda > 1$,
(ii) $Cz - Dz^\lambda \geq (\lambda-1)\lambda^{\lambda/(1-\lambda)}C^{\lambda/(\lambda-1)}D^{1/(1-\lambda)}$ *for* $0 < \lambda < 1$.

2. Main results

For convenience, we introduce the notation

$$i_s = \max\{i : t_0 < \theta_i < s\}.$$

For $s, t \notin \{\theta_i\}$ with $s < t$, we define an operator $\mathcal{H}_n : C([s,t],\mathbb{R}) \to \mathbb{R}$ as

$$\mathcal{H}_n[h] := \begin{cases} 0, & i_{s_n} = i_{t_n} \\ \mathcal{F}^{[n]}_{i_{s_n}+1}(\theta_{i_{s_n}+1}, s_n)[h] + \sum_{i=i_{s_n}+2}^{i_{t_n}} \mathcal{F}^{[n]}_i(\theta_i, \theta_{i-1})[h], & i_{s_n} < i_{t_n} \end{cases}$$

where

$$\mathcal{F}^{[n]}_j(c_1, c_2)[\phi] = \{1 - \Phi_\alpha(b_j/a_j)\}(c_1 - c_2)^{-\alpha} r(c_1)|\phi(c_1)|^{\alpha+1}. \tag{2.1}$$

Denote by $\mathcal{D}$ the set of admissible functions

$$\mathcal{D}(a,b) = \{u \in C^1[a,b] : u(t) \not\equiv 0, u(a) = u(b) = 0\}.$$

Our first result is as follows.

Theorem 2.1. *Suppose that for any given $T \geq t_0$, there exist intervals $I_1 = [s_1, t_1]$, $I_2 = [s_2, t_2] \subset [T, \infty)$, such that $s_j, t_j \notin \{\theta_i\}$, $j = 1, 2$ and*

(a) $q(t), q_k(t) \geq 0\ \forall t \in \{I_1 \cup I_2\} \setminus \{\theta_i\}$, $(k = 1, 2, \ldots, n)$;

(b) $e(t) \begin{cases} \leq 0, & t \in I_1 \setminus \{\theta_i\} \\ \geq 0, & t \in I_2 \setminus \{\theta_i\} \end{cases}$; $\forall i \in \mathbb{N}$.

If there exists $u \in \mathcal{D}(s_j, t_j)$ such that

$$\int_{s_j}^{t_j} \left[\left\{ q(t) + \zeta_n |e(t)|^{\eta_0} \prod_{k=1}^{n} q_k^{\eta_k}(t) \right\} |u(t)|^{\alpha+1} - r(t)|u'(t)|^{\alpha+1} \right] \mathrm{d}t + \mathcal{H}_j[u] > 0 \tag{2.2}$$

for $j = 1, 2$, where $\zeta_n = \prod_{k=0}^{n} \eta_k^{-\eta_k}$ with $\eta_0 = 1 - \sum_{k=1}^{n} \eta_k$, and $\eta_1, \ldots, \eta_n$ are positive constants satisfying conditions of Lemma 1.1, then Eq. (1.1) is oscillatory.

Proof. Suppose that there exists a nonoscillatory solution $x(t)$ of (1.1) so that $x(t) \neq 0$ for all $t \geq t_*$ for some $t_* \geq t_0$, where t_0 may depend on the solution $x(t)$. Let

$$\nu(t) := -\frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))}, \quad t \geq t_*. \tag{2.3}$$

It follows that for $t \geq t_*$ and $t \neq \theta_i$,

$$\nu'(t) = q(t) + \sum_{k=1}^{n} q_k(t)\Phi_{\beta_k - \alpha}(x(t)) - \frac{e(t)}{\Phi_\alpha(x(t))} + \alpha \frac{|\nu(t)|^{1+1/\alpha}}{r^{1/\alpha}(t)} \tag{2.4}$$

and for $\theta_i \geq t_*$

$$\Delta\nu(\theta_i) = -\xi_i \nu(\theta_i), \quad \xi_i = 1 - \Phi_\alpha(b_i/a_i). \tag{2.5}$$

By the arithmetic–geometric mean inequality, see [23],

$$\sum_{k=0}^{n} \eta_k \mu_k \geq \prod_{k=0}^{n} \mu_k^{\eta_k}, \quad \mu_k \geq 0,\ (k = 0, 1, \ldots, n) \tag{2.6}$$

where $\eta_k > 0$, $k = 1, 2, \ldots, n$, are chosen to satisfy conditions of Lemma 1.1 with $\eta_0 = 1 - \sum_{k=1}^{n} \eta_k > 0$ for the given $\alpha, \beta_1, \beta_2, \ldots, \beta_{n-1}$ and β_n.

Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq t_*$. When $x(t)$ is eventually negative, the proof follows the same argument using the interval I_2 instead of I_1. By assumption, we can choose $s_1, t_1 \geq t_*$ such that $q(t), q_k(t) \geq 0$ and $e(t) \leq 0$ for all $t \in I_1 \setminus \{\theta_i\}$ and for all $k = 1, 2, \ldots, n$, $i \in \mathbb{N}$. It follows that Eq. (2.4) becomes

$$\nu'(t) = q(t) + \left\{ \sum_{k=1}^{n} q_k(t)x^{\beta_k}(t) + |e(t)| \right\} x^{-\alpha}(t) + \alpha \frac{|\nu(t)|^{1+1/\alpha}}{r^{1/\alpha}(t)}, \quad t \neq \theta_i. \tag{2.7}$$

Now, using inequality (2.6) with

$$\mu_0 = \eta_0^{-1}|e(t)| \quad \text{and} \quad \mu_k = \eta_k^{-1} q_k(t) x^{\beta_k}(t), \quad k = 1, 2, \ldots, n,$$

we obtain

$$\nu'(t) \geq q(t) + \eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{k=1}^{n}\eta_k^{-\eta_k}q_k^{\eta_k}(t) + \alpha\frac{|\nu(t)|^{1+1/\alpha}}{r^{1/\alpha}(t)}$$
$$= \widetilde{q}(t) + \alpha\frac{|\nu(t)|^{1+1/\alpha}}{r^{1/\alpha}(t)}, \quad t \neq \theta_i \tag{2.8}$$

where

$$\widetilde{q}(t) = q(t) + \zeta_n|e(t)|^{\eta_0}\prod_{k=1}^{n}q_k^{\eta_k}(t).$$

If $i_{s_1} < i_{t_1}$, then there are impulse moments in $[s_1, t_1]$: $\theta_{i_{s_1}+1}, \theta_{i_{s_1}+2}, \ldots, \theta_{i_{t_1}}$. Multiplying both sides of (2.8) by $|u(t)|^{\alpha+1}$ and integrating over I_1 gives

$$\int_{s_1}^{t_1}\widetilde{q}(t)|u(t)|^{\alpha+1}\mathrm{d}t \leq -\alpha\int_{s_1}^{t_1}\frac{|\nu(t)|^{1+1/\alpha}}{r^{1/\alpha}(t)}|u(t)|^{\alpha+1}\mathrm{d}t + \int_{s_1}^{t_1}|u(t)|^{\alpha+1}\nu'(t)\mathrm{d}t.$$

Employing the integration by parts formula in the last integral and using $u(s_1) = u(t_1) = 0$, we have

$$\int_{s_1}^{t_1}\widetilde{q}(t)|u(t)|^{\alpha+1}\mathrm{d}t \leq -\alpha\int_{s_1}^{t_1}\frac{|\nu(t)|^{1+1/\alpha}}{r^{1/\alpha}(t)}|u(t)|^{\alpha+1}\mathrm{d}t$$
$$-(\alpha+1)\int_{s_1}^{t_1}\Phi_\alpha(u(t))u'(t)\nu(t)\mathrm{d}t - \sum_{i=i_{s_1}+1}^{i_{t_1}}|u(\theta_i)|^{\alpha+1}\Delta\nu(\theta_i). \tag{2.9}$$

In view of (2.5), we obtain

$$\int_{s_1}^{t_1}\{\widetilde{q}(t)|u(t)|^{\alpha+1} - r(t)|u'(t)|^{\alpha+1}\}\mathrm{d}t$$
$$\leq \sum_{i=i_{s_1}+1}^{i_{t_1}}\xi_i\nu(\theta_i)|u(\theta_i)|^{\alpha+1} - \int_{s_1}^{t_1}\left[r(t)|u'(t)|^{\alpha+1} + \alpha\frac{|\nu(t)|^{1+1/\alpha}}{r^{1/\alpha}(t)}|u(t)|^{\alpha+1} - (\alpha+1)|u(t)|^{\alpha}|u'(t)||\nu(t)|\right]\mathrm{d}t. \tag{2.10}$$

Note that the terms in brackets in (2.10) are nonnegative due to (1.6) with

$$\gamma = 1/\alpha, \qquad z_1 = \alpha^{\alpha/(\alpha+1)}\frac{|u(t)|^{\alpha}}{r^{1/(\alpha+1)}(t)}|\nu(t)|, \qquad z_2 = \{\alpha r(t)\}^{\alpha/(\alpha+1)}|u'(t)|^{\alpha}.$$

It follows that

$$\int_{s_1}^{t_1}\{\widetilde{q}(t)|u(t)|^{\alpha+1} - r(t)|u'(t)|^{\alpha+1}\}\mathrm{d}t \leq \sum_{i=i_{s_1}+1}^{i_{t_1}}\xi_i\nu(\theta_i)|u(\theta_i)|^{\alpha+1}. \tag{2.11}$$

On the other hand, for $t \in (s_1, \theta_{i_{s_1}+1}]$,

$$(r(t)\Phi_\alpha(x'(t)))' = e(t) - q(t)\Phi_\alpha(x(t)) - \sum_{k=1}^{n}q_k(t)\Phi_{\beta_k}(x(t))$$
$$= e(t) - q(t)x^{\alpha}(t) - \sum_{k=1}^{n}q_k(t)x^{\beta_k}(t) \leq 0$$

which means that $r(t)\Phi_\alpha(x'(t))$ is nonincreasing on $(s_1, \theta_{i_{s_1}+1}]$. Now, for any $t \in (s_1, \theta_{i_{s_1}+1}]$, we have

$$x(t) - x(s_1) = x'(\varepsilon)(t - s_1), \quad \varepsilon \in (s_1, t).$$

Since $x(s_1) > 0$ and the function $\Phi_\alpha(.)$ is an increasing function,

$$\Phi_\alpha(x(t)) > \Phi_\alpha(x'(\varepsilon))\Phi_\alpha(t - s_1) \geq \frac{r(t)\Phi_\alpha(x'(t))}{r(\varepsilon)}(t - s_1)^{\alpha}, \quad \varepsilon \in (s_1, t).$$

From the fact that $r(t)$ is nondecreasing, we have

$$-\frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))} \geq -r(\varepsilon)(t - s_1)^{-\alpha} \geq -r(\theta_{i_{s_1}+1})(t - s_1)^{-\alpha}.$$

Letting $t \to \theta^-_{i_{s_1}+1}$, we obtain

$$\nu(\theta_{i_{s_1}+1}) \geq -r(\theta_{i_{s_1}+1})(\theta_{i_{s_1}+1} - s_1)^{-\alpha}. \tag{2.12}$$

In a similar way, we proceed on $(\theta_{i-1}, \theta_i]$ to get

$$\nu(\theta_i) = -\frac{r(\theta_i)\Phi_\alpha(x'(\theta_i))}{\Phi_\alpha(x(\theta_i))} \geq -r(\theta_i)(\theta_i - \theta_{i-1})^{-\alpha} \tag{2.13}$$

for $i = i_{s_1+2}, \ldots, i_{t_1}$. Using (1.3), (2.12) and (2.13), we have

$$\begin{aligned}\sum_{i=i_{s_1}+1}^{i_{t_1}} \xi_i \nu(\theta_i)|u(\theta_i)|^{\alpha+1} &= \xi_{i_{s_1}+1}\nu(\theta_{i_{s_1}+1})|u(\theta_{i_{s_1}+1})|^{\alpha+1} + \sum_{i=i_{s_1}+2}^{i_{t_1}} \xi_i \nu(\theta_i)|u(\theta_i)|^{\alpha+1} \\ &\leq -\left\{\mathcal{F}^{[1]}_{i_{s_1}+1}(\theta_{i_{s_1}+1}, s_1)[u] + \sum_{i=i_{s_1}+2}^{i_{t_1}} \mathcal{F}^{[1]}_i(\theta_i, \theta_{i-1})[u]\right\} \\ &= -\mathcal{H}_1[u].\end{aligned}$$

It follows from (2.11) that

$$\int_{s_1}^{t_1} \{\widetilde{q}(t)|u(t)|^{\alpha+1} - r(t)|u'(t)|^{\alpha+1}\}\mathrm{d}t \leq -\mathcal{H}_1[u]. \tag{2.14}$$

If $i_{s_1} = i_{t_1}$, then there is no impulse moment in $[s_1, t_1]$, and that (2.11) yields

$$\int_{s_1}^{t_1} \{\widetilde{q}(t)|u(t)|^{\alpha+1} - r(t)|u'(t)|^{\alpha+1}\}\mathrm{d}t \leq 0 = -\mathcal{H}_1[u]. \tag{2.15}$$

Both inequalities (2.14) and (2.15) contradict with our assumption (2.2). This completes the proof. □

The next theorem is for the case $e(t) \equiv 0$.

Theorem 2.2. *Suppose that for any given $T \geq t_0$, there exist intervals $I_1 = [s_1, t_1] \subset [T, \infty)$, such that $s_1, t_1 \notin \{\theta_i\}$, and $q(t)$, $q_k(t) \geq 0$ for all $t \in I_1 \setminus \{\theta_i\}$, $k = 1, 2, \ldots, n$, $i \in \mathbb{N}$.*

If there exists $u \in \mathcal{D}(s_1, t_1)$ such that

$$\int_{s_1}^{t_1} \left[\left\{q(t) + \sigma_n \prod_{k=1}^{n} q_k^{\eta_k}(t)\right\}|u(t)|^{\alpha+1} - r(t)|u'(t)|^{\alpha+1}\right]\mathrm{d}t + \mathcal{H}_1[u] > 0, \tag{2.16}$$

where $\sigma_n = \prod_{k=1}^{n} \eta_k^{-\eta_k}$, and $\eta_1, \eta_2, \ldots, \eta_n$ are positive constants satisfying conditions of Lemma 1.2, then Eq. (1.1) with $e(t) \equiv 0$ is oscillatory.

Proof. The proof is based on applying Lemma 1.2 with $e(t) \equiv 0$, and similar to that of Theorem 2.1. □

In our last theorem, we allow the functions $q_k(t)$ to be negative for $k = m+1, m+2, \ldots, n$.

Theorem 2.3. *Suppose that for any given $T \geq t_0$, there exist intervals $I_1 = [s_1, t_1]$, $I_2 = [s_2, t_2] \subset [T, \infty)$, such that $s_p, t_p \notin \{\theta_i\}$, $p = 1, 2$ and*

(a) $q(t), q_k(t) \geq 0\ \forall t \in \left\{I_1 \cup I_2\right\} \setminus \{\theta_i\}$, $(k = 1, 2, \ldots, m)$;

(b) $e(t) \begin{cases} < 0, & t \in I_1 \setminus \{\theta_i\} \\ > 0, & t \in I_2 \setminus \{\theta_i\} \end{cases}$; $\forall i \in \mathbb{N}$.

If there exist $u \in \mathcal{D}(s_p, t_p)$ and positive numbers γ_k, $k = 1, \ldots, m$, and τ_j, $j = m+1, \ldots, n$, such that

$$\int_{s_p}^{t_p} \left\{\widehat{q}(t)|u(t)|^{\alpha+1} - r(t)|u'(t)|^{\alpha+1}\right\}\mathrm{d}t + \mathcal{H}_p[u] > 0, \quad (p = 1, 2) \tag{2.17}$$

where

$$\begin{aligned}\widehat{q}(t) &= q(t) + \sum_{k=1}^{m} \beta_k(\beta_k - \alpha)^{\alpha/\beta_k - 1}\alpha^{-\alpha/\beta_k} q_k^{\alpha/\beta_k}(t)\{\gamma_k|e(t)|\}^{1-\alpha/\beta_k} \\ &\quad - \sum_{j=m+1}^{n} \beta_j(\alpha - \beta_j)^{\alpha/\beta_j - 1}\alpha^{-\alpha/\beta_j} \check{q}_j^{\alpha/\beta_j}(t)\{\tau_j|e(t)|\}^{1-\alpha/\beta_j},\end{aligned}$$

with

$$\sum_{k=1}^{m}\gamma_k+\sum_{j=m+1}^{n}\tau_j=1$$

and $\breve{q}_j(t)=\max\{-q_j(t),0\}$, *$j=m+1,\ldots,n$, then Eq.* (1.1) *is oscillatory.*

Proof. Suppose that there exists a nonoscillatory solution $x(t)$ of (1.1) so that $x(t)\neq 0$ for all $t\geq t_*$ for some $t_*\geq t_0$. We define $\nu(t)$ as in (2.3) and obtain (2.4), and (2.5). As in the proof of Theorem 2.1, we may assume that $x(t)>0$ for all $t\geq t_0$. By assumption, we can choose $s_1,t_1\geq t_*$ such that $q(t),q_k(t)\geq 0$ and $e(t)<0$ for all $t\in I_1\setminus\{\theta_i\}$ and for all $k=1,2,\ldots,m$, $i\in\mathbb{N}$. It follows from rearranging (2.7) that

$$\nu'(t)=q(t)+\alpha\frac{|\nu(t)|^{1+1/\alpha}}{r^{1/\alpha}(t)}+\left[\sum_{k=1}^{m}\left\{q_k(t)x^{\beta_k}(t)+\gamma_k|e(t)|\right\}+\sum_{j=m+1}^{n}\left\{q_j(t)x^{\beta_j}(t)+\tau_j|e(t)|\right\}\right]x^{-\alpha}(t),\quad t\neq\theta_i. \tag{2.18}$$

Applying Lemma 1.3(i) to the first summation in (2.18) with $\lambda=\beta_k/\alpha>1$, $z=x^{\alpha}(t)$, $A=q_k(t)$ and $B=\gamma_k|e(t)|$, we obtain

$$q_k(t)x^{\beta_k}(t)+\gamma_k|e(t)|\geq\beta_k(\beta_k-\alpha)^{\alpha/\beta_k-1}\alpha^{-\alpha/\beta_k}q_k^{\alpha/\beta_k}(t)\{\gamma_k|e(t)|\}^{1-\alpha/\beta_k}x^{\alpha}(t). \tag{2.19}$$

The second summation in (2.18) can be made smaller by applying Lemma 1.3(ii) with $\lambda=\beta_j/\alpha\in(0,1)$, $D=\breve{q}_j(t)$, $C=\lambda(1-\lambda)^{1/\lambda-1}(\breve{q}_j(t))^{1/\lambda}(\tau_j|e(t)|)^{1-1/\lambda}$ and $z=x^{\alpha}(t)$. We see that

$$\begin{aligned}q_j(t)x^{\beta_j}(t)+\tau_j|e(t)|&\geq\tau_j|e(t)|-\breve{q}_j(t)x^{\beta_j}(t)\\&\geq-\beta_j(\alpha-\beta_j)^{\alpha/\beta_j-1}\alpha^{-\alpha/\beta_j}\breve{q}_j^{\alpha/\beta_j}(t)\{\tau_j|e(t)|\}^{1-\alpha/\beta_j}x^{\alpha}(t).\end{aligned} \tag{2.20}$$

Using (2.5), (2.18), (2.19) and (2.20), we obtain

$$\begin{aligned}&\nu'(t)\geq\widehat{q}(t)+\alpha\frac{|\nu(t)|^{1+1/\alpha}}{r^{1/\alpha}(t)}, && t\neq\theta_i\\&\Delta\nu(t)=-\xi_i\nu(t), && t=\theta_i.\end{aligned} \tag{2.21}$$

The remainder of the proof is the same as that of Theorem 2.1. The proof is complete. □

Remark 1. If the function r is not nondecreasing, then Theorem 2.1 is still valid if the term $r(c_1)$ is replaced by r_j^* in (2.1) where

$$r_j^*=\max\{r(t):t\in I_j=[s_j,t_j]\},\quad j=1,2.$$

In this case, if we take $q(t)\equiv 0$, $n=1$ and $\alpha=1$, then we recover [16, Theorem 2.1]. We also note that the condition $b_i\geq a_i>0$ given in [16] is not necessary; the condition (1.3) suffices.

Remark 2. If the impulses are dropped in (1.1) i.e. $a_i\equiv b_i\equiv 1$, and $\alpha=1$, then our results reduce to Theorems 1, 2 and 3 given by Sun and Wong in [17].

Example 2.4. Consider the impulsive equation

$$\begin{aligned}&x''+\sigma_0x+\sigma_1|x|x+\sigma_2|x|^{-1/2}x=\sin(\pi t), && t\neq\theta_i\\&x(\theta_i^+)=(-1)^i\sigma_3x(\theta_i),\qquad x'(\theta_i^+)=(-1)^i\sigma_4x'(\theta_i)\end{aligned} \tag{2.22}$$

where $\theta_i=(2i-1)/8$, $i\in\mathbb{N}$, and that σ_k, $k=0,1,2$, are nonnegative constants and $\sigma_4/\sigma_3\geq 1$. We can take $\eta_1=4/9$ and $\eta_2=2/9$ to see that Lemma 1.1 holds.

Let $u(t)=\sin(\pi t)$, and choose $s_1=2m-1$, $t_1=s_2=2m$ and $t_2=2m+1$. For any given $t_*\geq 0$ we may choose $m\in\mathbb{N}$ sufficiently large so that $2m\geq t_*$. Then conditions (a) and (b) of Theorem 2.1 are satisfied. Moreover

$$\begin{aligned}&\int_{s_j}^{t_j}\left[\left\{\sigma_0+\zeta_2|\sin(\pi t)|^{1/3}\sigma_1^{4/9}\sigma_2^{2/9}\right\}\sin^2(\pi t)-\pi^2\cos^2(\pi t)\right]\mathrm{d}t\\&=\frac{\sigma_0}{2}+3^{5/3}2^{8/9}\frac{\Gamma(2/3)}{7\sqrt{\pi}\,\Gamma(7/6)}\sigma_1^{4/9}\sigma_2^{2/9}-\frac{\pi^2}{2},\quad(j=1,2)\end{aligned} \tag{2.23}$$

where Γ is the "Gamma function", and that

$$\begin{aligned}\mathcal{H}_1[\sin(\pi t)]&=\mathcal{F}_{8m-3}^{[1]}(2m-7/8,2m-1)[\sin(\pi t)]+\sum_{i=8m-2}^{8m}\mathcal{F}_i^{[1]}((2i-1)/8,(2i-3)/8)[\sin(\pi t)]\\&=(10-\sqrt{2})(1-\sigma_4/\sigma_3).\end{aligned} \tag{2.24}$$

One can also calculate that $\mathcal{H}_2[\sin(\pi t)] = (10 - \sqrt{2})(1 - \sigma_4/\sigma_3)$. Thus (2.2) holds if

$$\sigma_0 + c_0 \sigma_1^{4/9} \sigma_2^{2/9} > c_1 \frac{\sigma_4}{\sigma_3} + c_2$$

where $c_0 \approx 2.71882$, $c_1 \approx 17.1716$ and $c_2 \approx -7.30197$, which by Theorem 2.1 is sufficient for oscillation of (2.22).

Example 2.5. Consider the impulsive equation

$$\begin{aligned} &((k_0 + |\sin 2t|)\Phi_3(x'(t)))' + k_1 |\sin 2t| \Phi_3(x(t)) + k_2 \sin^2(t/2)\Phi_4(x(t)) + k_3 \cos^2(t/2)\Phi_2(x(t)) = 0, \quad t \neq \theta_i \\ &x(\theta_i^+) = (-1)^i k_4 x(\theta_i), \qquad x'(\theta_i^+) = (-1)^i k_5 x'(\theta_i) \end{aligned} \tag{2.25}$$

where $\theta_i = (2i - 1)\pi/8$, $i \in \mathbb{N}$, and that k_0 is positive, k_1, k_2, k_3 are nonnegative constants and $k_5/k_4 \geq 1$. It is enough to take $\eta_1 = \eta_2 = 1/2$ so that Lemma 1.2 holds.

Let $u(t) = \sin t$, and choose $s_1 = n\pi$ and $t_1 = (n+1)\pi$. For any given $t_* \geq 0$, we may choose $n \in \mathbb{N}$ sufficiently large so that $n \geq t_*$. Then

$$\begin{aligned} &\int_{n\pi}^{(n+1)\pi} \left[\left\{ k_1 |\sin 2t| + \sigma_2 \sqrt{k_2 k_3} |\sin(t/2)\cos(t/2)| \right\} \sin^4 t - \{k_0 + |\sin 2t|\} \cos^4 t \right] \mathrm{d}t \\ &\quad = \frac{2}{3}(k_1 - 1) + \frac{16}{15}\sqrt{k_2 k_3} - \frac{3\pi}{8} k_0, \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} \mathcal{H}_1[\sin t] &= \mathcal{F}_{4n+1}^{[1]}((8n+1)\pi/8, n\pi)[\sin t] + \sum_{i=4n+2}^{4n+4} \mathcal{F}_i^{[1]}((2i-1)\pi/8, (2i-3)\pi/8)[\sin t] \\ &= \xi_{4n+1}(\pi/8)^{-3} r((n+1/8)\pi) \sin^4((n+1/8)\pi) + (\pi/4)^{-3} \sum_{i=4n+2}^{4n+4} \xi_i r((2i-1)\pi/8) \sin^4((2i-1)\pi/8) \\ &= \frac{4}{\pi^3}(33 - 14\sqrt{2})(2k_0 + \sqrt{2})(1 - (k_5/k_4)^3). \end{aligned} \tag{2.27}$$

Thus (2.16) holds if

$$\frac{2}{3}k_1 + \frac{16}{15}\sqrt{k_2 k_3} - \frac{3\pi}{8}k_0 > \frac{4}{\pi^3}(33 - 14\sqrt{2})(2k_0 + \sqrt{2}) \left\{ \left(\frac{k_5}{k_4} \right)^3 - 1 \right\} + \frac{2}{3}$$

which by Theorem 2.2 is sufficient for oscillation of (2.25).

Example 2.6. Consider the impulsive equation

$$\begin{aligned} &x'' + \sigma_0 x + \sigma_1 |x| x - \sigma_2 |x|^{-1/2} x = \sin(\pi t), \qquad t \neq \theta_i \\ &x(\theta_i^+) = (-1)^i \sigma_3 x(\theta_i), \qquad x'(\theta_i^+) = (-1)^i \sigma_4 x'(\theta_i) \end{aligned} \tag{2.28}$$

where $\theta_i = (2i-1)/32$, $i \in \mathbb{N}$, and that σ_k, $k = 0, 1, 2$, are nonnegative constants and $\sigma_4/\sigma_3 \geq 1$.

Let $u(t) = \sin(\pi t)$, and choose $s_1 = 4/3 + 2j$, $t_1 = 5/3 + 2j$, $s_2 = 7/3 + 2j$ and $t_2 = 8/3 + 2j$. For any given $t_* \geq 0$, we may choose $j \in \mathbb{N}$ sufficiently large so that $2j \geq t_*$. Then conditions (a) and (b) of Theorem 2.3 are satisfied. Moreover, taking $\gamma_1 = \tau_2 = 1/2$, we obtain

$$\begin{aligned} &\int_{s_p}^{t_p} \left[\sigma_0 + \sqrt{2}\sigma_1^{1/2} |\sin(\pi t)|^{5/2} - \pi^2 \cos^2(\pi t) \right] \mathrm{d}t \\ &\quad = \frac{\pi}{3}\sigma_0 + \frac{\sqrt{2}}{\pi}\sigma_1^{1/2} \int_{\pi/3}^{2\pi/3} \sin^{5/2} t \, \mathrm{d}t + \left(\frac{\sqrt{3}}{4} - \frac{\pi}{6} \right) \pi, \quad (p = 1, 2), \end{aligned} \tag{2.29}$$

$$\begin{aligned} \mathcal{H}_1[\sin(\pi t)] &= \mathcal{F}_{32j+22}^{[1]}(2j + 43/32, 2j + 4/3)[\sin(\pi t)] + \sum_{i=32j+23}^{32j+27} \mathcal{F}_i^{[1]}((2i-1)/32, (2i-3)/32)[\sin(\pi t)] \\ &= (1 - \sigma_4/\sigma_3)[32\cos^2(\pi/32) + 32\cos^2(3\pi/32) + 97\cos^2(5\pi/32)] \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} \mathcal{H}_2[\sin(\pi t)] &= \mathcal{F}_{32j+38}^{[2]}(2j + 75/32, 2j + 7/3)[\sin(\pi t)] + \sum_{i=32j+39}^{32j+43} \mathcal{F}_i^{[2]}((2i-1)/32, (2i-3)/32)[\sin(\pi t)] \\ &= (1 - \sigma_4/\sigma_3)[32\cos^2(\pi/32) + 32\cos^2(3\pi/32) + 16\cos^2(5\pi/32) + 96\cos^2(7\pi/32)]. \end{aligned} \tag{2.31}$$

Using (2.29)–(2.31) we can deduce that (2.17) holds if

$$\sigma_0 + c_0\sqrt{\sigma_1} > c_1\frac{\sigma_4}{\sigma_3} + c_2$$

where $c_0 \approx 0.402456$, $c_1 \approx 124.909573$ and $c_2 \approx -124.637816$, which by Theorem 2.3 is sufficient for oscillation of (2.28).

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