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Motivic integral of K3 surfaces over a non-archimedean field

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Abstract

We prove a formula expressing the motivic integral (Loeser and Sebag, 2003) [34] of a K3 surface over $\mathbb{C}((t))$ with semi-stable reduction in terms of the associated limit mixed Hodge structure. Secondly, for every smooth variety over a complete discrete valuation field we define an analogue of the monodromy pairing, constructed by Grothendieck in the case of abelian varieties, and prove that our monodromy pairing is a birational invariant of the variety. Finally, we propose a conjectural formula for the motivic integral of maximally degenerate K3 surfaces over an arbitrary complete discrete valuation field and prove this conjecture for Kummer K3 surfaces.
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1. Introduction

1.1. Motivic integral of a Calabi–Yau variety

Let R be a complete discrete valuation ring with fraction field K and perfect residue field k. By a Calabi–Yau variety X over K we mean a smooth projective scheme X over K, of pure

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dimension d, with trivial canonical bundle $\omega_X := \Omega^d_{X/K}$. In [34], Loeser and Sebag associated with any Calabi–Yau variety X over K a canonical element

$$\int_X \in K_0(\mathrm{Var}_k)_{\mathrm{loc}}$$

of the ring $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$, where $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ is obtained from the Grothendieck ring $K_0(\mathrm{Var}_k)$ of algebraic varieties over k by inverting the class $[\mathbb{A}^1_k]$ of the affine line.

The motivic integral $\int_X$ can be computed from a weak Néron model of X. Recall, that a weak Néron model of a smooth proper scheme X over K is a smooth scheme $\mathcal{V}$ of finite type over R together with an isomorphism $\mathcal{V} \otimes_R K \simeq X$ satisfying the following property: for every finite unramified extension $R' \supset R$ with fraction field K', the canonical map $\mathcal{V}(R') \to X(K')$ is bijective. According to [10, §3.5, Theorem 3], every smooth proper K-scheme X admits a weak Néron model. We note that a weak Néron model is almost never unique: for example, if $\overline{X}$ is a proper regular model of X over R, then the smooth locus $\overline{X}_{\mathrm{sm}}$ of $\overline{X}$ is a weak Néron model of X (see Lemma 2.10).

Given a Calabi–Yau variety X over K, a weak Néron model $\mathcal{V}$ of X, and a non-zero top degree differential form $\omega \in \Gamma(X, \omega_X)$, we can view ω as a rational section of the canonical bundle $\omega_{\mathcal{V}/R}$ on $\mathcal{V}$. The divisor of ω is supported on the special fiber V° of $\mathcal{V}$. Thus, we can write

$$\mathrm{div}\,\omega = \sum_i m_i V_i^\circ, \tag{1.1}$$

where $V_1^\circ, \ldots, V_s^\circ$ are the irreducible components of the special fiber V°. The motivic integral of X is defined by the formula[1]

$$\int_X := \sum_i [V_i^\circ]\left(m_i - \min_i m_i\right). \tag{1.2}$$

Here, given an element $[Z] \in K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ and an integer n, we write $[Z](n)$ for its Tate twist:

$$[Z](n) := [Z] \cdot [\mathbb{A}^1]^{-n}.$$

A key result proven by Loeser and Sebag [34, Theorem 4.4.1] is that the right-hand side of Eq. (1.2) is independent of the choice of $\mathcal{V}$ and ω.

If $k = \mathbb{F}_q$, the image of the motivic integral under the homomorphism

$$K_0(\mathrm{Var}_{\mathbb{F}_q})_{\mathrm{loc}} \to \mathbb{Z}_{(q)}, \qquad [Z] \rightsquigarrow |Z(\mathbb{F}_q)| \tag{1.3}$$

is equal to the volume $\int_{X(K)} |\omega|$, for an appropriately normalized $\omega \in \Gamma(X, \omega_X)$ [34, §4.6].

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[1] We note that our terminology and notation are different from those used by Loeser and Sebag. Notation for $\int_X$ in [34] is $[\overline{X}]$. The name "motivic integral" is reserved in [34] for a more general construction that associates with any smooth proper K-scheme X and a top degree differential form $\omega \in \Gamma(X, \omega_X)$ an element $\int_X \omega$ of a certain completion of the motivic ring $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$.

In this paper we express the motivic integral of K3 surfaces over $\mathbb{C}((t))$ with strictly semi-stable reduction in terms of the associated limit mixed Hodge structures. We also compute the motivic integral of some K3 surfaces over an arbitrary complete discrete valuation field. To our knowledge the only class of varieties, for which similar formulas were previously known, is the class of abelian varieties (see, e.g. [49, Exposé IX], [48,23,22]), where the computation is based on the theory of Néron models, and, in particular, for $K = \mathbb{C}((t))$, on the Hodge theoretic description of the special fiber of the Néron model. Unfortunately, K3 surfaces do not have a Néron model, in general, which makes our problem substantially more difficult.

Let us describe the organization of the paper in more detail.

1.2. Limit mixed Hodge structure

In Section 2 we explain some preliminary material, the most important of which is the notion of limit mixed Hodge structure associated with a variety over the field of formal Laurent series $\mathbb{C}((t))$. Schmid and Steenbrink associated with every smooth projective variety over the field K_{mer} of meromorphic functions on an open neighborhood of zero in the complex plane a mixed Hodge structure, called the limit mixed Hodge structure. In Section 2.2, using Log Geometry, we extend the Steenbrink–Schmid construction to smooth projective varieties over $\mathbb{C}((t))$.

1.3. Motivic integral of K3 surfaces over $\mathbb{C}((t))$

In order to state our first main result we need to introduce a bit of notation. Let X be a smooth projective K3 surface over $K = \mathbb{C}((t))$ and let $H^2(\lim X) = (H^2(\lim X, \mathbb{Z}), W_i^{\mathbb{Q}}, F^i)$ be the corresponding limit mixed Hodge structure (see Section 2.2). Assume that the monodromy acts on $H^2(\lim X, \mathbb{Z})$ by a unipotent operator. Then its logarithm N is known to be integral [19, Proposition 1.2]:

$$N : H^2(\lim X, \mathbb{Z}) \to H^2(\lim X, \mathbb{Z}). \tag{1.4}$$

Set $W_i^{\mathbb{Z}} = W_i^{\mathbb{Q}} \cap H^2(\lim X, \mathbb{Z})$. The morphisms

$$GrN^i : W_{i+2}^{\mathbb{Z}} / W_{i+1}^{\mathbb{Z}} \to W_{2-i}^{\mathbb{Z}} / W_{1-i}^{\mathbb{Z}}, \quad i = 1, 2, \tag{1.5}$$

are injective and have finite cokernels. Let $r_i(X, K)$ be their orders. In Section 3 we prove the following result.

Theorem 1. *Let X be a smooth projective K3 surface over $K = \mathbb{C}((t))$. Assume that X has a strictly semi-stable model over $R = \mathbb{C}[[t]]$ and that the operator N is not equal to 0. Let s be the smallest integer such that $N^s = 0$. Then s is either 2 or 3 and for every finite extension $K_e \supset K$ of degree e the motivic integral of the K3 surface $X_e = X \otimes_K K_e$ over K_e is given by the following formulas.*

(a) *If $s = 2$ then*

$$\int_{X_e} = 2\mathbb{Z}(0) - \left(e\sqrt{r_1(X, K)} + 1\right)\left[E(X)\right] + 20\mathbb{Z}(-1)$$

$$+ \left(e\sqrt{r_1(X, K)} - 1\right)\left[E(X)\right](-1) + 2\mathbb{Z}(-2), \tag{1.6}$$

> where $E(X)$ is the elliptic curve defined by the weight 1 Hodge structure on $W_1^{\mathbb{Z}} = W_1^{\mathbb{Q}} \cap$
> $H^2(\lim X, \mathbb{Z})$ and $\mathbb{Z}(n) := [\mathbb{A}^1]^{-n}$, $n \in \mathbb{Z}$.

(b) *If $s = 3$ then*

$$\int_{X_e} = \left(\frac{e^2 r_2(X, K)}{2} + 2 \right) \mathbb{Z}(0) + \left(20 - e^2 r_2(X, K) \right) \mathbb{Z}(-1)$$

$$+ \left(\frac{e^2 r_2(X, K)}{2} + 2 \right) \mathbb{Z}(-2). \tag{1.7}$$

Note, that if $N = 0$ the K3 surface X has a smooth proper model over R whose special fiber Y (and thus the motivic integral) is determined by the polarized pure Hodge structure $H^2(\lim X, \mathbb{Z})$.

Let us explain the idea of our proof assuming that $e = 1$. First, using the theory of Hilbert schemes and Artin's approximation theorem, we reduce the proof to the case when X is obtained by the restriction of a smooth family $\mathcal{X}$ of K3 surfaces over a smooth punctured complex curve $C = \overline{C} - a$ to the formal punctured neighborhood of the point $a \in \overline{C}$. The rest of the proof is based on a result of Kulikov [32,41] asserting the existence of a (non-unique) strictly semi-stable model $\overline{\mathcal{X}} \xrightarrow{\overline{\pi}} \overline{C}$ such that the log canonical bundle $\omega_{\overline{\mathcal{X}}/\overline{C}}(\log)$ is trivial over an open neighborhood of the special fiber Y. For any such model, we have

$$\int_X = [Y_{\mathrm{sm}}],$$

where $Y_{\mathrm{sm}} \subset Y$ is the smooth locus of Y. It is shown in [32] that the special fiber Y of a Kulikov model has a very special form. If $s = 2$ the Clemens polytope $\mathrm{Cl}(Y)$ of Y (see Section 2.1) is a partition of an interval and all but two irreducible components of Y are ruled surfaces fibered over elliptic curves, all of which are isomorphic to a single elliptic curve E. The two components corresponding to the boundary points of $\mathrm{Cl}(Y)$ are rational surfaces. If $s = 3$ then all the irreducible components of Y are rational surfaces and the Clemens polytope $\mathrm{Cl}(Y)$ is a triangulation of a sphere. Next, using results of Friedman and Scattone [19,18] we prove that the Steenbrink weight spectral sequence for Kulikov's model $\overline{\mathcal{X}} \xrightarrow{\overline{\pi}} \overline{C}$ (and therefore by the Weak Factorization Theorem [31, Theorem 9], for every strictly semi-stable model of X) degenerates integrally at the second term. Of course, the degeneration of the weight spectral sequence with rational coefficients is a corollary of Hodge Theory and holds in general, but the degeneration over $\mathbb{Z}$ is a special non-trivial property of K3 surfaces. This, combined with the generalized Picard–Lefschetz formula, implies that, for $s = 2$, the Hodge structure on $H^1(E)$ is isomorphic to that on $W_1^{\mathbb{Z}}$ and that the number of irreducible components of Y equals $\sqrt{r_1(X, K)} + 1$. Similarly, as proven in [19], for $s = 3$, the combinatorics of Y (i.e., the number of irreducible, components, double curves and triple points) is completely determined by the monodromy action on the integral lattice $H^2(\lim X, \mathbb{Z})$. This, together with a variant of A'Campo's formula (Proposition 2.9), completes the proof.

1.4. Monodromy pairing

In Section 4 we introduce a generalization of the invariant $r_2(X, K)$, that we defined in Section 1.3 for K3 surfaces over $\mathbb{C}((t))$, to the case of an arbitrary smooth variety over a complete

discrete valuation field. Our construction is based on the theory of analytic spaces over non-archimedean fields developed by Berkovich [5]. For a complete discrete valuation field K we denote by $\widehat{K}$ the completion of an algebraic closure of K. One of the key features of Berkovich's theory is that the underlying topological space $|X^{\mathrm{an}}_{\widehat{K}}|$ of the analytification of a scheme X over K has interesting topological invariants (in contrast with the space $X(\widehat{K})$ equipped with the usual topology, which is totally disconnected). In particular, if X is the generic fiber of a proper strictly semi-stable scheme $\overline{X}$ over R the space $|X^{\mathrm{an}}_{\widehat{K}}|$ is homotopy equivalent to the Clemens polytope of the special fiber Y. We denote by $\Gamma^m_C(X)$ the singular cohomology of the space $|X^{\mathrm{an}}_{\widehat{K}}|$ with coefficients in a ring C. In Theorem 3, we prove that, for every prime ℓ different from the characteristic of the residue field of K, and for every smooth scheme X, there is a canonical isomorphism of $\mathrm{Gal}(\overline{K}/K)$-modules

$$\gamma : \Gamma^m_{\mathbb{Q}_\ell}(X) \xrightarrow{\sim} \mathrm{Im}\big(H^m(X_{\overline{K}}, \mathbb{Q}_\ell)(m) \xrightarrow{N^m} H^m(X_{\overline{K}}, \mathbb{Q}_\ell)\big), \tag{1.8}$$

where N is the logarithm of the monodromy operator. In particular, the dimension of the vector space on the right-hand side of (1.8) is independent of ℓ. Let us note, that a different description of the space $\Gamma^m_{\mathbb{Q}_\ell}(X)$ in the case of finite residue field was obtained earlier by Berkovich [8].

If d is the dimension of X, we use (1.8) to define a non-degenerate pairing

$$\Gamma^d_{\mathbb{Q}}(X) \otimes \Gamma^d_{\mathbb{Q}}(X) \to \mathbb{Q}. \tag{1.9}$$

In the special case when X is proper, the pairing (1.8) is given by the formula

$$(x, y) = (-1)^{\frac{d(d-1)}{2}} \langle \gamma(x), y' \rangle, \tag{1.10}$$

where $y' \in H^d(X_{\overline{K}}, \mathbb{Q}_\ell)$ is an element such that $N^d y' = \gamma(y)$ and $\langle , \rangle$ is the Poincaré pairing on $H^d(X_{\overline{K}}, \mathbb{Q}_\ell)$. We prove in Theorem 4 that (1.10) is independent of ℓ and positive. Moreover, the groups $\Gamma^m_C(X)$ and the monodromy pairing (1.9) are birational invariants of X.

We define a numeric (birational) invariant $r_d(X, K)$ of X to be the discriminant of the dual pairing

$$\Gamma_d(X) \otimes \Gamma_d(X) \to \mathbb{Q}, \tag{1.11}$$

where $\Gamma_d(X)$ is $\mathrm{Hom}(\Gamma^d_{\mathbb{Z}}(X), \mathbb{Z})$.

In Remark 4.5, we define for a polarized projective variety X and any integer m a more general positive pairing $\Gamma^m_{\mathbb{Q}}(X) \otimes \Gamma^m_{\mathbb{Q}}(X) \to \mathbb{Q}$ which in the case of semi-stable abelian variety A and its dual A' boils down, after some identifications, to the monodromy pairing $\Gamma_1(A) \otimes \Gamma_1(A') \to \mathbb{Z}$ defined by Grothendieck [49, Exp. IX]. In particular, the number $r_d(A, K)$ is non-zero if and only if A is completely degenerate in which case $r_d(A, K)$ is equal to $d!|\pi_0(\mathcal{V}(A) \otimes \bar{k})|$, where $\mathcal{V}(A)$ is the Néron model of A.

1.5. Motivic integral of maximally degenerate K3 surfaces

We say that a d-dimensional Calabi–Yau variety over a complete discrete valuation field K is maximally degenerate if $\Gamma_{\mathbb{Q}}^d(X) \neq 0$. According to (1.8), X is maximally degenerate if and only if for some (and, hence, for any) prime $\ell \neq \operatorname{char} k$ the map

$$H^d(X_{\overline{K}}, \mathbb{Q}_\ell)(m) \xrightarrow{N^d} H^d(X_{\overline{K}}, \mathbb{Q}_\ell)$$

is not zero.[2] We conjecture that for every maximally degenerate K3 surface over K there exists a finite extension $K' \supset K$ such that, for every finite extension $L \supset K$ of ramification index e containing K', we have

$$\int_{X_L} = \left(\frac{e^2 r_2(X, K)}{2} + 2\right)\mathbb{Q}(0) + \left(20 - e^2 r_2(X, K)\right)\mathbb{Q}(-1) + \left(\frac{e^2 r_2(X, K)}{2} + 2\right)\mathbb{Q}(-2).$$

If $\operatorname{char} k = 0$ our conjecture follows from part (b) of Theorem 1. In Section 5 we prove this conjecture in the case of Kummer K3 surfaces over an arbitrary complete discrete valuation field K with $\operatorname{char} k \neq 2$ by constructing explicitly a poly-stable formal model of the analytic space X^{an}.

The groups $\Gamma_{\mathbb{Z}}^d(X)$ that we used to define the invariant $r_d(X, K)$ can be interpreted as the weight 0 part of the limit motive of X (Remark 4.2). It would be interesting to define geometrically the limit 1-motive attached to X and use it to compute the motivic integral for K3 surfaces which are not maximally degenerate.

2. Preliminaries

2.1. Clemens polytope and nerve of a strictly semi-stable scheme

Let R be a complete discrete valuation ring with residue field k and fraction field K. Recall that a scheme $\overline{X}$ of finite type over $\operatorname{spec} R$ is strictly semi-stable if every point $x \in \overline{X}$ has a Zariski neighborhood $x \in U \subset \overline{X}$ such that the morphism $U \to \operatorname{spec} R$ factors through an étale morphism

$$U \to \operatorname{spec} R[T_0, \ldots, T_d]/(T_0 \cdots T_r - t), \quad 0 \leqslant r \leqslant d,$$

for a uniformizer t of K. If k is perfect, $\overline{X}$ is a strictly semi-stable scheme if and only if it is regular and flat over R, the generic fiber $X = \overline{X} \times_R K$ is smooth over K and the special fiber $Y = \overline{X} \times_R k$ is a reduced strictly normal crossing divisor on $\overline{X}$.

Let $\overline{X}$ be a strictly semi-stable scheme. Then the irreducible components $V_1, \ldots, V_m$ of Y as well as the schemes

$$Y^{(q)} = \coprod_{i_0 < \cdots < i_q} V_{i_0} \cap \cdots \cap V_{i_q} \tag{2.1}$$

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[2] There is an extensive literature on maximally degenerate Calabi–Yau varieties over $\mathbb{C}((t))$. See e.g. [35,33].

are smooth. It is convenient to encode the combinatorial structure of Y by a certain topological space. To do this we need to introduce some terminology.

By an abstract triangulated set we mean a contravariant functor $\tilde{\Delta} \to$ Sets, where $\tilde{\Delta}$ is the category whose objects are finite totally ordered sets $[q] := \{0, \ldots, q\}$, $q \in \mathbb{Z}_{\geqslant 0}$, and whose morphisms are strictly increasing maps. Thus, giving an abstract triangulated set $S_\bullet$ amounts to giving a set S_q of "q-simplices" for each $q \in \mathbb{Z}_{\geqslant 0}$ together with "boundary maps" $\delta_j : S_q \to S_{q-1}$, $j = 0, \ldots, q$, subject to certain conditions.[3] We shall write $|S_\bullet|$ for the realization of $S_\bullet$ [20, §1.1].

Given a strictly semi-stable scheme $\overline{X}$ consider the abstract triangulated set whose q-dimensional simplices are indexed by the set $\pi_0(Y_{\bar{k}}^{(q)})$. The boundary maps $\delta_j : \pi_0(Y_{\bar{k}}^{(q)}) \to \pi_0(Y_{\bar{k}}^{(q-1)})$, $j = 0, \ldots, q$, are given by the maps

$$\pi_0(V_{i_0,\bar{k}} \cap \cdots \cap V_{i_q,\bar{k}}) \to \pi_0(V_{i_0,\bar{k}} \cap \cdots \cap V_{i_{j-1},\bar{k}} \cap V_{i_{j+1},\bar{k}} \cap \cdots \cap V_{i_q,\bar{k}})$$

induced by the injections

$$V_{i_0} \cap \cdots \cap V_{i_q} \hookrightarrow V_{i_0} \cap \cdots \cap V_{i_{j-1}} \cap V_{i_{j+1}} \cap \cdots \cap V_{i_q}.$$

The realization of this triangulated set is a topological space which we call (following [31]) the Clemens polytope of Y and denote by $\mathrm{Cl}(Y)$. Although the abstract triangulated set we constructed depends upon the choice of ordering on the set of irreducible components V_i, the homeomorphism type of the topological space $\mathrm{Cl}(Y)$ does not.

Proposition 2.1. *Let $\overline{X}$ be a strictly semi-stable model of X over $\mathrm{spec}\, R$ with special fiber Y; then for every abelian group C,*

$$H_{\mathrm{sing}}^*\big(\mathrm{Cl}(Y), C\big) \cong H_{\mathrm{Zar}}^*(Y_{\bar{k}}, \underline{C}).$$

Proof. To simplify our notation we assume that $k = \bar{k}$. Consider the complex

$$i_{0*}\underline{C} \xrightarrow{\partial_0} i_{1*}\underline{C} \xrightarrow{\partial_1} \cdots,$$

where $i_q : Y^{(q)} \hookrightarrow Y$. The differentials ∂_i are characterized by the property that the induced map on global sections $\Gamma(i_{q*}\underline{C}) = C[\pi_0(Y^{(q)})] \to C[\pi_0(Y^{(q+1)})] = \Gamma(i_{q+1*}\underline{C})$ equals $\sum_j (-1)^j \delta_j^*$. This complex is a resolution of $\underline{C}$. Since each $Y^{(q)}$ is a disjoint union of smooth irreducibles and constant sheaves on irreducibles are flabby it follows that the sheaves $i_{q*}\underline{C}$ are flabby. Thus the complex of global sections

$$\Gamma(i_{0*}\underline{C}) \xrightarrow{\partial_0} \Gamma(i_{1*}\underline{C}) \xrightarrow{\partial_1} \cdots$$

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[3] The category of abstract triangulated sets can be viewed as a full subcategory of the category of simplicial sets: if $S_\bullet'$ is a simplicial set such that the boundary of each non-degenerate simplex of $S_\bullet'$ is non-degenerate then non-degenerate simplices of $S_\bullet'$ together with the boundary maps form an abstract triangulated set. This yields an equivalence between the full subcategory of the category of simplicial sets whose objects satisfy the above property and the category of abstract triangulated sets [20, §1.6].

computes the Zariski cohomology $H^*_{\mathrm{Zar}}(Y, \underline{C})$. On the other hand, this complex is the simplicial complex of $\mathrm{Cl}(Y)$. $\square$

Assume that $\overline{X}$ is a proper semi-stable scheme over R. Then, by the Proper Base Change theorem (e.g., [15]), for every torsion abelian group C we have canonical morphisms of $\mathrm{Gal}(\overline{K}/K)$-modules

$$H^*_{\mathrm{Zar}}(Y_{\bar{k}}, \underline{C}) \to H^*_{et}(Y_{\bar{k}}, \underline{C}) \cong H^*_{et}(\overline{X}_{R^{sh}}, \underline{C}) \to H^*_{et}(X_{\overline{K}}, \underline{C}), \tag{2.2}$$

where R^{sh} denotes a strict henselization of R. Applying (2.2) to $C = \mathbb{Z}/\ell^n\mathbb{Z}$ and passing to the limit, we obtain a canonical morphism

$$H^*_{\mathrm{sing}}\big(\mathrm{Cl}(Y), \mathbb{Z}_\ell\big) \to H^*(X_{\overline{K}}, \mathbb{Z}_\ell). \tag{2.3}$$

We will see in Section 4 that the groups $H^*_{\mathrm{sing}}(\mathrm{Cl}(Y), C)$ and the morphism (2.3) depend only on the generic fiber X and not on the choice of proper strictly semi-stable model $\overline{X}$.

Remark 2.2. Let us explain the relation of the notion of Clemens polytope to a more general notion of nerve of a scheme, introduced in [7]. For a reduced scheme Y over k, let $\mathrm{Nor}(Y) \subset Y$ be the normal locus of Y, which is an open subset of Y, and let $Y^{[0]} = Y$, $Y^{[i+1]} = Y^{[i]} \setminus \mathrm{Nor}(Y^{[i]})$, $i \geqslant 0$. The irreducible components of $Y^{[i]} \setminus Y^{[i+1]}$ are called strata of Y. The set, $\mathrm{Str}(Y)$, of all strata has a natural partial order: for strata $x, y \in \mathrm{Str}(Y)$, we say that $x \leqslant y$ if y is contained in the closure of x. We denote by $N(Y)$ the nerve of the partially ordered set $\mathrm{Str}(Y)$. If $\overline{X}$ is a strictly semi-stable scheme over R, the triangulated space $|N(Y \otimes \bar{k})|$ is obtained from $\mathrm{Cl}(Y)$ by subdivision. In particular, the spaces $|N(Y \otimes \bar{k})|$ are $\mathrm{Cl}(Y)$ homeomorphic.

2.2. The limit mixed Hodge structure associated with a variety over $\mathbb{C}((t))$

In [45], Steenbrink associated with every smooth projective variety over the field K_{mer} of meromorphic functions on an open neighborhood of zero in the complex plane a mixed Hodge structure, called the limit mixed Hodge structure. Another construction of the same mixed Hodge structure had been given earlier by Schmid [44]. In this section, we explain how to extend the Steenbrink–Schmid construction to smooth projective varieties over the field of formal Laurent series $K = \mathbb{C}((t))$. A rough idea: generalizing a construction by Steenbrink [47] we attach a mixed Hodge structure to every projective normal crossing (not necessarily reduced) log scheme over the log point. Applying this construction to the special fiber Y of a normal crossing model $\overline{X}$ of X over $R = \mathbb{C}[[t]]$ we get our $H^m(\lim X)$. We then prove independence of the choice of a model and functoriality.

We shall summarize the properties of our construction in the following theorem.

Theorem 2. *For every non-negative integer m, there exists a contravariant functor*

$$SmPr_K \to \widetilde{MHS},$$
$$X \rightsquigarrow H^m(\lim X) = \big(H^m(\lim X, \mathbb{Z}), W_i^{\mathbb{Q}}, F_i, T\big) \tag{2.4}$$

from the category of smooth projective varieties over $K = \mathbb{C}((t))$ to the category of mixed Hodge structures equipped with an endomorphism T of the underlying abelian group with the following properties.

(a) *If we write $T_{\mathbb{Q}} = SU$ for the factorization of the endomorphism $T_{\mathbb{Q}} \in \operatorname{End}(H^m(\lim X, \mathbb{Q}))$ into the product of semi-simple and unipotent endomorphisms, S and U respectively, such that $ST_{\mathbb{Q}} = T_{\mathbb{Q}}S$ and $UT_{\mathbb{Q}} = T_{\mathbb{Q}}U$, then $N = \log U$ is a morphism of rational mixed Hodge structures*

$$N : H^m(\lim X) \otimes \mathbb{Q} \to H^m(\lim X) \otimes \mathbb{Q}(-1)$$

and S is a finite order automorphism of $H^m(\lim X) \otimes \mathbb{Q}$.

(b) *The functor (2.4) is compatible with base change. That is, if $K_e = \mathbb{C}((t^{\frac{1}{e}})) \supset K$ is a finite extension and $X \rightsquigarrow X_{K_e}$ is the base change functor, we have a functorial isomorphism*

$$\left(H^m(\lim X, \mathbb{Z}), W_i^{\mathbb{Q}}, F^i, T^e \right) \simeq \left(H^m(\lim X_{K_e}, \mathbb{Z}), W_i^{\mathbb{Q}}, F^i, T \right).$$

(c) *If $\overline{X}$ is a strictly semi-stable scheme over $R = \mathbb{C}[\![t]\!]$, X and Y are the generic and special fibers of $\overline{X}$ respectively, and $Y^{(q)} \hookrightarrow Y$ is the closed subscheme defined in (2.1), one has the weight spectral sequence $E_r^{pq}(\overline{X})$ which converges to $H^*(\lim X)$ in the category of mixed $\mathbb{Z}$-Hodge structures with the first term given by the formula:*

$$E_1^{pq}(\overline{X}) = \bigoplus_{i, i-p \geqslant 0} H^{q+2p-2i}\left(Y^{(2i-p)}\right)(p - i).$$

The sequence $E_r^{pq}(\overline{X}) \otimes \mathbb{Q}$ degenerates at E_2 terms.

(d) *If $\mathcal{X}$ is a smooth projective variety over K_{mer} the limit mixed Hodge structure $H^m(\lim(\mathcal{X} \otimes_{K_{\mathrm{mer}}} K))$ is canonically isomorphic to the one constructed by Schmid and Steenbrink [44–46].*

Proof. Let

$$(\overline{X}, M_{\overline{X}}) \to (\operatorname{spec} R, M_R = R - 0)$$

be a proper smooth morphism of fine and saturated (fs for short) log schemes [25, §1]. Assume that the log structure on $(\overline{X}, M_{\overline{X}})$ is vertical, i.e., the induced log structure on $j : X = \overline{X} \otimes_R K \hookrightarrow \overline{X}$ is trivial. A basic example of this situation is a regular proper R-scheme $\overline{X}$ such that its reduced special fiber Y_{red} is a normal crossing divisor on $\overline{X}$ endowed with the log structure

$$M_{\overline{X}} = j_* \mathcal{O}_X^* \cap \mathcal{O}_{\overline{X}}. \tag{2.5}$$

The special fiber $Y = \overline{X} \otimes_R \mathbb{C}$ with the induced log structure is a proper smooth log scheme over the log point

$$\pi : (Y, M_Y) \to (\operatorname{spec} \mathbb{C})_{\log}.$$

Following [28, §1] we consider the associated map of topological spaces

$$\pi : Y^{\log} \to (\operatorname{spec} \mathbb{C})^{\log} = S^1,$$

where $S^1 \subset \mathbb{C}$ is the unit circle. The map π is a locally trivial fibration over S^1 [39, Theorem 5.1]. Let $\exp(2\pi i \tau) : \mathbb{R}^1 \to S^1$ be the universal cover, and let $\tilde{Y}^{\log}$ be the fiber product $Y^{\log} \times_{S^1} \mathbb{R}^1$. The topological space $\tilde{Y}^{\log}$ carries a canonical automorphism that takes a point $(y, a) \in Y^{\log} \times_{S^1} \mathbb{R}^1$ to $(y, a + 2\pi i)$. We will write T_Y for the induced automorphism of the cohomology group $H^m(\tilde{Y}^{\log}, \mathbb{Z})$. The following lemma implies that the cohomology of $\tilde{Y}^{\log}$ depends only on the generic fiber of $\overline{X}$.

Lemma 2.3. *Let* $f : (\overline{X}, M_{\overline{X}}) \to (\overline{X}', M_{\overline{X}'})$ *be a log morphism of smooth proper vertical fs log schemes over* $(\operatorname{spec} R, M_R)$. *Assume that the induced morphism of the generic fibers* $f_K : \overline{X} \otimes K \to \overline{X}' \otimes K$ *is an isomorphism. Then, for every non-negative integer m, the morphism*

$$f^* : H^m(\tilde{Y}'^{\log}, \mathbb{Z}) \to H^m(\tilde{Y}^{\log}, \mathbb{Z})$$

is an isomorphism.

Proof. Let n be a positive integer. The comparison theorems of Kato and Nakayama (see, e.g. [25, Theorem 5.9, Corollary 8.4]) imply the existence of the commutative diagram below:

$$
\begin{array}{ccc}
H^m(\tilde{Y}'^{\log}, \mathbb{Z}/n\mathbb{Z}) & \overset{\sim}{\longrightarrow} & H^m_{et}(\overline{X}' \otimes \overline{K}, \mathbb{Z}/n\mathbb{Z}) \\
\Big\downarrow {\scriptstyle f^*} & & \Big\downarrow {\scriptstyle f^*_{\overline{K}}} \\
H^m(\tilde{Y}^{\log}, \mathbb{Z}/n\mathbb{Z}) & \overset{\sim}{\longrightarrow} & H^m_{et}(\overline{X} \otimes \overline{K}, \mathbb{Z}/n\mathbb{Z})
\end{array}
\qquad (2.6)
$$

Since the groups $H^m(\tilde{Y}'^{\log}, \mathbb{Z})$, $H^m(\tilde{Y}^{\log}, \mathbb{Z})$ are finitely generated the lemma follows. $\quad\square$

Let us explain how the formation $(T_Y, H^m(\tilde{Y}^{\log}, \mathbb{Z}))$ is compatible with base change. For a positive integer e, the fs log scheme $(\operatorname{spec} R_e = \operatorname{spec} \mathbb{C}[\![t^{\frac{1}{e}}]\!], M_{R_e} = R_e - 0)$ is smooth over $(\operatorname{spec} R, M_R)$. Let $(\overline{X}_e, M_{\overline{X}_e})$ be the fiber product

$$(\overline{X}, M_{\overline{X}}) \otimes_{(\operatorname{spec} R, M_R)} (\operatorname{spec} R_e, M_{R_e})$$

in the category of fs log schemes.[4] As the functor $(Y, M_Y) \rightsquigarrow Y^{\log}$ commutes with fiber products we have a Cartesian diagram of topological spaces

$$
\begin{array}{ccc}
Y_e^{\log} & \longrightarrow & Y^{\log} \\
\Big\downarrow & & \Big\downarrow \\
S^1 & \longrightarrow & S^1
\end{array}
\qquad (2.7)
$$

---

[4] Warning: the functor that takes a fs log scheme to the underlying scheme does not commute with the fiber products.

where the lower horizontal map is an e-fold cover. We get from (2.7) a canonical isomorphism

$$H^m\big(\tilde{Y}^{\log}, \mathbb{Z}\big) \xrightarrow{\sim} H^m\big(\tilde{Y}_e^{\log}, \mathbb{Z}\big) \tag{2.8}$$

that carries T_{Y_e} to T_Y^e.

Assume, in addition, that the log scheme $\pi : (Y, M_Y) \to (\operatorname{spec} \mathbb{C})_{\log}$ satisfies the following condition:

(U): for every closed point $y \in Y$, the cokernel of the morphism $\pi^* : \mathbb{Z} = K^*/R^* \to (M_Y^{gr}/\mathcal{O}_Y^*)_y$ is torsion free.

In [26, Theorems 6.3 and 7.1], Illusie, Kato and Nakayama proved that under the above assumption the relative log de Rham cohomology $H^m(\overline{X}, \Omega^*_{\overline{X}/R}(\log))$ is a free R-module, the residue of the logarithmic Gauss–Manin connection on $H^m(\overline{X}, \Omega^*_{\overline{X}/R}(\log))$ is nilpotent, the Hodge spectral sequence, defined by the "stupid" filtration $\sigma_{\geqslant \bullet}$ on $\Omega^*_{\overline{X}/R}(\log)$, degenerates at the E_1 term and the Hodge filtration

$$H^m\big(\overline{X}, \sigma_{\geqslant j}\Omega^*_{\overline{X}/R}(\log)\big) \hookrightarrow H^m\big(\overline{X}, \Omega^*_{\overline{X}/R}(\log)\big)$$

splits (i.e. the associated graded R-module is free). Moreover, there is a canonical isomorphism[5]

$$H^m\big(\tilde{Y}^{\log}, \mathbb{C}\big) \simeq H^m\big(Y, \Omega^*_{Y/\mathbb{C}}(\log)\big) \tag{2.9}$$

compatible with the base change $Y \rightsquigarrow Y_e$. Set

$$F^j H^m\big(Y, \Omega^*_{Y/\mathbb{C}}(\log)\big) := H^m\big(Y, \sigma_{\geqslant j}\Omega^*_{Y/\mathbb{C}}(\log)\big) \hookrightarrow H^m\big(Y, \Omega^*_{Y/\mathbb{C}}(\log)\big).$$

As an immediate corollary of the Illusie–Kato–Nakayama results we get the following statement.

Lemma 2.4.

(a) *Let* $f : (\overline{X}, M_{\overline{X}}) \to (\overline{X}', M_{\overline{X}'})$ *be a log morphism of smooth proper vertical fs log schemes over* $(\operatorname{spec} R, M_R)$ *satisfying the condition* (U). *Assume that the induced morphism of generic fibers* $f_K : \overline{X} \otimes K \to \overline{X}' \otimes K$ *is an isomorphism. Then, for every non-negative integer* m, *the morphism*

$$f^* : F^\bullet H^m\big(Y', \Omega^*_{Y'/\mathbb{C}}(\log)\big) \to F^\bullet H^m\big(Y, \Omega^*_{Y/\mathbb{C}}(\log)\big)$$

is a filtered isomorphism.

(b) *For a smooth proper vertical fs log scheme* $(\overline{X}, M_{\overline{X}})$ *satisfying the condition* (U) *and a positive integer* e *the canonical morphism*

$$F^\bullet H^m\big(Y, \Omega^*_{Y/\mathbb{C}}(\log)\big) \to F^\bullet H^m\big(Y_e, \Omega^*_{Y_e/\mathbb{C}}(\log)\big)$$

is a filtered isomorphism.

---

[5] The isomorphism (2.9) depends on the choice of a uniformizer of R. Our choice is t.

Assume, in addition, that $\overline{X}$ is projective. Let

$$W_\bullet = W_\bullet H^m\big(\tilde{Y}^{\log}, \mathbb{Q}\big) \subset H^m\big(\tilde{Y}^{\log}, \mathbb{Q}\big)$$

be the monodromy filtration defined by the nilpotent endomorphism $N_Y = \log T_Y$ of $H^m(\tilde{Y}^{\log}, \mathbb{Q})$:

$$N_Y W_i \subset W_{i-2},$$
$$GrN_Y^i : W_{i+m}^{\mathbb{Q}}/W_{i+m-1}^{\mathbb{Q}} \xrightarrow{\sim} W_{m-i}^{\mathbb{Q}}/W_{m-i-1}^{\mathbb{Q}}.$$

Lemma 2.5. *For every smooth projective vertical fs log scheme* $(\overline{X}, M_{\overline{X}})$ *satisfying the condition* (U) *the triple* $(H^m(\tilde{Y}^{\log}, \mathbb{Z}), W_i H^m(\tilde{Y}^{\log}, \mathbb{Q}), F^j H^m(Y_{\mathrm{an}}, \Omega^*_{Y_{\mathrm{an}}/\mathbb{C}}(\log)))$ *together with the isomorphism* (2.9) *constitutes a mixed Hodge structure.*

Proof. By the semi-stable reduction theorem [30, p. 198] we can find an integer e, a projective strictly semi-stable scheme $\overline{X}'$ over R_e and log morphism

$$f : \big(\overline{X}', M_{\overline{X}'}\big) \to (\overline{X}_e, M_{\overline{X}_e}),$$

where $M_{\overline{X}'}$ is given by (2.5) and f is an isomorphism over the generic point of R_e. Lemmas 2.3 and 2.4 reduce the proof to the case when $\overline{X} = \overline{X}'$. In this case our assertion is proven in [29, pp. 405–406] and [47, §5.6]. $\square$

To construct the functor (2.4) we define an auxiliary subcategory SS_R of the category of schemes over R whose objects are regular projective R-schemes $\overline{X}$ such that the reduced special Y_{red} is a strict normal crossing divisor on $\overline{X}$. Let $\mathcal{S} \subset \mathrm{Mor}(SS_R)$ be the subset that consists of morphisms $f : \overline{X} \to \overline{X}'$ such that $f_K : \overline{X} \otimes K \simeq \overline{X}' \otimes K$.

Lemma 2.6. *The set* $\mathcal{S}$ *is a left multiplicative system in* $\mathrm{Mor}(SS_R)$ [27, §7]. *Moreover, the functor*

$$SS_R \to SmPr_K$$

that takes $\overline{X}$ *to* $\overline{X} \otimes K$ *exhibits the category* $SmPr_K$ *as the localization of* SS_R *by* $\mathcal{S}$.

Proof. The lemma follows from the Hironaka theorem on resolution of singularieties immediately. $\square$

Thus, by the universal property of the localization giving a functor from the category $SmPr_K$ to another category is equivalent to giving a functor from SS_R that takes every morphism in $\mathcal{S}$ to an isomorphism. We define a functor $\Psi : SS_R \to \widetilde{MHS}$ as follows. Let $\overline{X}$ be an object of SS_R, and let $M_{\overline{X}}$ be the canonical log structure given by the formula (2.5). For sufficiently divisible integer e the log scheme (Y_e, M_{Y_e}) satisfies the property (U). We set

$$\Psi(\overline{X}) = \big(H^m\big(\tilde{Y}^{\log}, \mathbb{Z}\big)$$
$$\simeq H^m\big(\tilde{Y}_e^{\log}, \mathbb{Z}\big), W_i H^m\big(\tilde{Y}_e^{\log}, \mathbb{Q}\big), F^j H^m\big(Y_e, \Omega^*_{Y_e/\mathbb{C}}(\log)\big), T_Y\big). \qquad (2.10)$$

The right-hand side of (2.10) is independent of e and is naturally promoted to a contravariant functor $\Psi : SS_R \to \widetilde{MHS}$. By Lemma 2.3 Ψ takes every morphism in $\mathcal{S}$ to an isomorphism. The functor (2.4) is constructed. Let us check the required properties of (2.4).

(a) The only non-trivial statement is that S preserves the Hodge filtration on $H^m(Y_e, \Omega^*_{Y_e/\mathbb{C}}(\log))$. Consider the action of the group $\mathbb{Z}/e\mathbb{Z}$ on the log scheme $(\overline{X}_e, M_{\overline{X}_e})$ induced by the Galois action on R_e. The restriction of this action to $(Y_e, M_{\overline{X}_e})$ yields an action on $H^m(Y_e, \Omega^*_{Y_e/\mathbb{C}}(\log))$. One easily checks that the action of the generator $1 \in \mathbb{Z}/e\mathbb{Z}$ on $H^m(Y_e, \Omega^*_{Y_e/\mathbb{C}}(\log))$ equals S. The compatibility with the Hodge filtration follows immediately.

(b) This follows from (2.8) and Lemma 2.4.

(c) Denote by $Y_{\mathrm{an}} = Y(\mathbb{C})$ the analytic space associated with Y. Let $\tilde{p} : \tilde{Y}^{\log} = Y^{\log} \times_{S^1} \mathbb{R}^1 \to Y_{\mathrm{an}}$ be the composition of the projection to the first factor and the canonical map $p : Y^{\log} \to Y_{\mathrm{an}}$. The complex $R\tilde{p}_* \mathbb{Z} \in D^b(Sh(Y_{\mathrm{an}}))$ has a canonical automorphism T_Y induced by the automorphism of the space $\tilde{Y}^{\log}$. In [29, pp. 405–406], Kawamata and Namikawa put a weight filtration on the complex $R\tilde{p}_* \mathbb{Q}$ and proved that this filtration yields the required spectral sequence with rational coefficients. Thus, we just need to lift the Kawamata–Namikawa filtration to $R\tilde{p}_* \mathbb{Z}$. The required canonical lifting is provided by the following result.

Lemma 2.7. *(Cf. [43, Proposition 2.7].) Assume that $\overline{X}$ is a strictly semi-stable scheme over R of relative dimension d.*

(a) *The complex $R\tilde{p}_* \mathbb{Z}$ is a $(-d)$-shifted perverse sheaf on Y (i.e., $R\tilde{p}_* \mathbb{Z}[d]$ is a perverse sheaf). Moreover, the canonical filtration $\tau_{\leqslant i} R\tilde{p}_* \mathbb{Z}$ is a filtration by $(-d)$-shifted perverse subsheaves and it coincides with the filtration on $R\tilde{p}_* \mathbb{Z}$ by kernels of $(T - 1)^{i+1}$ (computed in the abelian category of $(-d)$-shifted perverse sheaves):*

$$\tau_{\leqslant i} R\tilde{p}_* \mathbb{Z} = \mathrm{Ker}\big((T - 1)^{i+1} : R\tilde{p}_* \mathbb{Z} \to R\tilde{p}_* \mathbb{Z}\big).$$

In particular, $(T - 1)^{d+1}$ is 0 on $R\tilde{p}_* \mathbb{Z}$.

(b) *Let $0 \subset W_{-d} R\tilde{p}_* \mathbb{Z} \subset \cdots \subset W_i R\tilde{p}_* \mathbb{Z} \subset W_d R\tilde{p}_* \mathbb{Z} = R\tilde{p}_* \mathbb{Z}$ be the monodromy filtration on $R\tilde{p}_* \mathbb{Z}$ viewed as an object of the abelian category of $(-d)$-shifted perverse sheaves equipped with the nilpotent endomorphism $T - 1$. Then, for every integer r, we have an isomorphism*

$$Gr_r^W R\tilde{p}_* \mathbb{Z} \simeq \bigoplus_{\substack{i-j=r \\ i,j \geqslant 0}} \bigwedge^{i+j+1} \big(M_{Y_{\mathrm{an}}}^{gr}/\mathcal{O}^*_{Y_{\mathrm{an}}}\big) \simeq \bigoplus_{i-j=r} a_{i+j*} \mathbb{Z}[-i-j],$$

where a_q denotes the embedding $Y^{(q)} \hookrightarrow Y$. The first isomorphism is canonical, the second one depends on the order of the set of irreducible components of Y.

(c) *The Verdier dual complex $\mathcal{D}_Y(R\tilde{p}_* \mathbb{Z})$ is quasi-isomorphic to $R\tilde{p}_* \mathbb{Z}[2 \dim X]$.*

Proof. For the first statement it suffices to prove that, for every prime number ℓ, the complex $\mathbb{Z}_\ell \otimes R\tilde{p}_* \mathbb{Z} = R\tilde{p}_* \mathbb{Z}_\ell$ has the corresponding properties. According to the comparison results of Kato and Nakayama (see, e.g. [25, Theorem 5.9, Corollary 8.4]) the complex $R\tilde{p}_* \mathbb{Z}_\ell$ is quasi-isomorphic to the complex of nearby cycles $R\Psi \mathbb{Z}_\ell$ computed using the étale topology. The results for $R\Psi \mathbb{Z}_\ell$ are proven in [43, Lemma 2.5 and Corollary 2.6]. The proof of the second statement is parallel to the proof of the analogous result for $R\Psi \mathbb{Z}_\ell$ [43, Proposition 2.7].

For the last statement of the lemma observe that $R\tilde{p}_*\mathbb{Z}$ is quasi-isomorphic to $Rp_{1*}\mathbb{Z}$, where $p_1 : Y_1^{\log} \to Y_{an}$ is the restriction of the map $Y^{\log} \to Y_{an} \times S^1$ to the fiber over $Y_{an} \times \{1\}$. As the map p_1 is proper, we have

$$\mathcal{D}_Y(Rp_{1*}\mathbb{Z}) \simeq Rp_{1*}\mathcal{D}_{Y_1^{\log}}\mathbb{Z}.$$

Finally, a simple local computation shows that $\mathcal{D}_{Y_1^{\log}}\mathbb{Z} \simeq \mathbb{Z}[2\dim X]$. $\quad\square$

(d) The last assertion of Theorem 2 is proven in [26, Theorem 8.3], [46, Appendix]. The proof of Theorem 2 is now completed. $\quad\square$

Remark 2.8. We expect that the functor (2.4) extends to the category of smooth quasi-compact rigid analytic varieties over $\mathbb{C}((t))$ (cf. [3]).

We finish this subsection by recalling a variant of the Picard–Lefschetz formula for semi-stable degenerations. Let $\overline{X}$ be a projective strictly semi-stable scheme over R of relative dimension d, and let Y be its special fiber. The simplicial complex that computes the homology of the Clemens polytope $\mathrm{Cl}(Y)$ coincides with the complex

$$E_1^{-d,2d}(\overline{X})(d) \to E_1^{-d+1,2d}(\overline{X})(d) \to \cdots \to E_1^{0,2d}(\overline{X})(d),$$

where $E_r^{pq}(\overline{X})$ is the weight spectral sequence from Theorem 2. From this we get a canonical morphism

$$H^m(\lim X, \mathbb{Z}) \to E_2^{-d+m,2d}(\overline{X})(d) \simeq H_m(\mathrm{Cl}(Y)). \tag{2.11}$$

As the weight spectral sequence degenerates rationally in E_2 terms (2.11) yields an isomorphism

$$H_m(\mathrm{Cl}(Y)) \otimes \mathbb{Q} \overset{\gamma}{\simeq} Gr_{2d}^{W\mathbb{Q}} H^m(\lim X)(d). \tag{2.12}$$

We apply this to $m = d$. If

$$\langle,\rangle : W_0^{\mathbb{Q}} H^d(\lim X) \otimes Gr_{2d}^{W\mathbb{Q}} H^d(\lim X)(d) \to \mathbb{Q}$$

denotes the pairing induced by Poincaré duality (Lemma 2.7(c)) then, for every

$$x = \sum_{v \in \pi_0(Y^{(d)})} a_v v, \qquad y = \sum_{v \in \pi_0(Y^{(d)})} b_v v \in H_d(\mathrm{Cl}(Y)) \otimes \mathbb{Q},$$

we have

$$(-1)^{\frac{d(d-1)}{2}} \langle GrN^d \gamma(x), \gamma(y) \rangle = \sum_v a_v b_v. \tag{2.13}$$

This follows from compatibility of the weight spectral sequence with Poincaré duality and the monodromy action [43, Corollary 2.6 and Proposition 2.15].

2.3. Motivic Serre invariant

Let R be a complete discrete valuation ring with perfect residue field k and fraction field K. The motivic Serre invariant of a smooth proper variety X over K is the class of the special fiber V^0 of a weak Néron model $\mathcal{V}$ of X in the quotient ring

$$K_0(\mathrm{Var}_k)_{\mathrm{loc}} \to K_0(\mathrm{Var}_k)_{\mathrm{loc}}/\big(\mathbb{Z}(1) - \mathbb{Z}\big).$$

It is shown in [34, Theorem 4.5.1] that the motivic Serre invariant $S(X)$ is well defined, i.e., independent of the choice of $\mathcal{V}$. If X is a Calabi–Yau variety $S(X)$ equals the image of the motivic integral $\int_X$ in the quotient ring.

Let $K = \mathbb{C}((t))$. In the following proposition, which is a refinement of A'Campo's formula for the Euler characteristic of the motivic integral,[6] we denote by $S^H(X)$ the image of $S(X)$ under the ring homomorphim

$$K_0(\mathrm{Var}_{\mathbb{C}})_{\mathrm{loc}}/\big(\mathbb{Z}(1) - \mathbb{Z}\big) \to K_0(MHS)/\big(\mathbb{Z}(1) - \mathbb{Z}\big) \tag{2.14}$$

that takes the class of a variety Z to the virtual mixed Hodge structure $\sum (-1)^i [H_c^i(Z, \mathbb{Z})]$.

Proposition 2.9. *Let X be a smooth projective variety over $\mathbb{C}((t))$. Assume that X has a projective strictly semi-stable model $\overline{X}$ over $\mathbb{C}[[t]]$. Then $S^H(X)$ is equal to the class of $\sum (-1)^i [H^i(\lim X)]$.*

Proof. We start with the following general (and well known) observation.

Lemma 2.10. *Let R be a complete discrete valuation ring with perfect residue field k and fraction field K, and let $\overline{X}$ be a proper flat scheme over R. Assume that $\overline{X}$ is regular and that the generic fiber $X = \overline{X} \otimes_R K$ is smooth over K. Then the smooth locus $\overline{X}_{\mathrm{sm}}$ of the morphism $\overline{X} \to \mathrm{spec}\, R$ is a weak Néron model of X.*

Proof. Since X is smooth we have that $\overline{X}_{\mathrm{sm}} \otimes_R K = X$. Let $R' \supset R$ be a finite unramified extension with fraction field K'. We need to show that every morphism $x : \mathrm{spec}\, K' \to X$ extends to an R-morphism $\bar{x} : \mathrm{spec}\, R' \to \overline{X}_{\mathrm{sm}}$. As $\overline{X}$ is proper over R, x extends to an R-morphism $\bar{x} : \mathrm{spec}\, R' \to \overline{X}$. We claim that $\bar{x}$ takes the closed point of $\mathrm{spec}\, R'$ to a smooth point, y, of the special fiber $Y = \overline{X} \otimes_R k$. Since k is perfect, it suffices to check that y is a regular point of Y [21, II, Corolalry 5.3]. Indeed, let $\mathcal{O}_{\overline{X},y}$ (resp. $\mathcal{O}_{Y,y}$) be the local ring of $\overline{X}$ (resp. Y) at y and let $m_{\overline{X},y} \subset \mathcal{O}_{\overline{X},y}$ (resp. $m_{Y,y} \subset \mathcal{O}_{Y,y}$) be the maximal ideal. We have a surjective morphism

$$m_{\overline{X},y}/m_{\overline{X},y}^2 \twoheadrightarrow m_{Y,y}/m_{Y,y}^2 \tag{2.15}$$

of finite-dimensional vector spaces over $\mathcal{O}_{\overline{X},y}/m_{\overline{X},y}$. Let us show that the image in $m_{\overline{X},y}/m_{\overline{X},y}^2$ of a uniformizer $t \in R$ is not equal to 0. Indeed, we have a morphism $\mathcal{O}_{\overline{X},y} \xrightarrow{\bar{x}^*} R'$ induced by $\bar{x}$ such that the composition $R \to \mathcal{O}_{\overline{X},y} \xrightarrow{\bar{x}^*} R'$ is the identity morphism. Since K' is unramified over K, t is also a uniformizer for R'. Therefore, t does not belong to $m_{\overline{X},y}^2$. We proved that

---

[6] Related results were obtained by Nicaise [40].

the image of t in $m_{\overline{X},y}/m_{\overline{X},y}^2$ is not 0. On the other hand, its image in $m_{Y,y}/m_{Y,y}^2$ is 0. Hence, morphism (2.15) is not injective and, therefore,

$$\dim m_{\overline{X},y}/m_{\overline{X},y}^2 > \dim m_{Y,y}/m_{Y,y}^2.$$

On the other hand, since $\overline{X}$ is regular, we have that $\dim m_{\overline{X},y}/m_{\overline{X},y}^2$ equals the Krull dimension of $\mathcal{O}_{\overline{X},y}$. Thus, $\dim m_{Y,y}/m_{Y,y}^2 \leqslant \dim \mathcal{O}_{\overline{X},y} - 1 = \dim \mathcal{O}_{Y,y}$. Hence, Y is regular and, therefore, smooth at point y. It follows that the map $\overline{x} : \operatorname{spec} R' \to \overline{X}$ factors through $\overline{X}_{sm} \subset \overline{X}$. $\quad\square$

We now come back to the proof of Proposition 2.9. According to the above lemma the smooth locus $\mathcal{V}$ of $\overline{X}$ is a weak Néron model of X. Using notation of (2.1) and the inclusion–exclusion formula we find

$$\left[V^0\right] = \sum_{j=0}^{\dim X} \left((-1)^j (j+1)\left[Y^{(j)}\right]\right).$$

On the other hand, by part (c) of Theorem 2 the class $\sum(-1)^i[H^i(\lim X)]$ is equal to the image under (2.14) of the class

$$[\lim X] = \sum_{j=0}^{\dim X} \left((-1)^j \left[Y^{(j)}\right] \sum_{a=0}^{j} \mathbb{Z}(-a)\right).$$

Comparing the two formulas we complete the proof of Proposition 2.9. $\quad\square$

Let $\chi : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$ be the ring homomorphism defined by

$$\chi\left([Z]\right) = \sum(-1)^i \dim H_c^i(Z, \mathbb{C}).$$

Notice that since $\chi(\mathbb{Z}(1) - \mathbb{Z}) = 0$, χ factors uniquely through $K_0(\mathrm{Var}_{\mathbb{C}})_{\mathrm{loc}}/(\mathbb{Z}(1) - \mathbb{Z})$. We have the following corollary of Proposition 2.9.

Corollary 2.11. *(Cf. A'Campo [1].) Let X be a smooth projective variety over $K = \mathbb{C}((t))$. Assume that X has a projective strictly semi-stable model $\overline{X}$ over $\mathbb{C}[[t]]$. Then*

$$\chi\left(S(X)\right) = \sum(-1)^i \dim H^i(\lim X, \mathbb{C}).$$

In the rest of this subsection, we explain an analogue of the above proposition for the finite residue field case. Let K be a local field with residue field $k = \mathbb{F}_q$, and let

$$K_0(\mathrm{Var}_{\mathbb{F}_q})_{\mathrm{loc}}/\left(\mathbb{Z}(1) - \mathbb{Z}\right) \to \mathbb{Z}/(q-1)$$

be the homomorphism induced by (1.3). The image of $S(X)$ in $\mathbb{Z}/(q-1)$ is the classical Serre invariant which we denote by $S^q(X)$.

Proposition 2.12. *Let X be a smooth proper variety over K. Assume that X has a proper strictly semi-stable model over the ring of integers R. Then the Serre invariant of X is given by the formula*

$$\sum_j \mathrm{Tr}\big(F^{-1}, H^j(X_{\overline{K}}, \mathbb{Q}_\ell)\big) \tag{2.16}$$

where $F \in \mathrm{Gal}(\overline{K}/K)$ is a lifting of the Frobenius automorphism $Fr \in \mathrm{Gal}(\overline{k}/k)$ and ℓ is a prime number different from the characteristic of k.

Proof. This can be proved as its Hodge analogue above using the ℓ-adic weight spectral sequence. We give a different proof. Let $\overline{X}$ be a strictly semi-stable model of X. Then the Serre invariant of X equals $|Y_{\mathrm{sm}}(k)|$ modulo $(q-1)$. On the other hand, if $\Psi(\mathbb{Q}_\ell)$ is the complex of nearby cycles (viewed as a complex of ℓ-adic sheaves on Y), by the Grothendieck–Lefschetz formula we have

$$\sum_j (-1)^j \mathrm{Tr}\big(F^{-1}, H^j(X_{\overline{K}}, \mathbb{Q}_\ell)\big)$$

$$= \sum_j (-1)^j \mathrm{Tr}\big(F^{-1}, H^j(Y_{\bar{k}}, \Psi(\mathbb{Q}_\ell))\big)$$

$$= \sum_{y \in Y(k)} \sum_i (-1)^i \mathrm{Tr}\big(F^{-1}, \mathcal{H}^i(\Psi(\mathbb{Q}_\ell))_y\big). \tag{2.17}$$

If $y \in Y_{\mathrm{sm}}(k)$, the corresponding internal sum equals 1. If $y \in Y_{\mathrm{sing}}(k)$ then $\mathcal{H}^i(\Psi(\mathbb{Q}_\ell))_y \simeq \bigwedge^i T(-i)$, where T is a vector space with the trivial action of $\mathrm{Gal}(\overline{K}/K)$ [49, Exposé I, Theorem 3.3]. Thus, for $y \in Y_{\mathrm{sing}}(k)$, we have

$$\sum_i (-1)^i \mathrm{Tr}\big(F^{-1}, \mathcal{H}^i(\Psi(\mathbb{Q}_\ell))_y\big) \equiv \sum_i (-1)^i \dim \bigwedge^i T \equiv 0 \mod (q-1).$$

It follows that the right-hand side of (2.17) is equal to $|Y_{\mathrm{sm}}(k)|$ modulo $(q-1)$ which is the Serre invariant of X. $\square$

3. Motivic integral of K3 surfaces over $\mathbb{C}((t))$

In this section we will prove Theorem 1 stated in the introduction. Without loss of generality we may assume that the ramification index e is equal to 1. Indeed, by Theorem 2 part (b), the formulas (1.6) and (1.7) for the pair $(X/K, e)$ are equivalent to those for the pair $(X_{K_e}/K_e, 1)$. If X admits a strictly semi-stable model over R then X_{K_e} admits a strictly semi-stable model over R_e [43, Lemma 1.11]. We will write r_i for $r_i(X, K)$.
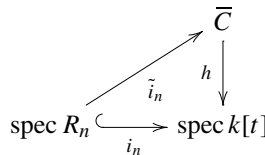
3.1. Approximation of varieties over the formal disk

We will need the following version of Artin's Approximation Theorem.

Proposition 3.1. *Let k be a field of characteristic 0, and let $\overline{X}$ be a projective strictly semi-stable scheme over $R = k[\![t]\!]$. For every positive integer n there exist*

(1) *a smooth curve $\overline{C}$ over k with a point $a \in \overline{C}(k)$,*
(2) *an étale morphism $h : \overline{C} \to \mathbb{A}^1_k = \operatorname{spec} k[t]$ that carries a to 0,*
(3) *a flat projective scheme $\overline{\mathcal{X}}$ over $\overline{C}$,*
(4) *an isomorphism of schemes over $R_n = \operatorname{spec} k[t]/t^{n+1}$:*

$$\overline{X} \times_{\operatorname{spec} R} \operatorname{spec} R_n \simeq \overline{\mathcal{X}} \times_{\overline{C}} \operatorname{spec} R_n.$$

Here $\operatorname{spec} R_n$ is viewed as a scheme over $\overline{C}$ via the unique morphism $\tilde{i}_n : \operatorname{spec} R_n \to \overline{C}$ that carries the point 0 to a and makes the following diagram commutative

$$
\begin{array}{ccc}
 & & \overline{C} \\
 & \tilde{i}_n \nearrow & \downarrow h \\
\operatorname{spec} R_n & \xrightarrow{\ i_n\ } & \operatorname{spec} k[t]
\end{array}
$$

If $\overline{C}, a, h, \overline{\mathcal{X}}$ are as above, the scheme $\overline{\mathcal{X}}$ is regular in an open neighborhood of its special fiber Y' and Y' is a reduced divisor on $\overline{\mathcal{X}}$ with strict normal crossings. In addition, if X is a d-dimensional Calabi–Yau variety the collection $\overline{C}, a, h, \overline{\mathcal{X}}$ can be chosen so that the line bundle $\Omega^d_{\mathcal{X}/C}$ is trivial and

$$\int_X = \int_{\mathcal{X} \times_C \operatorname{spec} K'} . \tag{3.1}$$

Here we set $C = \overline{C} - a$, $\mathcal{X} = \overline{\mathcal{X}} \times_{\overline{C}} C$, and K' denotes the fraction field of the completed local ring $R' = \hat{\mathcal{O}}_{\overline{C},a}$.

Proof. Choose an embedding $\overline{X} \hookrightarrow \mathbb{P}^n_R$ and let $\nu : \operatorname{spec} R \to \operatorname{Hilb}(\mathbb{P}^n_R)$ be the corresponding morphism to the Hilbert scheme. Using Artin's Formal Approximation Theorem (see e.g., [10, §3.6]) on the morphism ν we obtain (1)–(4). Next, we claim that the scheme $\overline{X}' = \overline{\mathcal{X}} \times_{\overline{C}} \operatorname{spec} R'$ is regular. As $\overline{X}'$ is proper over R' and the set of its regular points is open [16, IV, 6.12.5] it suffices to show that the local ring of any point of the special fiber Y' is regular which in turn follows from property (4) and the regularity of $\overline{X}$. Moreover, Y' being isomorphic to the special fiber of a strictly semi-stable scheme $\overline{X}$ is a strict normal crossing divisor on $\overline{X}'$ and on $\overline{\mathcal{X}}$. Note that under our assumption that $\operatorname{char} k = 0$ this implies strict semi-stability of $\overline{X}'$.

Suppose that X is a Calabi–Yau variety. Then the divisor of any non-zero relative log form $\omega \in H^0(\overline{X}, \Omega^d_{\overline{X}/R}(\log))$ is supported on the special fiber Y of $\overline{X}$. Write $\operatorname{div}(\omega) = \sum_i n_i[V_i]$, where V_i are the irreducible components of Y. Assume that the quadruple $\overline{C}, a, h, \overline{\mathcal{X}}$ satisfies properties (1)–(4) with $n \geqslant \sum_i n_i$. To prove the last assertion of the proposition, formula (3.1),

we will show that there exists a section $\omega' \in H^0(\overline{X}', \Omega^d_{\overline{X}'/R'}(\log))$ whose divisor is supported on the special fiber Y' of $\overline{X}'$ and such that via the isomorphism $Y \simeq Y'$ from (4)

$$\operatorname{div}(\omega) = \operatorname{div}(\omega'). \tag{3.2}$$

Indeed, by Lemma 4.1 from [29], for every proper strictly semi-stable scheme $\overline{X}$ over R the R-module $H^0(\overline{X}, \Omega^d_{\overline{X}/R}(\log))$ is free and, in addition, we have

$$H^0\big(\overline{X}, \Omega^d_{\overline{X}/R}(\log)\big) \otimes_R R_n \xrightarrow{\sim} H^0\big(\overline{X} \otimes R_n, \Omega^d_{\overline{X} \otimes R_n/R_n}(\log)\big).$$

Applying this result to $\overline{X}$ and $\overline{X}'$ we find that $H^0(\overline{X}, \Omega^d_{\overline{X}/R}(\log))$ and $H^0(\overline{X}', \Omega^d_{\overline{X}'/R'}(\log))$ are free modules of rank 1 over R and R' respectively and that (4) induces an isomorphism

$$\theta : H^0\big(\overline{X}, \Omega^d_{\overline{X}/R}(\log)\big) \otimes_R R_n \xrightarrow{\sim} H^0\big(\overline{X}', \Omega^d_{\overline{X}'/R'}(\log)\big) \otimes_R R_n.$$

(The R-action on $H^0(\overline{X}', \Omega^d_{\overline{X}'/R'}(\log))$ comes via the isomorphism $R \xrightarrow{\sim} R'$ induced by h.) We claim that a section $\omega' \in H^0(\overline{X}', \Omega^d_{\overline{X}'/R'}(\log))$ such that $\theta(\omega \otimes 1) = \omega' \otimes 1$ does the job. Our claim is local: it suffices to show that, for a closed point $b \in \overline{X}$ and local regular functions $f, g \in \mathcal{O}_{\overline{X},b}$ such that $\operatorname{div}(f)$ is supported on Y, $\sum_i \operatorname{ord}_{V_i} f \leqslant n$, and $f - g \in (t^{n+1})$, one has $\operatorname{div}(f) = \operatorname{div}(g)$. Let x_i be a system of local parameters at b such that $t = x_1 \cdots x_m$. Then, locally around b, we have $f = x_1^{n_1} \cdots x_m^{n_m} u$, where u is invertible and $\sum_i n_i \leqslant n$. If $n_1 > 0$, $g \in f + (t^{n+1})$ is divisible by x_1 and $\frac{f}{x_1} - \frac{g}{x_1} \in (t^n)$. Arguing by induction we see that g is divisible by $x_1^{n_1} \cdots x_m^{n_m}$ and

$$\frac{f}{x_1^{n_1} \cdots x_m^{n_m}} - \frac{g}{x_1^{n_1} \cdots x_m^{n_m}} \in (t).$$

In particular, $g = x_1^{n_1} \cdots x_m^{n_m} u'$ for some invertible u'.

To complete the proof of the proposition let us explain how (3.2) implies (3.1). Suppose that the pair $\overline{X}, \omega' \in H^0(\overline{X}', \Omega^d_{\overline{X}'/R'}(\log))$ is chosen such that Eq. (3.2) holds. Then, in particular, ω' restricts to a non-vanishing differential form on X'. Thus, X' is a Calabi–Yau variety. Secondly, by Lemma 2.10 the schemes $\overline{X}_{\mathrm{sm}}$ and $\overline{X}'_{\mathrm{sm}}$ are weak Néron models of X and X' respectively. Moreover, by property (4) and (3.2) there exists an isomorphism between the special fibers of $\overline{X}_{\mathrm{sm}}$ and $\overline{X}'_{\mathrm{sm}}$ that carries $\operatorname{div}(\omega)$ to $\operatorname{div}(\omega')$. Using (1.2) formula (3.1) follows. $\square$

3.2. Kulikov model

It is enough to prove Theorem 1 in the case where X is the restriction of a strictly semi-stable family over a complex curve. Indeed, apply Proposition 3.1 to a strictly semi-stable model $\overline{X}$ of X. As the limit mixed Hodge structure of a strictly semi-stable scheme depends only on its special fiber together with its log structure which, in turn, is determined by its first infinitesimal neighborhood $\overline{X} \otimes_R R/t^2$, the formulas (1.6), (1.7) for X are equivalent to those for $\mathcal{X} \times_C \operatorname{spec} K'$.

Let X be a K3 surface over K, which is the restriction of a strictly semi-stable family over a complex curve. In [32, Theorem 2], Kulikov demonstrated that X has a projective strictly semi-stable model $\overline{X}$ over R such that the log canonical bundle $\Omega^2_{\overline{X}/R}(\log)$ is trivial and the special fiber Y is of one of the following types (depending on the number s defined in Theorem 1):

(I) ($s = 1$) Y is a smooth K3 surface.
(II) ($s = 2$) Y is a chain of smooth surfaces $V_0, \ldots, V_m$ ruled by elliptic curves, with smooth rational surfaces on either end and each double curve $V_i \cap V_{i+1}$ is a smooth elliptic curve.
(III) ($s = 3$) Y is a union of smooth rational surfaces whose pairwise intersections are smooth rational curves and the Clemens polytope of Y is a triangulation of S^2.

In addition, for $s = 2$, Friedman showed in [18, Theorem 2.2] that a Kulikov model can be chosen so that all the ruled elliptic surfaces in Y are minimal, i.e., $\mathbb{P}^1$-fibrations over an elliptic curve. We shall call such model *special*.

If $\overline{X}$ is a Kulikov model, we have

$$\int_X = [Y_{\mathrm{sm}}]. \tag{3.3}$$

Indeed, by Lemma 2.10 the smooth locus $\overline{X}_{\mathrm{sm}}$ of $\overline{X}$ is a weak Néron model of X. Moreover, since the log canonical bundle $\Omega^2_{\overline{X}/R}(\log)$ is trivial, the bundle $\Omega^2_{\overline{X}_{\mathrm{sm}}/R}$ (which is isomorphic to the restriction of $\Omega^2_{\overline{X}/R}(\log)$ to $\overline{X}_{\mathrm{sm}}$) is also trivial. If $\omega \in \Gamma(X_{\mathrm{sm}}, \Omega^2_{\overline{X}_{\mathrm{sm}}/R})$ is a trivializing section, the numbers m_i appearing in formula (1.1) are all equal to 0. Thus, by formula (1.2) the motivic integral $\int_X$ is equal to the sum of classes of the irreducible components of Y_{sm}. Since Y_{sm} is smooth its irreducible components are pairwise disjoint and, hence, the sum of its classes is equal to $[Y_{\mathrm{sm}}]$.

3.3. Type II degeneration

Suppose that $\overline{X}$ is a type II special Kulikov model. Let $V_0, \ldots, V_m$ be the irreducible components of Y such that V_0 and V_m are rational surfaces, and let $C_i = V_i \cap V_{i+1}$ be the double curves.

Lemma 3.2.

(1) *Let $E_1, \ldots, E_{m-1}$ be ruling elliptic curves for $V_1, \ldots, V_{m-1}$. Then $C_i \cong E_i \cong E_j \cong C_j$ for all i and j.*
(2) *At least one of the rational components, V_0 or V_m, is not minimal.*

Proof. (1) We will first prove that $E_1 \cong E_2$. Let C_1 and C_2 be elliptic curves given by the intersection $V_1 \cap V_2$ and $V_2 \cap V_3$ respectively. We have the following diagram
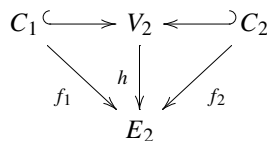
Notice that the maps f_1 and f_2 cannot be constant since this would imply the existence of injections of C_1 and C_2 into rational curves. Thus f_1 and f_2 must be finite. The triviality of the log canonical bundle $\Omega_Y^2(\log)$ implies that for the canonical class K_{V_2} we have $K_{V_2} = -[C_1] - [C_2]$. On the other hand, the restriction of K_{V_2} to a smooth fiber, $h^{-1}(a)$, of the map $h : V_2 \to E_2$ is isomorphic to $K_{h^{-1}(a)}$. As $h^{-1}(a)$ is a smooth rational curve, we have that $\deg(K_{V_2}|_{h^{-1}(a)}) = -2$ which implies the degree of the divisor $-[C_1] - [C_2]$ intersected with the fiber $h^{-1}(a)$ is -2. Hence the images of C_1 and C_2 in V_2 have only one intersection point with a generic fiber which implies f_1 and f_2 are one-to-one and $C_1 \cong E_2 \cong C_2$. We then apply the same method of proof to show that $C_2 \cong E_3 \cong C_3$ and so on.

(2) We claim that for a minimal ruled elliptic surface V_i and two disjoint sections $C_{i-1}, C_i \subset V_i$, we have

$$\left([C_{i-1}]\right)^2_{V_i} = -\left([C_i]\right)^2_{V_i}.$$

Indeed, the Néron–Severi group of V_i is generated by the class $[C_i]$ of C_i and the class $[\mathbb{P}^1]$ of a smooth fiber of the map $V_i \to E_i$. If $[C_{i-1}] = [C_i] + c[\mathbb{P}^1]$, we have

$$0 = \left(c[\mathbb{P}^1]\right)^2_{V_i} = \left([C_{i-1}] - [C_i]\right)^2_{V_i} = \left([C_{i-1}]\right)^2_{V_i} + \left([C_i]\right)^2_{V_i}.$$

On the other hand, since Y is the special fiber of a semi-stable degeneration, we have for every i

$$\left([C_i]\right)^2_{V_i} = -\left([C_i]\right)^2_{V_{i+1}}.$$

Combining the two formulas we see that $([C_0])^2_{V_0} = -([C_{m-1}])^2_{V_m}$. In particular, at least for one of the rational components, say V_0, the self-intersection of the double curve lying on it is non-positive. Thus, $(K_{V_0})^2_{V_0} = (-[C_0])^2_{V_0} \leqslant 0$. Using Noether's formula [4, I.14] it follows that V_0 is not minimal. $\quad\square$

Let E be an elliptic curve such that $E \cong C_i$ for all i. Then we get from (3.3)

$$\int_X = \sum_{i=0}^{m}[V_i] - 2\sum_{i=0}^{m-1}[C_i] = \sum_{i=0}^{m}[V_i] - 2m[E].$$

Since V_0 and V_m are both rational surfaces we have $[V_0] = \mathbb{Z} + a_0\mathbb{Z}(-1) + \mathbb{Z}(-2)$ and $[V_m] = \mathbb{Z} + a_m\mathbb{Z}(-1) + \mathbb{Z}(-2)$. Each V_i for $1 \leqslant i \leqslant m - 1$ is birationally equivalent to $\mathbb{P}^1 \times E$. Thus, by [4, II.11], $[V_i] = [E \times \mathbb{P}^1] + a_i\mathbb{Z}(-1)$ for $1 \leqslant i \leqslant m - 1$. Letting $a = \sum_{i=0}^{m} a_i$ we have

$$\int_X = 2\mathbb{Z} + a\mathbb{Z}(-1) + (m-1)[E] \cdot [\mathbb{P}^1] + 2\mathbb{Z}(-2) - 2m[E]$$

$$= 2\mathbb{Z} + a\mathbb{Z}(-1) + (m-1)[E] + (m-1)[E](-1) + 2\mathbb{Z}(-2) - 2m[E]$$

$$= 2\mathbb{Z} + a\mathbb{Z}(-1) - (m+1)[E] + (m-1)[E](-1) + 2\mathbb{Z}(-2).$$

Using Corollary 2.11 and the fact that the Euler characteristic of a K3 surface is 24 it follows that $a = 20$. Thus we have the formula

$$\int_X = 2\mathbb{Z} - (m+1)[E] + 20\mathbb{Z}(-1) + (m-1)[E](-1) + 2\mathbb{Z}(-2).$$

Now we want to express the number of double curves m and the class of the elliptic curve $[E]$ in terms of the limit mixed Hodge structure $H^2(\lim X)$. First, we show that the *integral* weight spectral sequence E_r^{pq} from Theorem 2 degenerates at the second term. Since it degenerates rationally it will suffice to show that the E_2 terms are torsion free. The non-trivial portion of the first term of the spectral sequence is

$$\bigoplus_{i=0}^{m-1} H^2(C_i)(-1) \xrightarrow{\delta_4} \bigoplus_{i=0}^{m} H^4(V_i),$$

$$\bigoplus_{i=0}^{m-1} H^1(C_i)(-1) \xrightarrow{\delta_3} \bigoplus_{i=1}^{m-1} H^3(V_i),$$

$$\bigoplus_{i=0}^{m-1} H^0(C_i)(-1) \xrightarrow{\delta_2} \bigoplus_{i=0}^{m} H^2(V_i) \xrightarrow{\delta_2'} \bigoplus_{i=0}^{m-1} H^2(C_i),$$

$$\bigoplus_{i=1}^{m-1} H^1(V_i) \xrightarrow{\delta_1} \bigoplus_{i=0}^{m-1} H^1(C_i),$$

$$\bigoplus_{i=0}^{m} H^0(V_i) \xrightarrow{\delta_0} \bigoplus_{i=0}^{m-1} H^0(C_i). \tag{3.4}$$

The first and the last complexes compute (co)homology of the Clemens polytope of Y and, hence, are quasi-isomorphic to $\mathbb{Z}$. Consider the middle complex. The map δ_2 is injective since $\delta_2 \otimes \mathbb{Q}$ is. Let us prove that δ_2' is surjective. For every $(u_0, \ldots, u_m) \in \bigoplus_{i=0}^{m} H^2(V_i)$, we have

$$\delta_2'(u_0, \ldots, u_m) = \big((u_0)|_{C_0} - (u_1)|_{C_0}, \ldots, (u_{m-1})|_{C_{m-1}} - (u_m)|_{C_{m-1}}\big).$$

For every $1 \leqslant i \leqslant m-1$ the restriction morphisms $H^2(V_i) \to H^2(C_i)$, $H^2(V_i) \to H^2(C_{i-1})$ are surjective because V_i is ruled over C_i and over C_{i-1}. By part (2) of Lemma 3.2 one of the rational surfaces, say V_0, is not minimal. If D is a smooth rational -1-curve on V_0, we have

$$-1 = (K_{V_0} \cdot D)_{V_0} = (-C_0 \cdot D)_{V_0}.$$

In particular, the restriction morphism $H^2(V_0) \to H^2(C_0)$ is surjective. Surjectivity of δ_2' follows. Thus, the third complex in (3.4) has non-trivial cohomology only in the middle degree. As the complex is self-dual, the middle cohomology group must be torsion free. Consider the fourth complex. Identifying $H^1(C_i)$ with $H^1(E) =: H$, we find that the fourth complex is isomorphic to

$$H^{\oplus m-1} \xrightarrow{\delta_1} H^{\oplus m}$$

with the differential given by the formula

$$\delta_1(u_1, \ldots, u_{m-1}) = (u_1, u_2 - u_1, \ldots, u_{m-1} - u_{m-2}, -u_{m-1}).$$

In particular, it has non-trivial cohomology group only in a single degree and this group is isomorphic to H. The second complex in (3.4) is dual to the fourth one. This completes the proof of degeneration.

Since the spectral sequence degenerates at E_2 and the E_2 terms are torsion free it follows that $W_1^{\mathbb{Z}} = \text{Coker}(\delta_1) \cong H = H^1(E, \mathbb{Z})$. Thus $W_1^{\mathbb{Z}}$ determines the elliptic curve E.

It remains to prove that $m^2 = r_1$.[7] Indeed, we have the following commutative diagram of abelian groups

$$
\begin{array}{ccccc}
W_3^{\mathbb{Z}} = E_2^{-1,3} \simeq H & \xrightarrow{\Delta} & H^{\oplus m} & \xrightarrow{\delta_3} & H^{\oplus m-1} \\
& & \Big\downarrow{\text{Id}} & \searrow^{N} & \\
H^{\oplus m-1} & \xrightarrow{\delta_1} & H^{\oplus m} & \xrightarrow{\Sigma} & H \simeq E_2^{1,0} = W_1^{\mathbb{Z}}
\end{array}
$$

where Δ is the diagonal map, Σ is the summation map, and δ_3 is given by the formula

$$\delta_3(u_0, \ldots, u_{m-1}) = (u_1 - u_0, \ldots, u_{m-1} - u_{m-2}).$$

It follows that

$$N = \Sigma \circ \Delta = m \, \text{Id},$$

and thus we have

$$r_1 := \left| \text{Coker}\big(W_3^{\mathbb{Z}} \xrightarrow{N} W_1^{\mathbb{Z}}\big) \right| = \left| \text{Coker}(H \xrightarrow{m} H) \right| = m^2.$$

This completes this proof of the theorem for type II degenerations.

3.4. Type III degeneration

Suppose that $\overline{X}$ is a type III Kulikov degeneration. In [19, Proposition 7.1], Friedman and Scattone proved that the number of triple points of Y is equal to r_2. Then since the Clemens polytope of Y is a triangulation of S^2 it follows that the number of double curves in Y is equal to $\frac{3}{2}r_2$ and using Euler's formula for triangulations of a sphere we have that the number of irreducible components of Y equals $\frac{r_2}{2} + 2$. We know that each irreducible component V_i of Y is a smooth rational surface and each C_j is a smooth rational curve. Thus for each C_j we have $[C_j] = \mathbb{Z} + \mathbb{Z}(-1)$ and since every non-singular rational surface can be obtained by blowing up

---

[7] This fact is stated without proof in [19].

either the projective plane or a Hirzebruch surface it follows that $[V_i] = \mathbb{Z} + a_i\mathbb{Z}(-1) + \mathbb{Z}(-2)$ for some $a_i \in \mathbb{Z}_{\geqslant 0}$. Let $a = \sum_i a_i$. Then, we have

$$
\begin{aligned}
\int_X &= \sum_{i \in \pi_0(Y^{(0)})} [V_i] - 2 \sum_{j \in \pi_0(Y^{(1)})} [C_j] + 3r_2\mathbb{Z} \\
&= \left(\frac{r_2}{2} + 2\right)\mathbb{Z} + a\mathbb{Z}(-1) + \left(\frac{r_2}{2} + 2\right)\mathbb{Z}(-2) - 3r_2\big(\mathbb{Z} + \mathbb{Z}(-1)\big) + 3r_2\mathbb{Z} \\
&= \left(\frac{r_2}{2} + 2\right)\mathbb{Z}(-2) + (a - 3r_2)\mathbb{Z}(-1) + \left(\frac{r_2}{2} + 2\right)\mathbb{Z}.
\end{aligned}
$$

Finally, using Proposition 2.9 it follows that

$$
a - 3r_2 = 20 - r_2.
$$

Remark 3.3. We claim that in notation of Section 3.4 the canonical map (2.11)

$$
W_4^{\mathbb{Z}}/W_3^{\mathbb{Z}} \xrightarrow{\gamma} H_2\big(\mathrm{Cl}(Y)\big) \tag{3.5}
$$

is an isomorphism. Indeed, let x be a generator of $W_4^{\mathbb{Z}}/W_3^{\mathbb{Z}}$, and let

$$
\gamma(x) = \sum_{i \in \pi_0(Y^{(2)})} b_i \delta_i,
$$

where δ_i are 2-simplices of $\mathrm{Cl}(Y)$. Then, since $\gamma(x) \in H_2(\mathrm{Cl}(Y))$, the boundary of the 2-dimensional chain $\sum_{i \in \pi_0(Y^{(2)})} b_i\delta_i$ is 0. As the δ_i form a triangulation of a compact connected manifold it follows that all the numbers $|b_i|$ are equal one to the other.[8] If b denotes their common value, we have by the Picard–Lefschetz formula (2.13)

$$
-\big\langle GrN^2\gamma(x), \gamma(x)\big\rangle = \sum_{i \in \pi_0(Y^{(2)})} b_i^2 = \big|\pi_0\big(Y^{(2)}\big)\big|b^2.
$$

The number at the left-hand side of the above formula equals r_2. Thus by Friedman–Scattone's result $b = 1$ and therefore $\gamma(x)$ is a generator of $H_2(\mathrm{Cl}(Y))$.

It follows from a general result of Berkovich explained in the next section that the group $H_2(\mathrm{Cl}(Y))$ and morphism (3.5) are independent of the choice of a strictly semi-stable model $\overline{X}$. Thus, it is an isomorphism for every such model.

4. The monodromy pairing

Let K be a complete discrete valuation field, and let $\widehat{K}$ be the completion of an algebraic closure $\overline{K}$ of K. In [5], Berkovich developed a theory of analytic spaces over K. The underlying topological space $|X_{\widehat{K}}^{\mathrm{an}}|$ of the analytification of a scheme X over K has interesting topological

---

[8] Indeed, every 1-simplex ϵ of the triangulation has precisely two 2-simplices, say δ_i and δ_j, adjacent to it. Thus, in order to have the coefficient at ϵ of the boundary of $\gamma(x)$ vanish $|b_i|$ must be equal to $|b_j|$.

invariants (in contrast with the space $X(\widehat{K})$ equipped with the usual topology, which is totally disconnected). In particular, if X is the generic fiber of a proper strictly semi-stable scheme $\overline{X}$ over R the space $|X^{\mathrm{an}}_{\widehat{K}}|$ is homotopy equivalent to the Clemens polytope of the special fiber Y [7, §5]. In this section we construct a positive pairing on the singular cohomology group $H^m(|X^{\mathrm{an}}_{\widehat{K}}|, \mathbb{Q})$ that generalizes Grothendieck's monodromy pairing in the case of abelian varieties. Applications to motivic integrals are discussed in the last section.

4.1. Cohomology of the analytic space associated with a smooth scheme

Let R be a complete discrete valuation domain, K its fraction field, k the residue field, and let $I \subset G = \mathrm{Gal}(\overline{K}/K)$ be the inertia subgroup. We denote by s and η the closed and generic points of $\mathrm{spec}\, R$ respectively. For a prime number ℓ different from $\mathrm{char}\, k$, we have a canonical surjection [49, I, §0.3]

$$\chi : I \to \mathbb{Z}_\ell(1)(\bar{k}).$$

If $\rho : G \to \mathrm{Aut}(V)$ is a finite rank $\mathbb{Z}_\ell$-representation of G there is a canonical G-homomorphism:

$$N : V \otimes \mathbb{Q}_\ell(1) \to V \otimes \mathbb{Q}_\ell,$$

defined as follows. The composition $\mathrm{Log} \circ \rho$ with the ℓ-adic logarithm $\mathrm{Aut}(V) \to \mathrm{End}(V \otimes \mathbb{Q}_\ell)$ restricted to the inertia subgroup I factors through χ. The map $\mathbb{Z}_\ell(1) \to \mathrm{End}(V \otimes \mathbb{Q}_\ell)$ yields N.

Denote by $\widehat{K}$ the completion of the algebraic closure $\overline{K}$ with respect to the unique valuation $\overline{K}^* \to \mathbb{Q}$ extending the valuation on K. For a smooth scheme X of finite type over K, let $X^{\mathrm{an}}_{\widehat{K}}$ be the $\widehat{K}$-analytic space associated with $X \otimes_K \widehat{K}$ [5, §3.4], and let $|X^{\mathrm{an}}_{\widehat{K}}|$ be the underlying topological space. According to [24, Theorem 9.1], [7, Theorem 13.1.7] $|X^{\mathrm{an}}_{\widehat{K}}|$ is a paracompact locally contractible topological space homotopy equivalent to a finite CW complex. In particular, the singular cohomology groups

$$\Gamma^m_C(X) = H^m\big(\big|X^{\mathrm{an}}_{\widehat{K}}\big|, C\big)$$

with coefficients in a ring C are finitely generated C-modules. The action of the Galois group G on $|X^{\mathrm{an}}_{\widehat{K}}|$ induces one on $\Gamma^m_C(X)$. In [24, Theorem 13.1.8], Hrushovski and Loeser proved that there exists a finite normal extension $K' \supset K$ such that the morphism

$$H^m\big(\big|X^{\mathrm{an}}_{K'}\big|, C\big) \to H^m\big(\big|X^{\mathrm{an}}_{\widehat{K}}\big|, C\big) = \Gamma^m_C(X)$$

is an isomorphism.[9] It follows, that the action of G on $\Gamma^m_C(X)$ factors through a finite quotient $G = \mathrm{Gal}(\overline{K}/K) \twoheadrightarrow \mathrm{Gal}(K'/K)$.

---

[9] This result was announced in [7, Theorem 10.1], however the proof in [7] is not correct: the assertion on p. 82 that a proper hyper-covering of a scheme X induces a hyper-covering of the topological space $|X^{\mathrm{an}}|$ is false. Example: take the hyper-covering associated with the r-fold étale cover $\mathbb{G}_m \to \mathbb{G}_m$. If the associated simplicial topological space over $|\mathbb{G}^{\mathrm{an}}_m|$ were a hyper-covering one would get an isomorphism between the cohomology of the contractible space $|\mathbb{G}^{\mathrm{an}}_m|$ and the group cohomology $H^*(\mathbb{Z}/r\mathbb{Z}, A)$. In fact, $\Gamma^*_A(X)$ is an interesting example of cohomology theory that does not have the étale descent property.

Theorem 3. *For every smooth variety X and every prime number $\ell \neq \operatorname{char} k$, the canonical morphism* [6, *Theorem* 7.5.4]; [7, *Theorem* 3.2]

$$\gamma : \Gamma^m_{\mathbb{Z}_\ell}(X) \to H^m\big(X^{\mathrm{an}}_{\overline{K}}, \mathbb{Z}_\ell\big) \simeq H^m(X_{\overline{K}}, \mathbb{Z}_\ell) \tag{4.1}$$

induces an isomorphism of G-modules

$$\Gamma^m_{\mathbb{Q}_\ell}(X) \xrightarrow{\sim} \operatorname{Im}\big(H^m(X_{\overline{K}}, \mathbb{Q}_\ell)(m) \xrightarrow{N^m} H^m(X_{\overline{K}}, \mathbb{Q}_\ell)\big). \tag{4.2}$$

We will write $N^m H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ for the right-hand side of (4.2).

Proof. Without loss of generality we may assume that k is separably closed and that X is irreducible. We first prove the theorem assuming that X is projective and has a strictly semi-stable model $\overline{X}$ over R. In this case, according to a key result of Berkovich [7, §5], $\Gamma^m_C(X)$ is isomorphic to the singular cohomology of the Clemens polytope of the special fiber of $\overline{X}$. On the other hand, we consider the weight filtration W_i on $H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ [42,43]. Interpreting the cohomology of the Clemens polytope as the weight zero part of $H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ we find that (4.2) is equivalent to a special case of Deligne's monodromy conjecture which asserts that, for every integer $0 \leqslant i \leqslant m$, the morphism

$$N^i : Gr_W^{m+i} H^m(X_{\overline{K}}, \mathbb{Q}_\ell)(i) \to Gr_W^{m-i} H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$$

is an isomorphism. We prove Deligne's conjecture for $i = m$ using the method of Steenbrink (who proved it for all i and $k = \mathbb{C}$). To prove the theorem for arbitrary smooth X we show that the functors $\Gamma^m_{\mathbb{Q}_\ell}$ and $N^m H^m$, first, admit transfers for finite morphisms and, second, take every dominant open embedding $U \hookrightarrow X$ to an isomorphism. Finally, we use de Jong's alteration result to complete the proof.

Step 1. Assuming that X has a projective strictly semi-stable model $\overline{X}$ over R, $X \simeq \overline{X} \otimes_R K$. Denote by D_i, $i = 1, 2, \ldots, s$, the irreducible components of the special fiber $Y = \overline{X} \otimes k$;

$$Y^{(q)} = \bigsqcup_{I \subset \{1, \ldots, s\}, |I| = q+1} \bigcap_{i \in I} D_i,$$

and by $\pi_0(Y^{(q)})$ the set of connected components of $Y^{(q)}$. We have a commutative diagram

$$
\begin{array}{ccc}
\Gamma^m_{\mathbb{Q}_\ell}(X) & \xrightarrow{\quad\gamma\quad} & H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \\
\simeq \downarrow & \nearrow{\scriptstyle\rho} & \\
H^m_{\mathrm{sing}}(\mathrm{Cl}(Y), \mathbb{Q}_\ell) \simeq E_2^{m,0}(\overline{X}) & &
\end{array}
\tag{4.3}
$$

where $E_2^{m,0}(\overline{X})$ is the weight zero term of the weight spectral sequence converging to $H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ [43]. According to [38] the weight spectral sequence degenerates at E_2; in particular the morphism ρ is injective. Since the range of the weight filtration on $H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ is at most $2m$ and N shifts the filtration by 2, we have

$$N^m H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \subset \operatorname{Im}(\rho). \tag{4.4}$$

Consider the commutative diagram

$$
\begin{array}{ccc}
E_2^{-m,2m}(\overline{X})(m) & \twoheadleftarrow & H^m(X_{\overline{K}}, \mathbb{Q}_\ell)(m) \\
\downarrow {\scriptstyle \overline{N}^m} & & \downarrow {\scriptstyle N^m} \\
E_2^{m,0}(\overline{X}) & \hookrightarrow & H^m(X_{\overline{K}}, \mathbb{Q}_\ell)
\end{array}
\tag{4.5}
$$

The upper horizontal arrow in this diagram is the projection to the weight $2m$ quotient. We will prove, following the method of [45, §5], that, for every m, one has

$$
\overline{N}^m : E_2^{-m,2m}(\overline{X})(m) \xrightarrow{\sim} E_2^{m,0}(\overline{X}).
\tag{4.6}
$$

This trivially holds for $d := \dim X < m$ because in this case both sides of (4.6) equal 0. Let us prove (4.6) for $m = d$. Consider the following commutative diagram [43]

$$
\begin{array}{ccccccc}
E_1^{d-1,0} & \longrightarrow & H^0(Y^{(d)}, \mathbb{Q}_\ell) = E_1^{d,0} & \longrightarrow & E_2^{d,0} \to 0 \\
& & \uparrow & & \\
& & \Big| & & \\
0 \to E_2^{-d,2d}(d) & \longrightarrow & H^0(Y^{(d)}, \mathbb{Q}_\ell) = E_1^{-d,2d}(d) & \longrightarrow & E^{-d+1,2d}(d)
\end{array}
$$

where

$$
E_1^{d-1,0} = H^0\big(Y^{(d-1)}, \mathbb{Q}_\ell\big), \qquad E^{-d+1,2d}(d) = H^2\big(Y^{(d-1)}, \mathbb{Q}_\ell\big)(1),
$$

the diagonal morphism is $\overline{N}^d$, and the vertical arrow is the identity morphism. The rows of the above diagram are exact and dual to one another. In particular, we have a non-degenerate paring

$$
\langle,\rangle : E_2^{d,0} \otimes E_2^{-d,2d}(d) \to \mathbb{Q}_\ell
$$

that identifies $E_2^{-d,2d}(d)$ with $H_d(\mathrm{Cl}(Y)) \otimes \mathbb{Q}_\ell$. Next, consider the symmetric form

$$
E_2^{-d,2d}(d) \otimes E_2^{-d,2d}(d) \to \mathbb{Q}_\ell, \qquad x \otimes y \mapsto \langle \overline{N}^d x, y \rangle.
\tag{4.7}
$$

We claim that (4.7) is non-degenerate. In fact, if

$$
x = \sum_{v \in \pi_0(Y^{(d)})} a_v v, \qquad y = \sum_{v \in \pi_0(Y^{(d)})} b_v v \in E_2^{-d,2d}(\overline{X})(d) \subset \mathbb{Q}_\ell\big[\pi_0\big(Y^{(d)}\big)\big],
$$

we have

$$
\langle N^d x, y \rangle = \sum a_v b_v.
$$

Thus (4.7) comes by extension of scalars from a positive form

$$
H_d\big(\mathrm{Cl}(Y), \mathbb{Q}\big) \otimes H_d\big(\mathrm{Cl}(Y), \mathbb{Q}\big) \to \mathbb{Q}.
\tag{4.8}
$$

This proves that the morphism (4.6) is injective; since $\dim E_2^{-d,2d} = \dim E_2^{d,0}$, it must be an isomorphism.

Assume that $0 < m < d$. Choose an embedding $\overline{X} \hookrightarrow \mathbb{P}_R^N$ and a generic hyperplane section $\overline{Z} = \overline{X} \cap \mathbb{P}_R^{N-d+m}$ of dimension m; $Z = X \cap \mathbb{P}_K^{N-d+m}$. Then $\overline{Z}$ is again strictly semi-stable and the embedding $i : \overline{Z} \hookrightarrow \overline{X}$ induces a morphism of spectral sequences $E_r^{p,q}(\overline{X}) \to E_r^{p,q}(\overline{Z})$. By the Hard Lefschetz Theorem the composition of the restriction morphism and the Poincaré pairing

$$H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \otimes H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \to H^m(Z_{\overline{K}}, \mathbb{Q}_\ell) \otimes H^m(Z_{\overline{K}}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(-m)$$

is non-degenerate. The induced isomorphism $H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \to (H^m(X_{\overline{K}}, \mathbb{Q}_\ell))^*(-m)$ takes $E_2^{m,0}(\overline{X}) \subset H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ to $(E_2^{-m,2m}(\overline{X}))^* \subset (H^m(X_{\overline{K}}, \mathbb{Q}_\ell))^*$. Thus

$$\dim E_2^{m,0}(\overline{X}) \leqslant \dim E_2^{-m,2m}(\overline{X}).$$

Let us show that (4.6) is injective. It is enough to check that in the commutative diagram

$$
\begin{array}{ccc}
E_2^{-m,2m}(\overline{X})(m) & \xrightarrow{\ \overline{N}^m\ } & E_2^{m,0}(\overline{X}) \\
\Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle i^*} \\
E_2^{-m,2m}(\overline{Z})(m) & \xrightarrow[\ \sim\ ]{\ \overline{N}^m\ } & E_2^{m,0}(\overline{Z})
\end{array}
\tag{4.9}
$$

the left downward arrow is an injection. We have

$$
\begin{array}{ccc}
E_2^{-m,2m}(\overline{X})(m) & \lhook\joinrel\longrightarrow & E_1^{-m,2m}(\overline{X})(m) = H^0(Y^{(m)}, \mathbb{Q}_\ell) \\
\Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle i^*} \\
E_2^{-m,2m}(\overline{Z})(m) & \longrightarrow & E_1^{-m,2m}(\overline{Z})(m) = H^0(Y^{(m)} \cap \mathbb{P}_k^{N-d+m}, \mathbb{Q}_\ell)
\end{array}
\tag{4.10}
$$

In this commutative diagram the upper horizontal arrow is an injection because the incoming differential $0 = E_1^{-m-1,2m}(\overline{X}) \xrightarrow{d_1} E_1^{-m,2m}(\overline{X})$ is trivial. The right downward arrow is an injection because $\mathbb{P}_k^{N-n+m}$ intersects every connected component of $Y^{(m)}$. This completes the proof of (4.6) and that of (4.2).

Step 2. Hrushovski and Loeser proved in [24, Theorem 13.1.8] that for every smooth variety X and an open dense subset $U \subset X$ the restriction morphism

$$\Gamma_C^m(X) \to \Gamma_C^m(U) \tag{4.11}$$

is an isomorphism. Let us show that the functor at the right-hand side of (4.2) has the same property:

$$N^m H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\ \sim\ } N^m H^m(U_{\overline{K}}, \mathbb{Q}_\ell). \tag{4.12}$$

We first prove (4.12) in the case when X is the generic fiber of a projective strictly semi-stable pair $(\overline{X}, \overline{Z} = \overline{Z}_f \cup Y)$ over R [13, §6.3] and $j : U \hookrightarrow X$ is the complement to $Z = \overline{Z} \otimes K$ in X.

Denote by T the special fiber $\overline{Z}_f \otimes k$ of the flat part of $\overline{Z}$ and by $\overline{j} : Y - T \hookrightarrow Y$ the embedding. The idea of the following argument (that goes back to Nakayama [38]) is the following. When the residue field k is finite (4.12) can be derived form the Weil conjectures, proven by Deligne, and the formula (4.6) proven in Step 1 (cf. [8, p. 672]). In general, the works of Fujiwara, Kato and Nakayama on logarithmic étale cohomology [25] imply that ℓ-adic cohomology groups of X and U depend only on the special fibers, Y and $Y - T$, endowed with their natural log structures (that, in turn, are determined by the first infinitesimal neighborhood of Y (resp. $Y - T$) in $\overline{X}$ (resp. $\overline{X} - \overline{Z}_f$)). Then, a specialization argument enables one to reduce to the finite field case. Let us explain the details.

For a scheme $S_{\log}$ over the log point $(\operatorname{spec} k)_{\log}$ we denote by $R\tilde{\epsilon}_*$ the functor from the derived category of ℓ-adic sheaves on the Kummer étale site, $S_{\log}^{ket}$, to the derived category of ℓ-adic sheaves on S equipped with an endomorphism of weight 2, i.e., a morphism $N : \mathcal{F} \to \mathcal{F}(-1)$ [25, §8, p. 308]. Consider the log structure on the scheme $\overline{X}$ associated with the divisor Y, and let $Y_{\log} = (Y, M_Y)$ be the special fiber with the induced log structure. According to [25, §8, Corollary 8.4.3] the action of the wild inertia $P \subset \operatorname{Gal}(\overline{K}/K)$ on the complexes of nearby cycles $\Psi \mathbb{Q}_\ell$, $\Psi R j_* \mathbb{Q}_\ell$ is trivial. Therefore we can and we will view the nearby cycles as objects of the derived category of ℓ-adic sheaves on Y endowed with an endomorphism N of weight 2. Then, we have

$$\Psi \mathbb{Q}_\ell \simeq R\tilde{\epsilon}_*(\mathbb{Q}_\ell),$$
$$\Psi R j_* \mathbb{Q}_\ell \simeq R\tilde{\epsilon}_*(R\overline{j}_* \mathbb{Q}_\ell).$$

We have to prove that the morphism

$$N^m H^m\big(Y_{\overline{k}}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)\big) \to N^m H^m\big(Y_{\overline{k}}, R\tilde{\epsilon}_*(R\overline{j}_* \mathbb{Q}_\ell)\big)$$

is an isomorphism. This will follow from a more general fact about log schemes over $(\operatorname{spec} k)_{\log}$.

Let $Y_{\log} = (Y, M_Y)$ be a fs log scheme over $(\operatorname{spec} k)_{\log}$, and let $T \hookrightarrow Y$ be closed sub-scheme. We will say that $(Y_{\log}, T)$ is a standard log strictly semi-stable pair if, for some integers $0 \leqslant a \leqslant b \leqslant d$, there is an isomorphism between $Y_{\log}$ and the special fiber the log scheme $\operatorname{spec} R[x_0, \ldots, x_d]/(x_0 \cdots x_a - \pi)$ (with the log structure defined by the divisor $\pi = 0$) that takes T to the subscheme given by the equation $x_{a+1} \cdots x_b = 0$. We will say that $(Y_{\log}, T)$ is a log strictly semi-stable pair if every point of Y has a Zariski neighborhood U such that $(U_{\log}, T \cap U)$ admits a strict étale morphism to a standard log strictly semi-stable pair. If this is the case, every irreducible component T_i of $T = T_1 \cup \cdots \cup T_n$ with the log structure induced from Y and $T_i \cap (T_1 \cup \cdots \cup T_{i-1}) \subset T_i$ is again a log strictly semi-stable pair.

Let $(Y_{\log}, T)$ be a proper log strictly semi-stable pair. In [38, §1], Nakayama constructed the weight spectral sequence E_r^{pq} converging to $H^m(Y \otimes \overline{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell))$ and proved that it degenerates in the E_2 terms. In particular, for every integer m, the canonical morphism

$$H^m_{\operatorname{sing}}\big(\operatorname{Cl}(Y)\big) \otimes \mathbb{Q}_\ell \simeq E_2^{m,0} \to H^m\big(Y \otimes \overline{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)\big)$$

is an embedding.

Lemma 4.1. *For every proper log strictly semi-stable pair $(Y_{\log}, T)$ the composition*

$$H^m_{\operatorname{sing}}\big(\operatorname{Cl}(Y_{\overline{k}}), \mathbb{Q}_\ell\big) \hookrightarrow H^m\big(Y_{\overline{k}}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)\big) \to H^m\big(Y_{\overline{k}}, R\tilde{\epsilon}_*(R\overline{j}_* \mathbb{Q}_\ell)\big) \tag{4.13}$$

is a monomorphism whose image contains $N^m H^m(Y_{\overline{k}}, R\tilde{\epsilon}_*(R\overline{j}_* \mathbb{Q}_\ell))$.

Proof. The specialization argument of Nakayama [38] reduces the statement to the case when k is a finite field; in the rest of the proof we will be assuming that this is the case. The vector spaces appearing in (4.13) carry an action of the Galois group $\mathrm{Gal}(\bar{k}/k)$. Let us look at the action of the Frobenius element $Fr \in \mathrm{Gal}(\bar{k}/k)$. For a finite-dimensional ℓ-adic representation V of $\mathrm{Gal}(\bar{k}/k)$ we denote by V_0 the largest invariant subspace of V such that all the eigenvalues of Fr on V_0 are roots of unity. Looking at the weight spectral sequence we see that

$$E_2^{m,0} = \big(H^m\big(Y_{\bar{k}}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)\big)\big)_0.$$

Thus, to prove the lemma it suffices to show the following:

(a) $$\big(H^m\big(Y_{\bar{k}}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)\big)\big)_0 \xrightarrow{\sim} \big(H^m\big(Y_{\bar{k}}, R\tilde{\epsilon}_*(R\bar{j}_*\mathbb{Q}_\ell)\big)\big)_0.$$ (4.14)

(b) The eigenvalues of Fr acting on $H^m(Y_{\bar{k}}, R\tilde{\epsilon}_*(R\bar{j}_*\mathbb{Q}_\ell))$ are Weil numbers of weights from 0 to $2m$.

Arguing by induction on $d = \dim Y$ we assume that the above assertions hold for log strictly semi-stable pairs of dimension less then d. Let $T_1, \ldots, T_n$ be irreducible components of T, let Y_j be the complement to $\bigcup_{i \leqslant j} T_i$ in Y. Consider the Gysin exact sequence

$$\cdots \to H^{m-2}\big((T_{j+1} \cap Y_j) \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)\big)(-1) \to H^m\big(Y_j \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)\big)$$
$$\to H^m\big(Y_{j+1} \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)\big) \to H^{m-1}\big((T_{j+1} \cap Y_j) \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)\big)(-1) \to \cdots.$$

By our induction assumption the boundary terms of the sequence have weights between 2 and $2m$. Induction on j proves the first claim (4.14). The second claim also follows from the above and from the fact that $H^m(Y \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell))$ has weights between 0 and $2m$. $\quad\square$

As we know from Step 1, for a projective strictly semi-stable scheme $\overline{X}$ over R, we have

$$N^m H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\sim} E_2^{m,0}.$$

This together with Lemma 4.1 completes the proof of (4.12) for strictly semi-stable pairs.

Before going further, recall that, for every generically finite surjective morphism $f : X' \to X$ of smooth connected varieties, the induced map

$$f^* : H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \to H^m\big(X'_{\overline{K}}, \mathbb{Q}_\ell\big)$$

is injective. In fact, the canonical isomorphism $\mathbb{Q}_\ell \xrightarrow{\sim} Rf^! \mathbb{Q}_\ell$ defines by adjunction a morphism

$$Rf_* \mathbb{Q}_\ell \xrightarrow{\sim} Rf_! \mathbb{Q}_\ell \to \mathbb{Q}_\ell.$$

In turn, the latter yields the transfer morphism

$$f_* : H^m\big(X'_{\overline{K}}, \mathbb{Q}_\ell\big) \to H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$$

such that the composition $f_* f^*$ equals multiplication by the degree of f over the generic point.

Let us return to the proof of (4.12). Without loss of generality we may assume that X is connected. Then, by de Jong's result [13, §6.3] we can find a proper generically finite surjective morphism $f : X' \to X$ such that X' is an open subscheme of a connected projective strictly semi-stable scheme $\overline{X}'$ over a finite extension $R' \supset R$ and such that $(\overline{X}', \overline{X}' - X')$ is a strictly semi-stable pair. Applying de Jong's result once again, we find a proper generically finite surjective morphism $g : \overline{X}'' \to \overline{X}'$, with connected $\overline{X}''$, such that $(\overline{X}'', \overline{X}'' - (fg)^{-1}(U))$ is a projective strictly semi-stable pair over some $R'' \supset R'$. Diagram:

$$
\begin{array}{ccccc}
(fg)^{-1}(U) & \hookrightarrow & X'' & \hookrightarrow & \overline{X}'' \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\
f^{-1}(U) & \hookrightarrow & X' & \hookrightarrow & \overline{X}' \\
\downarrow & & \downarrow{\scriptstyle f} & & \\
U & \hookrightarrow & X & &
\end{array}
\tag{4.15}
$$

We know that (4.12) is true for the embeddings $X' \hookrightarrow \overline{X}' \otimes K$ and $g^{-1}f^{-1}(U) \hookrightarrow \overline{X}'' \otimes K$.[10] Define a morphism $u : N^m H^m(U_{\overline{K}}, \mathbb{Q}_\ell) \to N^m H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ to be the composition

$$
N^m H^m(U_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{(fg)^*} N^m H^m\big((fg)^{-1}(U)_{\overline{K}}, \mathbb{Q}_\ell\big) \simeq N^m H^m\big(\overline{X}'' \otimes \overline{K}, \mathbb{Q}_\ell\big)
$$

$$
\xrightarrow{g_*} N^m H^m\big(\overline{X}' \otimes \overline{K}, \mathbb{Q}_\ell\big) \xrightarrow{\mathrm{Res}} N^m H^m\big(X' \otimes \overline{K}, \mathbb{Q}_\ell\big)
$$

$$
\xrightarrow{f_*} N^m H^m(X \otimes \overline{K}, \mathbb{Q}_\ell).
$$

An easy diagram chase shows that u divided by the degree of the morphism fg over the generic point is the two-sided inverse to the restriction morphism (4.12).

Step 3. Let $f : U' \to U$ be a finite surjective morphism of connected smooth varieties. Assume that the corresponding extension $\mathrm{Rat}(X) \subset \mathrm{Rat}(X')$ of the field of rational functions is normal and let G be its Galois group. Then, the pullback morphism f^* induces an isomorphism

$$
N^m H^m(U_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\sim} \big(N^m H^m\big(U'_{\overline{K}}, \mathbb{Q}_\ell\big)\big)^G.
$$

Let us show the functor $\Gamma_{\mathbb{Q}}^m$ has the same property:

$$
\Gamma_{\mathbb{Q}}^m(U) \xrightarrow{\sim} \big(\Gamma_{\mathbb{Q}}^m(U')\big)^G.
\tag{4.16}
$$

Indeed, by [6, Proposition 4.2.4], the cohomology of the topological space $|U_{\widehat{\overline{K}}}^{\mathrm{an}}|$ with rational coefficients coincides with the étale cohomology of the analytic space $U_{\widehat{\overline{K}}}^{\mathrm{an}}$ with coefficients in $\mathbb{Q}$. Next, since the functor of G-invariants is exact in any $\mathbb{Q}$-linear abelian category, we have

$$
\big(H_{et}^m\big(U'^{\mathrm{an}}_{\widehat{\overline{K}}}, \mathbb{Q}\big)\big)^G \simeq H_{et}^m\big(U_{\widehat{\overline{K}}}^{\mathrm{an}}, (f_*\mathbb{Q})^G\big).
$$

---

[10] Indeed, $(\overline{X}', \overline{X}' - X')$ is a strictly semi-stable pair over R'. Therefore, we have $N^m H^m(\overline{X}' \times_{R'} \overline{K}, \mathbb{Q}_\ell) \xrightarrow{\sim} N^m H^m(X' \times_{R'} \overline{K}, \mathbb{Q}_\ell)$. This implies that the morphism $N^m H^m(\overline{X}' \times_R \overline{K}, \mathbb{Q}_\ell) \to N^m H^m(X' \times_R \overline{K}, \mathbb{Q}_\ell)$ is an isomorphism as well.

We complete the proof of (4.16) by showing that the canonical morphism $\mathbb{Q} \to (f_*\mathbb{Q})^G$ is an isomorphism. In fact, the weak base change theorem [6, Theorem 5.3.1] reduces the statement to the case when $U'_{\widehat{K}}$ is a single point. In this case G acts transitively on points of $U'_{\widehat{K}}$ and our assertion follows.

Step 4. Now we can complete the proof of Theorem 3. We may assume that X is connected. Then, by [14, Theorem 5.9], there exists a proper generically finite surjective morphism $f : X' \to X$ such that the field extension $\mathrm{Rat}(X) \subset \mathrm{Rat}(X')$ is normal, X' is an open subscheme of a connected projective strictly semi-stable scheme $\overline{X}'$ over a finite extension $R' \supset R$. Let U be an open dense subset of X over which f is finite. By the result of Step 1 the theorem is true for $\overline{X}'$.[11] Then, by Step 2 it is true for $f^{-1}(U)$ and thus, by Step 3, for U. Applying the result of Step 2 once again we complete the proof of Theorem 3. $\square$

Remark 4.2. The groups $\Gamma^*_{\mathbb{Z}}(X)$ are related to the weight zero part of motivic vanishing cycles $\Psi(X) \in DM^{\mathrm{eff}}_{gm}(k)$ of X [2,3]. Namely, if $\mathrm{char}\, k = 0$, one has

$$\Gamma^m_{\mathbb{Z}}(X) \simeq \mathrm{Hom}_{DM^{\mathrm{eff}}_{gm}(\bar{k})}\big(\Psi(X), \mathbb{Z}[m]\big).$$

Remark 4.3. Assume that $K = \mathbb{C}((t))$. For every smooth projective X/K there is a canonical morphism (cf. [9, Theorem 5.1])

$$\Gamma^m_{\mathbb{Z}}(X) \to W^{\mathbb{Q}}_0 \cap H^m(\lim X, \mathbb{Z}) \tag{4.17}$$

that induces an isomorphism modulo torsion

$$\Gamma^m_{\mathbb{Q}}(X) \simeq W^{\mathbb{Q}}_0 H^m(\lim X). \tag{4.18}$$

Morphism (4.17) can be constructed as follows. Pick a finite extension $K' \supset K$ and strictly semi-stable model $\overline{X}_{R'}$ of $X_{K'} = X \otimes_K K'$ over the integral closure R' of R in K'. Then (4.17) is defined to be the composition

$$\Gamma^m_{\mathbb{Z}}(X) \simeq \Gamma^m_{\mathbb{Z}}(X_{K'}) \xrightarrow{\sim} H^m\big(\mathrm{Cl}(Y)\big) \to W^{\mathbb{Q}}_0 \cap H^m(\lim X_{K'}, \mathbb{Z}) \simeq W^{\mathbb{Q}}_0 \cap H^m(\lim X, \mathbb{Z}), \tag{4.19}$$

where Y is the special fiber of $\overline{X}_{R'}$ and the map $H^m(\mathrm{Cl}(Y)) \to W^{\mathbb{Q}}_0 \cap H^m(\lim X_{K'}, \mathbb{Z})$ comes from the weight spectral sequence (see Section 2.2). As the weight spectral sequence with rational coefficients degenerates at E_2 terms the above composition is an isomorphism up to torsion. The composition of (4.19) with the embedding $W^{\mathbb{Q}}_0 \cap H^m(\lim X, \mathbb{Z}) \hookrightarrow H^m(X_{\overline{K}}, \mathbb{Z}_\ell)$ equals the canonical morphism $\Gamma^m_{\mathbb{Z}}(X) \to H^m(X_{\overline{K}}, \mathbb{Z}_\ell)$ from Theorem 3. Thus, the morphism $\Gamma^m_{\mathbb{Z}}(X) \to W^{\mathbb{Q}}_0 \cap H^m(\lim X, \mathbb{Z})$ induced by (4.19) is independent of the choice of K' and $\overline{X}_{R'}$.

In general, morphism (4.17) is not bijective.

---

[11] Indeed, the result of Step 1 implies that the morphism $H^m(|X'^{an} \times_{K'} \widehat{K}|, \mathbb{Q}_\ell) \to N^m H^m(X' \times_{K'} \overline{K}, \mathbb{Q}_\ell)$ is an isomorphism. This implies that $H^m(|X'^{an} \times_K \widehat{K}|, \mathbb{Q}_\ell) \to N^m H^m(X' \times_K \overline{K}, \mathbb{Q}_\ell)$ is also an isomorphism.

We conjecture that for every smooth proper variety X over K, one has

$$\dim_{\mathbb{Q}} \Gamma_{\mathbb{Q}}^m(X) \leqslant \dim_K H^m(X, \mathcal{O}_X). \tag{4.20}$$

Conjecture (4.20) is motivated by the following result.

Proposition 4.4. *The inequality* (4.20) *is true if either of the following conditions holds.*

(a) $\operatorname{char} k = 0$.
(b) K *is a finite extension of* $\mathbb{Q}_p$.

Proof. When proving the first part of the proposition, we may assume that $R = \mathbb{C}[\![t]\!]$ and X is the generic fiber of a strictly semi-stable scheme $\overline{X}$ over R [24, Theorem 13.1.8]. In this case, we have

$$\Gamma_{\mathbb{Q}}^m(X) \simeq H_{\mathrm{Zar}}^m(Y, \mathbb{Q}) \simeq W_0^{\mathbb{Q}} H^m(\lim X),$$

where Y is the special fiber of $\overline{X}$. The first part of the proposition now follows from the inequality $\dim_{\mathbb{Q}} W_0^{\mathbb{Q}} \leqslant \dim_{\mathbb{C}} F^0/F^1 = \dim_K H^m(X, \mathcal{O}_X)$. For the second part, recall from [6, Theorem 1.1] that $\Gamma_K^m(X)$ is isomorphic to the subspace of the p-adic cohomology $H^m(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K$ that consists of smooth vectors, i.e., vectors whose stabilizer in G is open. Thus,

$$\dim_K \Gamma_K^m(X) \leqslant \dim_K \left(H^m(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \right)^G = \dim_K H^m(X, \mathcal{O}_X).$$

The last equality follows from the Hodge–Tate decomposition proven by Faltings [17]. □

4.2. The monodromy pairing

Let X be a smooth variety over a complete discrete valuation field K and $d = \dim X$. In this subsection we define a canonical positive symmetric form (that we shall call the monodromy pairing)

$$(\cdot, \cdot) : \Gamma_{\mathbb{Q}}^d(X) \otimes \Gamma_{\mathbb{Q}}^d(X) \to \mathbb{Q}. \tag{4.21}$$

The group $\Gamma_{\mathbb{Z}}^d(X)$ as well as the monodromy pairing depends only on the class of X modulo birational equivalence.

First, we define a pairing

$$(\cdot, \cdot)_\ell : N^d H^d(X_{\overline{K}}, \mathbb{Q}_\ell) \otimes N^d H^d(X_{\overline{K}}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell.$$

By [13, Theorem 4.1, Remark 4.2], there exists a proper generically finite surjective morphism $f : X' \to X$ such that X' is an open subscheme of a smooth projective variety $\tilde{X}'$ over a finite extension $K' \supset K$. Let r be the degree of f over the generic point. Consider the morphism

$$\overline{f}^* : N^d H^d(X_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{f^*} N^d H^d\big(X'_{\overline{K}}, \mathbb{Q}_\ell\big) \xleftarrow{\sim} N^d H^d\big(\tilde{X}'_{\overline{K}}, \mathbb{Q}_\ell\big).^{12}$$

The left arrow is an isomorphism by (4.12). Given $x, y \in N^d H^d(X_{\overline{K}}, \mathbb{Q}_\ell)$ we set

$$(x, y)_\ell = \frac{(-1)^{\frac{d(d-1)}{2}}}{r} \big\langle \overline{f}^*(x), \overline{f}^*(y') \big\rangle, \tag{4.22}$$

where $y' \in H^d(X_{\overline{K}}, \mathbb{Q}_\ell)(d)$ is an element such that $N^d y' = y$ and

$$\langle, \rangle : H^d\big(\tilde{X}'_{\overline{K}}, \mathbb{Q}_\ell\big) \otimes H^d\big(\tilde{X}'_{\overline{K}}, \mathbb{Q}_\ell\big)(d) \to \mathbb{Q}_\ell$$

is the Poincaré pairing. Let us check that $(\cdot, \cdot)_\ell$ is well defined. Indeed, if y'' is another element such that $N^d y'' = y$, then

$$\big\langle \overline{f}^*(x), \overline{f}^*(y' - y'') \big\rangle = \big\langle N^d \overline{f}^*(x'), \overline{f}^*(y' - y'') \big\rangle$$
$$= (-1)^d \big\langle \overline{f}^*(x'), N^d \overline{f}^*(y' - y'') \big\rangle = 0.$$

The independence of the choice of X', f and $\tilde{X}'$ follows from the fact that given another such triple X'', g and $\tilde{X}''$ we can find a smooth projective scheme over some finite extension of K that admits proper generically finite surjective morphisms to both $\tilde{X}'$ and $\tilde{X}''$.

Let us also remark that the pairing $(\cdot, \cdot)_\ell$ is symmetric.

Theorem 4. *For every smooth connected variety X of dimension d, the restriction* (4.21) *of the pairing $(\cdot, \cdot)_\ell$ to the subspace*

$$\Gamma^d_{\mathbb{Q}}(X) \hookrightarrow N^m H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$$

takes values in $\mathbb{Q}$ and is independent of $\ell \neq \operatorname{char} k$. Moreover, the pairing (4.21) *is positively defined (and, in particular, non-degenerate).*

Proof. Thanks to the birational invariance property of $\Gamma^d(X)$ (4.11) and de Jong's semi-stable reduction theorem [13, §6.3] it is enough to prove the theorem in the case when X is the generic fiber of a strictly semi-stable projective scheme $\overline{X}$ over R. In this case, using (4.3) we have a canonical isomorphism $\Gamma^d_{\mathbb{Q}}(X) \simeq H^d(\mathrm{Cl}(Y), \mathbb{Q})$ that identifies, by the Picard–Lefschetz formula (cf. (2.13)), the pairing $(\cdot, \cdot)_\ell$ restricted to $\Gamma^d_{\mathbb{Q}}(X)$ with the dual of the pairing (4.8). $\quad\square$

Remark 4.5. The construction of the monodromy pairing can be generalized as follows. For a pair (X, μ), where X is a smooth projective variety over K and $\mu \in H^2(X, \mathbb{Q}_\ell(1))$ is the class of an ample line bundle over X, and an integer $m \leqslant d$, we define a positive symmetric form

$$(\cdot, \cdot)_\mu : \Gamma^m_{\mathbb{Q}}(X) \otimes \Gamma^m_{\mathbb{Q}}(X) \to \mathbb{Q} \tag{4.23}$$

---

[12] We write $X'_{\overline{K}}$ for the fiber product of X' and $\operatorname{spec} \overline{K}$ over $\operatorname{spec} K$.

to be the composition

$$\Gamma_{\mathbb{Q}}^m(X) \otimes \Gamma_{\mathbb{Q}}^m(X) \to N^m H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \otimes N^m H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{(\cdot,\cdot)_{\ell,\mu}} \mathbb{Q}_\ell,$$

where $(x, N^m y')_{\ell,\mu} = (-1)^{\frac{d(d-1)}{2}} \langle x, y'\mu^{d-m} \rangle$. Let us prove that (4.23) is independent of ℓ and positive. Without loss of generality, we may assume that μ is the class of very ample line bundle L. Let $X \hookrightarrow \mathbb{P}_K^N$ be the corresponding embedding, and let $Z = X \cap \mathbb{P}_K^{N-d+m} \xhookrightarrow{i} X$ be a generic hyperplane section of dimension m. Then, $(\cdot,\cdot)_\mu$ equals the composition

$$\Gamma_{\mathbb{Q}}^m(X) \otimes \Gamma_{\mathbb{Q}}^m(X) \xrightarrow{i^* \otimes i^*} \Gamma_{\mathbb{Q}}^m(Z) \otimes \Gamma_{\mathbb{Q}}^m(Z) \xrightarrow{(\cdot,\cdot)} \mathbb{Q}.$$

By Theorem 3 and the Hard Lefschetz Theorem the restriction morphism $i^* : \Gamma_{\mathbb{Q}}^m(X) \to \Gamma_{\mathbb{Q}}^m(Z)$ is injective. Our claim follows from Theorem 4.

Remark 4.6. Assume that $K = \mathbb{C}((t))$. For a smooth projective d-dimensional scheme X over K the isomorphism

$$\Gamma_{\mathbb{Q}}^d(X) \simeq W_0^{\mathbb{Q}} H^d(\lim X)$$

from Remark 4.3 carries the monodromy pairing on $\Gamma_{\mathbb{Q}}^d(X)$ to the pairing

$$(\cdot,\cdot) : W_0^{\mathbb{Q}} H^d(\lim X) \otimes W_0^{\mathbb{Q}} H^d(\lim X) \to \mathbb{Q}$$

defined by the formula (cf. (4.22))

$$(x, y) = (-1)^{\frac{d(d-1)}{2}} \langle x, y' \rangle,$$

where $x \in W_0^{\mathbb{Q}}$, $y' \in W_{2d}^{\mathbb{Q}} / W_{2d-1}^{\mathbb{Q}}$ is such that $GrN^d(y') = y$, and $\langle \cdot, \cdot \rangle : W_0^{\mathbb{Q}} \otimes W_{2d}^{\mathbb{Q}} / W_{2d-1}^{\mathbb{Q}} \to \mathbb{Q}$ denotes the Poincaré pairing.

Example 4.7. Let A be a d-dimensional abelian variety over K with semi-stable reduction. According to [5, §6.5], after replacing K by a finite unramified extension, we can represent the analytic space A^{an} as the quotient of G^{an} by Λ, where G^{an} is the analytic group associated with a semi-abelian variety $0 \to \overline{T} \to \overline{G} \to \overline{B} \to 0$ over R and $\Lambda \xhookrightarrow{u} \overline{G}(K)$ a lattice. Moreover, the map $|G_{\overline{K}}^{\mathrm{an}}| \to |A_{\overline{K}}^{\mathrm{an}}|$ exhibits $|G_{\overline{K}}^{\mathrm{an}}|$ as a universal cover of $|A_{\overline{K}}^{\mathrm{an}}|$. In particular, $\Gamma_m(A) := H_m(|A_{\overline{K}}^{\mathrm{an}}|) \simeq \bigwedge^m \Lambda$. A polarization, μ, of A defines an isogeny $\mu_* : \Lambda \to \Xi$, where Ξ is the group of characters of $\overline{G}$. Using [12, Theorem 2.1], we see that the pairing

$$\Gamma_1(A) \otimes \Gamma_1(A) \to \mathbb{Q}$$

derived from (4.23) equals the pullback of Grothendieck's monodromy pairing

$$\Lambda \otimes \Xi \xrightarrow{u \otimes \mathrm{Id}} \overline{G}(K)/\overline{G}(R) \otimes \Xi \to \Xi^* \otimes \Xi \to \mathbb{Z}$$

via $\mathrm{Id} \otimes \mu_* : \Lambda \otimes \Lambda \to \Lambda \otimes \Xi$, divided by the degree $\mu^d \in \mathbb{Z}$ of the polarization.

4.3. A birational invariant

Let X be a smooth connected variety over a complete discrete valuation field K and $d = \dim X$. Assume that $\Gamma^d(X)_{\mathbb{Q}} \neq 0$. Let $\mathrm{Disc}(\cdot, \cdot) \in \mathbb{Q}^*$ be the discriminant of the monodromy pairing (4.21) relative to the lattice $\Gamma_{\mathbb{Z}}^d(X)/\Gamma_{\mathbb{Z}}^d(X)_{\mathrm{tor}} \subset \Gamma_{\mathbb{Q}}^d(X)$, and let

$$r_d(X, K) = \frac{1}{\mathrm{Disc}(\cdot, \cdot)}. \tag{4.24}$$

Since the group $\Gamma_{\mathbb{Z}}^d(X)$ and the monodromy pairing (4.21) are birational invariants of X so is the number $r_d(X, K)$. If $K \subset K'$ is a finite extension of ramification index e, we have

$$r_d(X \otimes K', K') = e^{d \dim \Gamma_{\mathbb{Q}}^d(X)} r_d(X, K).$$

In the remaining part of this section we shall relate the invariant $r_d(X, K)$ defined here to the one introduced in Section 1.3 for K3 surfaces over $\mathbb{C}((t))$.

Proposition 4.8. *Let X be a smooth projective K3 surface over $K = \mathbb{C}((t))$ and let $H^2(\lim X)$ be the corresponding limit mixed Hodge structure (see Section 2.2). Set $W_i^{\mathbb{Z}} := W_i^{\mathbb{Q}} \cap H^2(\lim X, \mathbb{Z})$. Assume that the monodromy acts on $H^2(\lim X, \mathbb{Z})$ by a unipotent operator and let $N : H^2(\lim X, \mathbb{Z}) \to H^2(\lim X, \mathbb{Z})$ be its logarithm (which is integral by [19, Proposition 1.2]). Then:*

(a) *The topological space $|X_{\widehat{K}}^{\mathrm{an}}|$ is contractible unless $N^2 \neq 0$. If $N^2 \neq 0$ the space $|X_{\widehat{K}}^{\mathrm{an}}|$ is homotopy equivalent to a 2-dimensional sphere and the canonical map (see Remark 4.3)*

$$\Gamma_{\mathbb{Z}}^2(X) \to W_0^{\mathbb{Z}} \tag{4.25}$$

is an isomorphism.

(b) *Assume that $N^2 \neq 0$. Then the number $r_2(X, \mathbb{C}((t)))$ defined by (4.24) is equal to the order of the following group*

$$\mathrm{Coker}\big(W_4^{\mathbb{Z}}/W_3^{\mathbb{Z}} \xrightarrow{GrN^2} W_0^{\mathbb{Z}}\big). \tag{4.26}$$

Proof. It is enough to prove the proposition in the case where X is the restriction of a strictly semi-stable family over a smooth curve. Indeed, at the expense of a finite extension of K we may choose a strictly semi-stable model $\overline{X}$ for X. The space $|X_{\widehat{K}}^{\mathrm{an}}|$ is homotopy equivalent to the Clemens polytope of the special fiber Y of $\overline{X}$ [7, §5]. Applying Proposition 3.1 to $\overline{X}$ we find a proper strictly semi-stable family $\overline{\mathcal{X}}$ over a smooth pointed curve $a \in \overline{C}$, whose fiber over the first infinitesimal neighborhood of point a is isomorphic to $\overline{X} \otimes_R R/t^2$ and whose generic fiber is a K3 surface. As the limit mixed Hodge structure of a strictly semi-stable scheme depends only on the first infinitesimal neighborhood of special fiber the validity of the proposition does not change if we replace X by $\mathcal{X} \times_C \mathrm{spec}\, K'$.

Thus, we may assume that X has a Kulikov model over $R = \mathbb{C}[[t]]$ (see Section 3.2). If $\overline{X}$ is a Kulikov model, then the Clemens polytope $\mathrm{Cl}(Y)$ of the special fiber of $\overline{X}$ is homeomorphic to a point or to an interval for type I or II degenerations and it is homeomorphic to a 2-dimensional

sphere for type III degenerations. This proves the first part of the proposition except for the claim that morphism (4.25) is an isomorphism. Using Berkovich's result [7, §5], the latter is equivalent the following assertion: the canonical morphism

$$H^2\big(\mathrm{Cl}(Y)\big) \to W_0^{\mathbb{Z}}$$

coming from the weight spectral sequence (see Theorem 2) is an isomorphism. In fact, the (equivalent) dual statement,

$$W_4^{\mathbb{Z}}/W_3^{\mathbb{Z}} \xrightarrow{\sim} H_2\big(\mathrm{Cl}(Y)\big)$$

is proven (using a deep result of Friedman and Scattone [19]) in Remark 3.3. This completes the proof of the first part of the proposition.

Part (b) of the proposition follows from the fact that (4.25) is an isomorphism and Remark 4.6. □

5. Motivic integral of maximally degenerate K3 surfaces over non-archimedean fields

5.1. The formula

Throughout this section R denotes a complete discrete valuation ring with fraction field K and perfect residue field k. We shall say that a smooth projective d-dimensional Calabi–Yau variety X over K is maximally degenerate if $\Gamma_{\mathbb{Q}}^d(X) \neq 0$. According to Theorem 3, X is maximally degenerate if and only if for some (and, therefore, for every) prime $\ell \neq \mathrm{char}\, k$ the map

$$H^d(X_{\overline{K}}, \mathbb{Q}_\ell)(m) \xrightarrow{N^d} H^d(X_{\overline{K}}, \mathbb{Q}_\ell)$$

is not zero.

Conjecture 1. *Let X be a smooth projective maximally degenerate K3 surface over K. Then:*

(a) *The topological space $|X_{\overline{K}}^{\mathrm{an}}|$ is homotopy equivalent to a 2-dimensional sphere. In particular, the group $\Gamma_{\mathbb{Z}}^m(X)$ is trivial for $m \neq 0, 2$ and isomorphic to $\mathbb{Z}$ otherwise.*
(b) *For every $\ell \neq \mathrm{char}\, k$ the lattice*

$$\mathbb{Z}_\ell \simeq \Gamma_{\mathbb{Z}_\ell}^2(X) \overset{(4.1)}{\hookrightarrow} H^2(X_{\overline{K}}, \mathbb{Z}_\ell)$$

is saturated, i.e.,

$$\Gamma_{\mathbb{Z}_\ell}^2(X) = \big(\Gamma_{\mathbb{Q}_\ell}^2(X)\big) \cap H^2(X_{\overline{K}}, \mathbb{Z}_\ell).$$

(c) *There exists a finite extension $K' \supset K$ such that for every finite extension $L \supset K'$ of ramification index e*

$$\int_{X_L} = \left(\frac{e^2 r_2(X, K)}{2} + 2\right)\mathbb{Q}(0) + (20 - e^2 r_2(X, K))\mathbb{Q}(-1) + \left(\frac{e^2 r_2(X, K)}{2} + 2\right)\mathbb{Q}(-2).$$

$$(5.1)$$

Remark 5.1. According to the first part of the conjecture, for every prime $\ell \neq \text{char}\, k$, the ℓ-primary factor of $r_2(X, K)$ has the following cohomological interpretation. If $r_2(X, K) = \ell^{a_\ell} r'$ and $(r', \ell) = 1$, then

$$a_\ell = -v_\ell\big(\text{Disc}(\cdot, \cdot)_\ell\big)$$

where $\text{Disc}(\cdot, \cdot)_\ell$ is the discriminant of the ℓ-adic monodromy pairing (4.3) restricted to $\text{Im}(N^2) \cap H^2_{et}(X_{\overline{K}}, \mathbb{Z}_\ell)$ and $v_\ell : \mathbb{Q}_\ell^* / \mathbb{Z}_\ell^* \to \mathbb{Z}$ is the valuation morphism.

Remark 5.2. According to Theorem 1 and Proposition 4.8, Conjecture 1 is true for $K = \mathbb{C}((t))$. Thus, it true for every K of equicharacteristic 0 [24, Theorem 13.1.8].

5.2. Kummer K3 surfaces

Throughout this subsection $\text{char}\, K \neq 2$. Let A be a 2-dimensional abelian variety over K. Then the group subscheme $A_2 := \text{Ker}(A \xrightarrow{[2]} A) \subset A$ of 2-torsion points is reduced of order 16. The quotient A/σ modulo the involution $A \xrightarrow{\sigma} A$, $\sigma(x) = -x$, is a projective surface, whose singular locus is precisely the image of A_2. A Kummer K3 surface X is the blow-up of A/σ at $A_2 \hookrightarrow A/\sigma$ which is smooth. Any translation invariant differential 2-form on A descends to a non-vanishing regular form ω on X. Equivalently, X can be viewed as the quotient of the variety Z obtained from A by blowing up at A_2.

Theorem 5. *Conjecture* 1 *is true if* X *is a Kummer K3 surface and* $\text{char}\, k \neq 2$.

Proof. Fix a prime number $\ell \neq \text{char}\, k$. By Theorem 3 since X is maximally degenerate and, for some finite extension $K' \supset K$, the $\text{Gal}(\overline{K}/K')$-module $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ is isomorphic to $H^2(A_{\overline{K}}, \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell(-1)^{\oplus 16}$ the abelian variety A is maximally degenerate. Thus, after replacing K by its finite extension we may assume that the analytic space, A^{an}, is the quotient of a split 2-dimensional torus T^{an} by a split lattice $\Lambda \subset T(K)$. We also assume that all the 2-torsion points of A are defined over K. Under these assumptions we will prove that the formula (5.1) is true for $L = K$ and therefore for all its finite extensions. To do this we shall construct a formal poly-stable model $\overline{X}$ for the analytic space X^{an}. By a general result of Berkovich [7, §5] the topological space $|X^{\text{an}}_{\overline{K}}|$ is homotopy equivalent to the realization of the nerve of the special fiber[13] $\overline{X} \times \bar{k}$, denoted by $|N(\overline{X} \times \bar{k})|$. On the other hand, the smooth locus of such a model is a weak Néron model of X^{an} and, thus, can be used to compute the motivic integral.

Let Ξ be the group of characters of T and Ξ^* the dual group. We have a canonical injective homomorphism $\rho : \Lambda \to \Xi^*$ given by the valuation on K. Choose bases $\{v_1, v_2\}$, $\{u_1, u_2\}$ for Λ and Ξ such that the matrix of ρ is diagonal

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

with positive m_i, and let $T \simeq \mathbb{G}_{m,K} \times \mathbb{G}_{m,K}$, $\Lambda \simeq \mathbb{Z}^2$ be the corresponding isomorphisms. Consider the standard "relatively complete" model $\overline{\mathbb{G}}_m$ of $\mathbb{G}_{m,K}$ over R in the sense of Mumford

---

[13] The notion of nerve of a scheme is recalled in Remark 2.2.

[37, §5]. We view $\overline{\mathbb{G}}_m$ as a formal scheme over R whose associated K-analytic space is $\mathbb{G}_{m,K}^{an}$ and which is equipped with an action of the multiplicative group K^*, extending the translation action on $\mathbb{G}_{m,K}^{an}$, and an involution that acts as $x \mapsto x^{-1}$ on $\mathbb{G}_{m,K}^{an}$. The special fiber of $\overline{\mathbb{G}}_m$ is a chain of projective lines $\mathbb{P}_k^1$; the action of K^* induces a simple and transitive action of $\mathbb{Z} \simeq K^*/R^*$ on the set of its irreducible components. The quotient of $\overline{\mathbb{G}}_m \times \overline{\mathbb{G}}_m$ by $\Lambda \subset T(K) = K^* \times K^*$ is a proper strictly poly-stable formal model of A^{an}. Its smooth locus is the formal Néron model of A^{an} [11, Definition 1.1 and Theorem 6.2]. In particular, our assumptions on A and K imply that the 2-torsion points of A define 16 sections of $(\overline{\mathbb{G}}_m \times \overline{\mathbb{G}}_m)/\Lambda$ over R meeting the special fiber at distinct smooth points. Let $\overline{Z}$ be the blow-up of $(\overline{\mathbb{G}}_m \times \overline{\mathbb{G}}_m)/\Lambda$ at these sections. By construction, the involution σ of A extends to an involution $\overline{\sigma}$ of $\overline{Z}$. The quotient $\overline{X} = \overline{Z}/\overline{\sigma}$ is again a proper strictly poly-stable formal scheme whose generic fiber is the analytic K3 surface X^{an}. In particular, every K'-point of X^{an}, where $K' \supset K$ is a finite unramified extension of K reduces to a nonsingular point of the special fiber of $\overline{X}$. It follows that the smooth locus $\overline{X}_{sm} \subset \overline{X}$ of $\overline{X}$ is a formal weak Néron model of X^{an} [11, Definition 1.3]. As the Néron top degree differential form on A induces a regular non-vanishing differential form on $\overline{X}_{sm}$ using [34, Theorem 4.4.1] we see that the motivic integral $\int_X$ equals the class $[Y_{sm}]$ of the smooth locus of the special fiber Y of $\overline{X}$. We shall show that $\int_X$ depends only on the order of the group $C = \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z}$ of connected components of the formal Néron model of A^{an}. Indeed, since all the 2-torsion points of A are defined over K, the numbers m_i are even. Thus, the involution σ has precisely 4 fixed points on C. It follows that Y_{sm} has $\frac{|C|}{2} + 2$ connected components. All the components of Y_{sm} are isomorphic to $\mathbb{G}_{m,k} \times \mathbb{G}_{m,k}$ except for those 4 that correspond to fixed points of σ on C. These 4 components are isomorphic to the blow-up of $\mathbb{G}_{m,k} \times \mathbb{G}_{m,k}$ at 4 points of order 2. Summarizing, we find

$$\int_X = \left(\frac{|C|}{2} + 2\right)\mathbb{Q}(0) + (20 - |C|)\mathbb{Q}(-1) + \left(\frac{|C|}{2} + 2\right)\mathbb{Q}(-2).$$

Thus, to complete the proof of the formula (5.1) we need to show that $|C| = r_2(X, K)$. Consider the commutative diagram induced by the morphism $f : \overline{Z} \to \overline{X}$ of the formal schemes

$$
\begin{array}{ccccccc}
H^2(|N(\overline{X} \times \overline{k})|, \mathbb{Z}) & \xrightarrow{\sim} & \Gamma_{\mathbb{Z}}^2(X) & \longrightarrow & H^2(X_{\widehat{K}}^{an}, \mathbb{Z}_\ell) & \simeq & H^2(X_{\overline{K}}, \mathbb{Z}_\ell) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
H_{Zar}^2(|N(\overline{Z} \times \overline{k})|, \mathbb{Z}) & \xrightarrow{\sim} & \Gamma_{\mathbb{Z}}^2(Z) & \longrightarrow & H^2(Z_{\widehat{K}}^{an}, \mathbb{Z}_\ell) & \simeq & H^2(Z_{\overline{K}}, \mathbb{Z}_\ell)
\end{array}
\tag{5.2}
$$

The topological space $|N(\overline{Z} \times \overline{k})|$ is homeomorphic to a real 2-dimensional torus[14]; the map $|N(\overline{Z} \times \overline{k})| \to |N(\overline{X} \times \overline{k})|$ induced by f identifies the target space with the quotient of the torus modulo the involution that takes a point to its inverse (with respect to the usual group structure

---

[14] Indeed, the scheme $\overline{Z} \times \overline{k}$ is isomorphic to a direct product of two (reducible) curves D_i, $i = 1, 2$, which, in turn, are isomorphic to m_i-gons of $\mathbb{P}_{\overline{k}}^1$'s. Thus, using that the formation $|N(\cdot)|$ commutes with products of poly-stable schemes over algebraically closed field [7, Proposition 3.14(ii) and Corollary 3.17], we find that $|N(\overline{Z} \times \overline{k})| \simeq |N(D_1)| \times |N(D_2)| \simeq S^1 \times S^1$.

on the real torus). In particular, $|N(\overline{X} \times \overline{k})|$ is homeomorphic to a 2-dimensional sphere. This proves the first part of the theorem. Moreover, we have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\;\sim\;} & \Gamma_{\mathbb{Z}}^2(X) \\
\Big\downarrow{\scriptstyle 2} & & \Big\downarrow{\scriptstyle f^*} \\
\mathbb{Z} & \xrightarrow{\;\sim\;} & \Gamma_{\mathbb{Z}}^2(Z) \quad \simeq \quad \Gamma_{\mathbb{Z}}^2(A)
\end{array}
\tag{5.3}
$$

where the isomorphism $\Gamma_{\mathbb{Z}}^2(A) \xrightarrow{\sim} \Gamma_{\mathbb{Z}}^2(Z)$ is induced by the morphism of formal schemes $\overline{Z} \to (\overline{\mathbb{G}}_m \times \overline{\mathbb{G}}_m)/\Lambda$ that identifies the nerves of their special fibers. On the other hand, since the morphism $Z \to X$ induced by f has degree 2, we have

$$
(x, y) = \frac{1}{2}\big(f^*(x), f^*(y)\big), \quad x, y \in \Gamma_{\mathbb{Z}}^2(X).
$$

Comparing this with (5.3) we find that

$$
r_2(X, K) = \frac{r_2(Z, K)}{2} = \frac{r_2(A, K)}{2}.
$$

It remains to show that $\frac{r_2(A,K)}{2} = |C|$. Consider the exact sequence of G-modules

$$
0 \to \Lambda^* \otimes \mathbb{Z}_\ell \to H^1(A_{\overline{K}}, \mathbb{Z}_\ell) \to \Xi \otimes \mathbb{Z}_\ell(-1) \to 0.
$$

The canonical morphism $\Gamma_{\mathbb{Z}}^1(A) \hookrightarrow H^1(A_{\overline{K}}, \mathbb{Z}_\ell)$ identifies $\Gamma_{\mathbb{Z}}^1(A)$ with $\Lambda^* \subset H^1(A_{\overline{K}}, \mathbb{Z}_\ell)$ [5, §6.5]. Let $\tilde{u}_1 \wedge \tilde{u}_2$ be an element of $H^2(A_{\overline{K}}, \mathbb{Z}_\ell)(2)$ that projects to $u_1 \wedge u_2 \in \bigwedge^2 \Xi \otimes \mathbb{Z}_\ell$. Then, we have

$$
N^2(\tilde{u}_1 \wedge \tilde{u}_2) = N\big(N(\tilde{u}_1) \wedge \tilde{u}_2 + \tilde{u}_1 \wedge N(\tilde{u}_2)\big) = 2N(\tilde{u}_1) \wedge N(\tilde{u}_2)
$$

$$
= 2m_1 m_2 \big(v_1^* \wedge v_2^*\big) \in \bigwedge^2 \Lambda^* \otimes \mathbb{Z}_\ell.
$$

It follows that the monodromy pairing on $\Gamma_{\mathbb{Z}}^2(A) \simeq \bigwedge^2 \Lambda^*$ is given by the formula

$$
\big(v_1^* \wedge v_2^*, v_1^* \wedge v_2^*\big) = -\frac{\langle v_1^* \wedge v_2^*, \tilde{u}_1 \wedge \tilde{u}_2 \rangle}{2m_1 m_2} = \frac{1}{2m_1 m_2}
$$

and therefore $r_2(A, K) = 2m_1 m_2 = 2|C|$. The proof of the formula (5.1) is completed.

Let us prove the second statement of the theorem. We will derive it from an analogous result for abelian varieties [5, §6.5] which asserts that the lattice

$$
\Gamma_{\mathbb{Z}_\ell}^2(A) \hookrightarrow H^2(A_{\overline{K}}, \mathbb{Z}_\ell)
$$

is saturated. It follows that the lattice

$$
\Gamma_{\mathbb{Z}_\ell}^2(A) \xrightarrow{\sim} \Gamma_{\mathbb{Z}_l}^2(Z) \hookrightarrow H^2(Z_{\overline{K}}, \mathbb{Z}_\ell) \simeq H^2(A_{\overline{K}}, \mathbb{Z}_\ell) \oplus \mathbb{Z}_\ell(-1)^{\oplus 16}
$$

is also saturated. We claim that in the commutative diagram

$$
\begin{array}{ccccc}
\mathbb{Z}_\ell & \xrightarrow{\sim} & \Gamma^2_{\mathbb{Z}_\ell}(X) & \longrightarrow & H^2(X_{\overline{K}}, \mathbb{Z}_\ell) \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
\mathbb{Z}_\ell & \xrightarrow{\sim} & \Gamma^2_{\mathbb{Z}_\ell}(Z) & \longrightarrow & H^2(Z_{\overline{K}}, \mathbb{Z}_\ell)
\end{array}
\tag{5.4}
$$

the vertical morphisms are isomorphisms up to 2-torsion. Indeed, the compositions $f_* f^*$ and $f^* f_*$ with the trace morphism $H^2(Z_{\overline{K}}, \mathbb{Z}_\ell) \xrightarrow{f_*} H^2(X_{\overline{K}}, \mathbb{Z}_\ell)$ are equal to $2\,\mathrm{Id}$. This proves the second part of the theorem for $\ell \neq 2$. For $\ell = 2$, we apply a result of Nikulin (see, e.g. [36, Lemma on p. 56]) that states the lattice

$$
H^2(A_{\overline{K}}, \mathbb{Z}_\ell) \hookrightarrow H^2(Z_{\overline{K}}, \mathbb{Z}_\ell) \xrightarrow{f_*} H^2(X_{\overline{K}}, \mathbb{Z}_\ell)
\tag{5.5}
$$

is saturated. As the lattice $\Gamma^2_{\mathbb{Z}_\ell}(X)$ equals the image of the saturated lattice $\Gamma^2_{\mathbb{Z}_\ell}(A) \hookrightarrow H^2(A_{\overline{K}}, \mathbb{Z}_\ell)$ under composition (5.5) it is saturated as well. $\quad\square$

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