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On Jordan decomposition of characters for $\mathrm{SU}(n, q)$

Marc Cabanes

Institut de Mathématiques de Jussieu, Université Paris 7, 175 rue du Chevaleret, Case 7012, F-75205 Paris Cedex 13, France

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ABSTRACT

As shown by Bonnafé, a step in proving a Jordan decomposition of characters of finite special linear groups is the parametrization of unipotent characters of centralizers of semi-simple elements in projective linear groups. We show the same kind of result in the case of finite special unitary groups. The proof leads to a mild adaptation of Bonnafé's methods expounded in [B99]. The outcome is a Jordan decomposition of characters compatible with Lusztig's twisted induction.

Introduction

Finite groups of Lie type deriving from reductive groups whose center is not connected (e.g. $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$ from $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$) have a representation theory which is less well understood than their connected center counterpart. However they are of great interest for finite group theory since the central coverings of finite simple groups of Lie type are of that kind.

It seems important to establish for those groups a Jordan decomposition of characters with properties similar to the ones known for reductive groups with connected center, for instance commutation with Lusztig's twisted induction of characters (see [L77,L84]). This was proved for finite special linear groups $\mathrm{SL}_n(q)$ by Bonnafé (see [B99], [B06, §27]). According to [B06, 32.1], the missing piece to prove the same statement for finite special unitary groups $\mathrm{SU}_n(q)$ is some parametrization of unipotent characters in (non-connected) centralizers of semi-simple elements in projective unitary groups. This is the main purpose of the present paper (see Theorem 4.9). We essentially review the arguments from the proof of the main statement (7.3.2) of [B99], pointing that many of them apply to the more general situation of a wreath product $\mathbf{G} = \mathbf{G}_1 \wr A$ whose base group $\mathbf{G}_1$ is reductive of arbitrary type.

E-mail address: cabanes@math.jussieu.fr.

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1. Notations and background

When $(v_i)_{i\in I}$ is a finite family of elements of a vector space over $\mathbb{Q}$, one denotes

$$\overline{\sum_{i\in I}} v_i := |I|^{-1}\sum_{i\in I} v_i.$$

1.1. Finite groups, central functions

When G is a group and H a subgroup (or element) of G, one uses the usual notations $\mathrm{C}_G(H)$, $\mathrm{N}_G(H)$ for the centralizer and normalizer, respectively. We define $\mathrm{W}_G(H)=\mathrm{N}_G(H)/H$ for any subgroup $H\subseteq G$.

When a group A acts on a set X, the subset of X of fixed points is denoted by X^A, while for any $x\in X$ its stabilizer in A is denoted by $A_x=\{a\in A\mid a.x=x\}$.

If G is a finite group, one denotes by $\mathcal{C}(G)$ the $\mathbb{C}$-vector space of class functions $G\to\mathbb{C}$ endowed with the scalar product $\langle f,f'\rangle_G=\overline{\sum}_{g\in G} f(g)\overline{f'(g)}$. The set of irreducible (ordinary, complex) characters $\mathrm{Irr}(G)$ is an orthonormal basis of $\mathcal{C}(G)$. The action of $\mathrm{Aut}(G)$ on $\mathcal{C}(G)$ is denoted by $\sigma.f(g):=f(\sigma^{-1}(g))$ for any $\sigma\in\mathrm{Aut}(G)$, $f\in\mathcal{C}(G)$, $g\in G$.

If $\phi:G\to G$ is a group automorphism, one may form the coset $G\phi$ in the natural semi-direct product $G\rtimes\langle\phi\rangle$. It is stable by G-conjugacy and the associated partition into G-orbits of $G\phi$ corresponds with a partition of G into so-called "ϕ-classes". One denotes by $\mathcal{C}(G\phi)$ the space of G-invariant functions $f:G\phi\to\mathbb{C}$ endowed with the scalar product $\langle f,f'\rangle_{G\phi}=\overline{\sum}_{g\in G} f(g\phi)\overline{f'(g\phi)}$ for $f,f'\in\mathcal{C}(G\phi)$. Note that each element of $\mathcal{C}(G\phi)$ is not only G-invariant but also $G\rtimes\langle\phi\rangle$-invariant (hence the restriction of an element of $\mathcal{C}(G\rtimes\langle\phi\rangle)$) since the image of $g\phi$ by ϕ is the same as the image by g^{-1} conjugacy.

Denote by $\mathbf{1}_{G\phi}\in\mathcal{C}(G\rtimes\langle\phi\rangle)$ the characteristic function of the subset $G\phi$.

When $G'\leqslant G$ is a ϕ-stable subgroup, one denotes by $\mathrm{Res}^{G\phi}_{G'\phi}:\mathcal{C}(G\phi)\to\mathcal{C}(G'\phi)$ the restriction map and its adjoint by $\mathrm{Ind}^{G\phi}_{G'\phi}:\mathcal{C}(G'\phi)\to\mathcal{C}(G\phi)$ which may be defined by $\mathrm{Ind}^{G\phi}_{G'\phi}(f')(g\phi)=|G|^{-1}\sum_x f'(xg\phi(x)^{-1}\phi)$ where the sum is over $x\in G$ such that $xg\phi(x)^{-1}\in G'$.

1.2. Wreath products and characters

If a group A acts on the group H by group automorphisms, one says that the semi-direct product $H\rtimes A$ is a wreath product if, and only if, H decomposes as a direct product $H=\prod_{i\in I}H_i$ with A acting on I, $a.H_i=H_{a.i}$ for any $a\in A$, $i\in I$, and a induces the identity on H_i as soon as $a.i=i$. Breaking $\prod_{i\in I}H_i$ along I/A, those semi-direct products $H\rtimes A$ are direct products of the situation described below.

Let X a finite set on which acts the finite group A, and let H_0 be another finite group. Denote by $\mathcal{M}(X,H_0)$ the set of maps $X\to H_0$ endowed with its natural group structure. Note that A acts by group automorphisms on $\mathcal{M}(X,H_0)$ and that $\mathcal{M}(X,H_0)\rtimes A$ is the typical wreath product usually denoted by $H_0\wr A$. For fixed points we have

$$\mathcal{M}(X,H_0)^A\cong\mathcal{M}(X/A,H_0) \tag{1.1}$$

by the composition with the canonical surjection $X\to X/A$.

Concerning conjugacy classes and irreducible characters of $\mathcal{M}(X,H_0)$, we have $\mathrm{Irr}(\mathcal{M}(X,H_0))\cong\mathcal{M}(X,\mathrm{Irr}(H_0))$ by a canonical map.

So, denoting $H:=\mathcal{M}(X,H_0)$, one can identify

$$\mathrm{Irr}\big(H^A\big)\cong\mathrm{Irr}(H)^A,\quad \chi_A\leftarrow\chi, \tag{1.2}$$

by $\mathrm{Irr}(\mathcal{M}(X,H_0)^A)\cong \mathrm{Irr}(\mathcal{M}(X/A,H_0))\cong \mathcal{M}(X/A,\mathrm{Irr}(H_0))\cong \mathcal{M}(X,\mathrm{Irr}(H_0))^A\cong \mathrm{Irr}(\mathcal{M}(X,H_0))^A = \mathrm{Irr}(H)^A$.

For characters in $\mathrm{Irr}(H)^A$, recall the canonical extension to the wreath product $H\rtimes A$ (see [B99, 2.3.1], [JK81, §4.3]).

Definition 1.3. For $\chi\in\mathrm{Irr}(H)$, let

$$\chi\rtimes A_\chi\in\mathrm{Irr}(H\rtimes A_\chi)$$

satisfying $(\chi\rtimes A_\chi)(ha)=(\chi\rtimes\langle a\rangle)(ha)=\chi_{\langle a\rangle}(\pi_a(h))$ for any $h\in H$, $a\in A_\chi$, $\chi_{\langle a\rangle}$ defined as in (1.2) and $\pi_a : H=\mathcal{M}(X,H_0)\to H^{\langle a\rangle}=\mathcal{M}(X/\langle a\rangle,H_0)$ associated to any section of the reduction map $X\to X/\langle a\rangle$.

Denote by $\chi\rtimes a$ the restriction of $\chi\rtimes\langle a\rangle$ to the coset $Ha\subseteq H\rtimes\langle a\rangle$.

Let us recall that $\chi\rtimes\langle a\rangle$, resp. $\chi\rtimes a$, is denoted as $\widetilde{\chi}$, resp. $\widetilde{\chi}_a$, in [B99] and [B06, §33].

Remark 1.4. Assume $f\in\mathcal{C}(H)^A$ is of the type $\bigotimes_{x\in X} f_x$ with $f_x\in\mathcal{C}(H_0)$ and $f_{a.x}=f_x$ for each $x\in X$, $a\in A$ (for instance all f_x's equal), so that f is defined by $f((h_x)_{x\in X})=\prod_{x\in X} f_x(h_x)$. Then, generalizing the above, one may define $f_A\in\mathcal{C}(H^A)$ and $f\rtimes A\in\mathcal{C}(H\rtimes A)$ extending f by $f_A((h_x)_{x\in X})=\prod_{x\in X/A} f_x(h_x)$ and $(f\rtimes A)(ha)=f_{\langle a\rangle}(\pi_a(h))$.

Another interesting property of wreath products $H_0\wr A=H\rtimes A$ is the simple description of $\mathcal{C}(Ha)$ for $a\in A$ (see [B99, 2.2.1]).

Proposition 1.5. *$(\chi\rtimes a)_{\chi\in\mathrm{Irr}(H)^{\langle a\rangle}}$ form an orthonormal basis of $\mathcal{C}(Ha)$.*

Clifford theorem (see for instance [N98, 8.9]) allows to parametrize $\mathrm{Irr}(H\rtimes A)$ in a very explicit way.

Definition 1.6. Let M be a group on which another group A acts by group automorphisms. One denotes by $\mathcal{I}(M,A)$ (resp. $\mathcal{I}^\wedge(M,A)$) the set of pairs (χ,α) with $\chi\in\mathrm{Irr}(M)$ and $\alpha\in\mathrm{Irr}(A_\chi)$ (resp. $\alpha\in A_\chi$). The group A acts on both sets by $\beta.(\chi,\alpha)=({}^\beta\chi,{}^\beta\alpha)$.

The corresponding quotients are denoted by $\overline{\mathcal{I}}(M,A)$ (resp. $\overline{\mathcal{I}}^\wedge(M,A)$).

The A-orbit of (χ,α) is denoted by $\chi*\alpha$.

Proposition 1.7. *In the situation of a wreath product $H\rtimes A$, one has a bijection $\overline{\mathcal{I}}(H,A)\xrightarrow{\sim}\mathrm{Irr}(H\rtimes A)$ defined by $\chi*\alpha\mapsto\mathrm{Ind}_{H.A_\chi}^{H.A}((\chi\rtimes A_\chi).\alpha)$.*

1.3. Non-connected reductive groups

In this section we give some terminology and basic tools on the representation theory of finite groups of Lie type obtained from non-connected reductive groups. The basic reference is [DM94] with generalizations given in [B99, §6].

Note that what follows is used essentially in cases where $\mathbf{G}=\mathbf{G}^\circ\rtimes A$ where $\mathbf{G}^\circ$ (neutral component) is a direct product of general linear groups and A acts by permutations of the summands ("wreath products"). A basic property of this case is that A induces "quasi-central automorphisms" (see [B99, 5.1.1]) of $\mathbf{G}^\circ$.

In this section, $(\mathbf{G},F)$ is a (possibly non-connected) reductive group defined over $\mathbb{F}_q$ with associated Frobenius endomorphism $F:\mathbf{G}\to\mathbf{G}$.

One calls parabolic subgroups of $\mathbf{G}$ the closed subgroups $\mathbf{P}\subseteq\mathbf{G}$ containing some Borel subgroup of $\mathbf{G}^\circ$ (note that in contrast to [DM94] we are not requiring $\mathbf{P}=\mathrm{N}_{\mathbf{G}}(\mathbf{P}^\circ)$). Then $\mathbf{P}^\circ$ is a parabolic subgroup of $\mathbf{G}^\circ$. Denoting by $\mathrm{R_u}$ the unipotent radical, one has $\mathbf{P}^\circ = \mathrm{R_u}(\mathbf{P}) \rtimes \mathbf{L}_0$ for $\mathbf{L}_0$ a so-called Levi subgroup of $\mathbf{G}^\circ$ [DM91, 1.15], and one gets $\mathbf{P} = \mathrm{R_u}(\mathbf{P}) \rtimes \mathrm{N}_{\mathbf{P}}(\mathbf{L}_0)$ (see [B99, §6.1]).

A group of the above kind $\mathbf{L} := \mathrm{N}_{\mathbf{P}}(\mathbf{L}_0)$ is called a *Levi subgroup* of $\mathbf{G}$. A *quasi-Borel* subgroup of $\mathbf{G}$ is the normalizer in $\mathbf{G}$ of any Borel subgroup of $\mathbf{G}$. All quasi-Borel subgroups of $\mathbf{G}$ are $\mathbf{G}^\circ$-conjugate. A *maximal quasi-torus* is any Levi subgroup of a quasi-Borel. It is clearly of the form $\mathrm{N}_{\mathbf{G}}(\mathbf{T}_0, \mathbf{B}_0)$ where $\mathbf{T}_0 \subseteq \mathbf{B}_0 \subseteq \mathbf{G}^\circ$ are a maximal torus and a Borel subgroup of $\mathbf{G}^\circ$. They are all $\mathbf{G}^\circ$-conjugate.

Proposition 1.8. *(See [DM94, 1.40], [B99, 6.1.5].) Assume that $\mathbf{G} = \mathbf{G}^\circ.\langle a\rangle$ where $a \in \mathbf{G}^F$ induces a quasi-central automorphism of $\mathbf{G}^\circ$. Then:*

- *There exists an $\langle a, F\rangle$-stable maximal torus $\mathbf{T}_0$ of $\mathbf{G}^\circ$* [DM94, 1.36.(ii)].
- *If $\mathbf{T}$ is an F-stable maximal quasi-torus which contains a, then $(\mathbf{T}^{\langle a\rangle})^\circ$ is a maximal torus of $(\mathbf{G}^{\langle a\rangle})^\circ$.*
- *Any F-stable maximal quasi-torus $\mathbf{T}$ of $\mathbf{G}$ has a $(\mathbf{G}^\circ.\langle a\rangle)^F$-conjugate $\mathbf{T}'$ which contains a, and $\mathbf{T} \mapsto (\mathbf{T}'^{\langle a\rangle})^\circ$ induces a bijection between $\mathbf{G}^F$-conjugacy classes of F-stable maximal quasi-tori of $\mathbf{G}$ and $(\mathbf{G}^{\langle a\rangle})^{\circ F}$-conjugacy classes of F-stable maximal tori of $(\mathbf{G}^{\langle a\rangle})^\circ$, which are in turn naturally parametrized by the F-classes* (*see Section* 1.1 *above*) *of $\mathrm{W}_{\mathbf{G}^\circ}(\mathbf{T}_0)^{\langle a\rangle}$.*

Maximal quasi-tori are called *of type $w \in \mathrm{W}_{\mathbf{G}^\circ}(\mathbf{T}_0)^{\langle a\rangle}$ with regard to $\mathbf{T}_0\langle a\rangle$* when they correspond with the F-class of w by the above.

When $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{G}$, the definition of a functor

$$\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} : \mathbb{Z}\,\mathrm{Irr}\big(\mathbf{L}^F\big) \to \mathbb{Z}\,\mathrm{Irr}\big(\mathbf{G}^F\big)$$

is similar to the connected case. The variety $\mathbf{Y} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathrm{R_u}(\mathbf{P})\}$ is acted on by $\mathbf{L}^F$ on the right and $\mathbf{G}^F$ on the left. So the virtual $\overline{\mathbb{Q}}_\ell$-module obtained from the ℓ-adic cohomology of $\mathbf{Y}$ with compact support $\mathrm{H_c}(\mathbf{Y}) = \sum_{i\in\mathbb{Z}}(-1)^i\mathrm{H}^i_{\mathrm{c}}(\mathbf{Y}, \overline{\mathbb{Q}}_\ell)$ is a virtual $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}]$-module. Each simple $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}]$-module can be realized over a finite extension of $\mathbb{Q}$, so we assume that the generalized character of $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$ we are considering is over $\mathbb{C}$. The associated tensor product functor is the sought $\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$.

Many properties generalizing the ones of the $\mathbf{G} = \mathbf{G}^\circ$ case hold.

Proposition 1.9. *Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with F-stable Levi subgroup $\mathbf{L}$.*

(i) *If $\mathbf{Q}$ is a parabolic subgroup of $\mathbf{G}$ with F-stable Levi subgroup $\mathbf{M}$ with $\mathbf{L} \subseteq \mathbf{M}$ and $\mathbf{P} \subseteq \mathbf{Q}$, then*

$$\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} = \mathrm{R}^{\mathbf{G}}_{\mathbf{M}\subseteq\mathbf{Q}} \circ \mathrm{R}^{\mathbf{M}}_{\mathbf{L}\subseteq\mathbf{P}\cap\mathbf{M}}.$$

(ii) *If $\mathbf{L}$ is a quasi-maximal torus, the functor $\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ does not depend on the choice of $\mathbf{P}$.*

(iii) *If $\chi \in \mathrm{Irr}(\mathbf{L}^F)$ and $g \in \mathbf{G}^F$, then*

$$\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}\chi(g) = \sum_x \big(\mathrm{R}^{\mathbf{G}^\circ\langle l_x\rangle}_{\mathbf{L}^\circ\langle l_x\rangle\subseteq\mathbf{P}^\circ\langle l_x\rangle}\mathrm{Res}^{\mathbf{L}^F}_{\mathbf{L}^{\circ F}\langle l_x\rangle}\chi\big) \circ \mathrm{ad}_x(g)$$

where the sum is over cosets $x\mathbf{L}^F.\mathbf{G}^{\circ F} \subseteq \mathbf{G}^F$ such that ${}^x g \in \mathbf{L}^F.\mathbf{G}^{\circ F}$ and where $l_x \in \mathbf{L}^F$ is such that ${}^x g \in l_x.\mathbf{G}^{\circ F}$.

Proof. (i) is [B99, 6.3.3]. For (ii) see [DM94, 4.5], [B99, Remarque 7.1]. (iii) is [DM94, 2.3.(i)]. □

We are mainly concerned with the set $\mathcal{E}(\mathbf{G}^F, 1)$ of so-called "unipotent" characters (see [DM91, 13.19]).

Definition 1.10. The unipotent characters of $\mathbf{G}^F$ are the irreducible components of the various $\operatorname{Ind}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F}(\chi)$ for $\chi \in \mathcal{E}(\mathbf{G}^{\circ F}, 1)$. They are also the irreducible components of the generalized characters $R_{\mathbf{T}}^{\mathbf{G}}(1)$ for $\mathbf{T}$ ranging over F-stable maximal quasi-tori of $\mathbf{G}$ (see [B99, 6.4.2]).

2. Wreath products of general linear groups

Adapting [B99, §7], one defines here certain groups $\mathbf{H}^F$ where $\mathbf{H}$ is reductive but non-connected in the form $\mathbf{H}^\circ \rtimes A$, a wreath product in the sense of §2.2 above. We also show that centralizers of semi-simple elements in projective linear or unitary groups are of that type.

Let q be a power of p, the characteristic of $\mathbb{F}$.

Notation 2.1. In the following, we will denote by $(\mathbf{H}, F)$ a finite product of pairs $(\mathbf{H}_i, F_i)$ where $\mathbf{H}_i = \mathrm{GL}_{n_i}(\mathbb{F}) \wr A_i = \mathrm{GL}_{n_i}(\mathbb{F})^{d_i} \rtimes A_i$ $(A_i \subseteq \mathfrak{S}_{d_i})$ while $F_i : \mathbf{H}_i \to \mathbf{H}_i$ is $\sigma_i F_i'$ for $\sigma_i \in \mathrm{N}_{\mathfrak{S}_{d_i}}(A_i)$ and $F_i' : \mathbf{H}_i \to \mathbf{H}_i$ is the map trivial on A_i and raising matrix entries to the q-th power composed with some power of the map sending matrices to their transpose-inverse. One denotes by A the product of the A_i's so that $\mathbf{H} = \mathbf{H}^\circ \rtimes A$, and by σ the product of the σ_i's.

Note that $\mathbf{H}^F = \mathbf{H}^{\circ F} \rtimes A^F$ where F acts by conjugacy by σ on A in $\prod_i \mathfrak{S}_{d_i}$.

Remark 2.2. The action of A on $\mathbf{H}^\circ$ is by quasi-central automorphisms (see [B99, 7.1.1]).

One denotes by $\mathbf{T}_0 \subseteq \mathbf{H}^\circ$ the product of subgroups of diagonal matrices in each $\mathrm{GL}_{n_i}(\mathbb{F})$, i.e. the F-stable diagonal torus of $\mathbf{H}^\circ$. One denotes the associated Weyl group

$$W^\circ = \mathrm{N}_{\mathbf{H}^\circ}(\mathbf{T}_0)/\mathbf{T}_0$$

which identifies with the product of the subgroups of permutation matrices in each summand $\mathrm{GL}_{n_i}(\mathbb{F})$. Note that it is stable by A and each element of W° is fixed by F_i', so that σ is the automorphism of W° induced by F.

Remark 2.3. The semi-direct product $W^\circ \rtimes A$ is to be considered as the Weyl group of the non-connected reductive group $\mathbf{H} = \mathbf{H}^\circ \rtimes A$. Note that $(W^\circ \rtimes A)^F = W^{\circ\langle\sigma\rangle} \rtimes A^{\langle\sigma\rangle}$ is a wreath product. On the other hand $\mathbf{H}^F$ may not be one. An example is $\mathbf{H}^\circ = (\mathrm{GL}_n(\mathbb{F}))^d$ with $\sigma = (1, \ldots, d)$, F_1 raising matrix entries to the q-th power, $A = \langle\sigma\rangle$. Then $\mathbf{H}^F = \mathrm{GL}_n(\mathbb{F}_{q^d}) \rtimes \langle F_1\rangle$.

Proposition 2.4.

(i) *A-stable maximal tori of $\mathbf{H}^\circ$ are conjugated under $\mathbf{H}^{\circ A}$.*
(ii) *Suppose A abelian and $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{H}$. Then $(\mathbf{L}^\circ.\mathbf{L}^F, F)$ is of the type described in Notation 2.1.*

Proof. (i) A maximal torus in a direct product $\mathbf{H}^\circ = \prod_{i\in I} \mathbf{H}_i^\circ$ of connected groups is the direct product of its intersections with the $\mathbf{H}_i$'s. A first consequence is that one may assume that $A \subseteq \mathfrak{S}_d$ is transitive on a direct product $\mathbf{H}^\circ = \mathbf{H}_1 \times \cdots \times \mathbf{H}_1$ (d summands), so that $\mathbf{H}^{\circ A} = \{(g, \ldots, g) \mid g \in \mathbf{H}_1\}$. An A-stable maximal torus will then be of the form $\mathbf{T}_1 \times \cdots \times \mathbf{T}_1$ for $\mathbf{T}_1$ a maximal torus of $\mathbf{H}_1$. Then our statement results from conjugacy of maximal tori in $\mathbf{H}_1$.

(ii) Note first that $\mathbf{L}^\circ = \mathbf{H}^\circ \cap \mathbf{L}$, so one can identify $\mathbf{L}/\mathbf{L}^\circ$ with a subgroup $A_{\mathbf{L}}$ of $A \cong \mathbf{H}/\mathbf{H}^\circ$. It is F-stable. By [B99, 6.2.2], some $\mathbf{H}^{\circ F}$-conjugate of $\mathbf{L}$ contains $A_{\mathbf{L}}^F$, so one can assume $\mathbf{L}^\circ.\mathbf{L}^F = \mathbf{L}^\circ \rtimes A_{\mathbf{L}}^F$.

Keeping $\mathbf{H}^\circ = \prod_{i\in I} \mathbf{H}_i^\circ$ and $\mathbf{L}^\circ = \prod_{i\in I} \mathbf{L}^\circ \cap \mathbf{H}_i^\circ$, each $\mathbf{L}^\circ \cap \mathbf{H}_i^\circ$ is a direct product of GL_m's. If $\mathbf{L} \cap \mathbf{H}_{i_0}^\circ$ decomposes as $\mathbf{M}_1 \times \cdots \times \mathbf{M}_t$, then for any $a \in A_{\mathbf{L}}^F$, $\mathbf{L}^\circ \cap \mathbf{H}_{a.i_0}^\circ = {}^a(\mathbf{L}^\circ \cap \mathbf{H}_{i_0}^\circ) = {}^a\mathbf{M}_1 \times \cdots \times {}^a\mathbf{M}_t$ and this decomposition only depends on $a.i_0$, since if $a.i_0 = a'.i_0$ then $a^{-1}a'$ is trivial on $\mathbf{H}_{i_0}^\circ$ and therefore

${}^a\mathbf{M}_1 = {}^{a'}\mathbf{M}_1, \ldots, {}^a\mathbf{M}_t = {}^{a'}\mathbf{M}_t$. This allows to write $\prod_{i\in A_{\mathbf{L}}^F.i_0} \mathbf{L}^\circ \cap \mathbf{H}_i^\circ$ as a product of GL's permuted by $A_{\mathbf{L}}^F$. Doing it for any $A_{\mathbf{L}}^F$-orbit in I gives the same for the whole of $\mathbf{L}$.

As for the action of F, since it is a Frobenius endomorphism, it can be described as a permutation σ of I and a Frobenius twisted or not on each factor $\mathbf{L}^\circ{}_i$ stabilized by a given power F^j. The permutations of I induced by $A_{\mathbf{L}}^F$ (see above) have to commute with σ since $A_{\mathbf{L}}^F$ commutes with F as endomorphisms of $\mathbf{H}^\circ$.

This gives our claim. □

The relevance of this kind of non-connected groups for the Jordan decomposition of characters is through the following.

Let $n \geqslant 1$ be an integer. Let $\widetilde{\mathbf{G}}^* = \mathrm{GL}_n(\mathbb{F})$, let $F^* : \widetilde{\mathbf{G}}^* \to \widetilde{\mathbf{G}}^*$ be the raising of matrix entries to the q-th power composed with some power of the transpose-inverse automorphism.

Let $\mathbf{G}^* = \mathrm{PGL}_n(\mathbb{F})$ be the quotient of $\widetilde{\mathbf{G}}^*$ by its center.

Proposition 2.5.

(i) *Let $\mathbf{L}^*$ be an F-stable Levi subgroup of $\widetilde{\mathbf{G}}^*$, then $(\mathrm{N}_{\widetilde{\mathbf{G}}^*}(\mathbf{L}^*), F^*)$ is isomorphic with some $(\mathbf{H}, F)$ as defined in Notation 2.1.*

(ii) *Let s be a semi-simple element in $(\mathbf{G}^*)^{F^*}$. Let $\mathbf{C}^*$ be the inverse image of $\mathrm{C}_{\mathbf{G}^*}(s)$ in $\widetilde{\mathbf{G}}^*$. Then $(\mathbf{C}^*, F^*)$ is isomorphic with some $(\mathbf{H}, F)$ as in Notation 2.1 with some cyclic $A \cong \mathbf{H}/\mathbf{H}^\circ$ of order prime to p.*

Proof. (i) We denote by $\mathbf{T}_0^*$ the torus of diagonal matrices in $\widetilde{\mathbf{G}}^*$ so that its normalizer is a semi-direct product of $\mathbf{T}_0^*$ with the subgroup $\mathcal{V}$ of all permutation matrices. Choose as generating set $S \subseteq \mathcal{V}$ the transposition matrices exchanging two consecutive elements in the canonical basis and denote by $\mathbf{L}_I$ the Levi subgroup corresponding to $I \subseteq S$. The hypothesis is that $\mathbf{L}^* = {}^g\mathbf{L}_I$ for some $g \in \widetilde{\mathbf{G}}^*$. Then $(\mathbf{L}^*, F^*)$ is isomorphic with $(\mathbf{L}_I, xF^*)$ for $x = g^{-1}F^*(g) \in \mathrm{N}_{\widetilde{\mathbf{G}}^*}(\mathbf{L}_I)$. But ${}^x\mathbf{T}_0^*$ and $\mathbf{T}_0^*$ are both maximal tori of $\mathbf{L}_I$, so $x = lw$, for $l \in \mathbf{L}_I$, $w \in \mathrm{N}_{\mathcal{V}}(\mathbf{L}_I)$. Applying Lang theorem to wF^* in $\mathbf{L}_I$, there is $h \in \mathbf{L}_I$ such that $h^{-1}wF^*(h)w^{-1} = l$ and therefore by conjugacy by h, $(\mathbf{L}^*, F^*) \cong (\mathbf{L}_I, xF^*) \cong (\mathbf{L}_I, wF^*)$. Then $(\mathrm{N}_{\widetilde{\mathbf{G}}^*}(\mathbf{L}^*), F^*) \cong (\mathrm{N}_{\widetilde{\mathbf{G}}^*}(\mathbf{L}_I), wF^*)$. But $\mathbf{L}_I$ is a product of $\mathrm{GL}_{n_i}(\mathbb{F})$'s in $\mathrm{GL}_n(\mathbb{F})$ for $\sum_i n_i = n$ and its normalizer is of the type studied before: take d_m the number of i's such that $n_i = m$. So that $\mathrm{N}_{\widetilde{\mathbf{G}}^*}(\mathbf{L}_I) = \mathbf{L}_I \rtimes A$ with $A \cong \prod_m \mathfrak{S}_{d_m}$ and $w \in A$ with F^* acting trivially on A since $A \subseteq \mathcal{V}$. Hence our claim.

(ii) $\mathrm{C}_{\mathbf{G}^*}(s)^\circ$ is an F^*-stable Levi subgroup of $\mathbf{G}^*$ (see for instance [CE04, 13.15]) and $\mathrm{C}_{\mathbf{G}^*}(s)$ normalizes it, so (i) gives our claim. For the cyclicity and p' order of $\mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}_{\mathbf{G}^*}(s)^\circ$, see for instance [CE04, 13.16.(ii)]. □

3. Parametrizing unipotent characters

We now take $(\mathbf{H}, F)$ as in Notation 2.1. Recall that $\mathbf{T}_0$ is the torus of diagonal matrices in $\mathbf{H}^\circ$ and that $W^\circ \cong \mathrm{N}_{\mathbf{H}^\circ}(\mathbf{T}_0)/\mathbf{T}_0$ is the subgroup of permutation matrices.

As in [B99, §7.4], one wants to parametrize $\mathcal{E}(\mathbf{H}^F, 1)$ (see §2.4) by $\mathrm{Irr}(W^{\circ F}.A^F)$. The model is the case of $\mathbf{H}^{\circ F}$, i.e. finite general linear or unitary groups, as treated by Lusztig and Srinivasan (see [DM91, §15.4]). Recall that for $w \in W^\circ$ we denote by $(\mathbf{T}_0)_w$ the torus of F-type w in $\mathbf{H}^\circ$ (see Proposition 1.8). From the case of general linear and unitary groups (Lusztig and Srinivasan [LS77]), corresponding to an irreducible $\mathbf{H}^{\circ\langle\sigma\rangle}$, one easily gets

Theorem 3.1. *(See [LS77, 2.2], [DM91, 15.8].) If $\eta \in \mathrm{Irr}(W^\circ)^F$, let*

$$R_\eta^\circ := \frac{1}{|W^\circ|} \sum_{w \in W^\circ} \eta(w) \mathrm{R}_{(\mathbf{T}_0)_w}^{\mathbf{H}^\circ}(1).$$

Then there is a sign $\epsilon_\eta \in \{-1, 1\}$ such that $\epsilon_\eta R_\eta^\circ \in \mathrm{Irr}(\mathbf{H}^{\circ F})$ and one has a bijection

$$\mathrm{Irr}\big(W^\circ\big)^F \to \mathcal{E}\big(\mathbf{H}^{\circ F}, 1\big),$$
$$\eta \mapsto \epsilon_\eta R_\eta^\circ.$$

Remark 3.2. When $\mathbf{H}^{\circ F}$ is a general linear group, the sign above is 1 (see [DM91, 15.8]). When it is a unitary group this sign can be determined from the degree formulas for R_η° (see [Cr85, 13.8]) obtained by changing $q \mapsto -q$ in the same formula for general linear groups. It does not depend on q.

The bijection and the formula for ϵ_η in the general case of $\mathbf{H}^{\circ F}$ is trivially obtained by direct product. Note however that ϵ_η takes into account the structure of the summand of $\mathbf{H}^{\circ F}$, not just of $W^{\circ F}$.

Looking now for a bijection $\mathrm{Irr}(W^{\circ F}.A^F) \xrightarrow{\sim} \mathcal{E}(\mathbf{H}^F, 1)$, one has for the left side a quite simple and canonical parametrization of characters from the $W^{\circ F}$ case thanks to Proposition 1.7. For the right side, one has to find a substitute to the $\chi \rtimes A_\chi$ extension of Definition 1.3 which is not available here since $\mathbf{H}^F = \mathbf{H}^{\circ F} \rtimes A^F$ is in general not a wreath product (see Remark 2.3).

Let $\eta \in \mathrm{Irr}(W^{\circ F})$ and $a \in A_\eta^F$. Having noted that $W^{\circ F} = W^{\circ\sigma}$ is again a direct product where a acts by permutation of components one has a natural bijection $\mathrm{Irr}(W^{\circ F})^{\langle a\rangle} \to \mathrm{Irr}(W^{\circ\langle F,a\rangle}) \to \mathrm{Irr}(W^{\circ\langle a\rangle})^F$ (1.2) using the fact that a and F commute. Denote by $\eta_a^F \in \mathrm{Irr}(W^{\circ\langle a\rangle})^F = \mathrm{Irr}(W^{\circ\langle a\rangle})^\sigma$ the image of η. Since F acts by the permutation σ of components in a decomposition of $W^{\circ\langle a\rangle}$ as a direct product, one has the associated extension $\eta_a^F \rtimes \langle\sigma\rangle \in \mathrm{Irr}(W^{\circ\langle a\rangle} \rtimes \langle\sigma\rangle)$ (see Definition 1.3).

Definition 3.3. Let $\widetilde{R}_\eta$ the element of $\mathcal{C}(\mathbf{H}^{\circ F}.A_\eta^F)$ such that for any $h \in \mathbf{H}^{\circ F}$, $a \in A_\eta^F$,

$$\widetilde{R}_\eta(ha) = \overline{\sum_{w \in W^{\circ\langle a\rangle}}} \big(\eta_a^F \rtimes \langle\sigma\rangle\big)(w\sigma)\mathrm{R}_{(\mathbf{T}_0.\langle a\rangle)_w}^{\mathbf{H}^\circ.\langle a\rangle}(1)(ha),$$

where $(\mathbf{T}_0.\langle a\rangle)_w$ denotes any maximal quasi-torus in $\mathbf{H}^\circ\langle a\rangle$ of type w with regard to $\mathbf{T}_0\langle a\rangle$ (see Proposition 1.8).

The fact that $\widetilde{R}_\eta$ is indeed a central function on $\mathbf{H}^{\circ F}.A_\eta^F$ is an easy consequence of the following

Lemma 3.4. *If $b \in A^F$ and $\eta \in \mathrm{Irr}(W^{\circ F})$, then ${}^bA_\eta^F = A_{{}^b\eta}^F$ and $b.\widetilde{R}_\eta = \widetilde{R}_{{}^b\eta}$.*

Proof. The first claim is clear. For the second claim, let us fix $a \in A_\eta^F$ and $w \in W^{\circ\langle a\rangle}$. It suffices to check that $b.\mathrm{R}_{(\mathbf{T}_0.\langle a\rangle)_w}^{\mathbf{H}^\circ.\langle a\rangle}(1) = \mathrm{R}_{(\mathbf{T}_0.\langle {}^ba\rangle)_{{}^bw}}^{\mathbf{H}^\circ.\langle {}^ba\rangle}(1)$. The equivariance of ℓ-adic cohomology yields $b.\mathrm{R}_{(\mathbf{T}_0.\langle a\rangle)_w}^{\mathbf{H}^\circ.\langle a\rangle}(1) = \mathrm{R}_{{}^b(\mathbf{T}_0.\langle a\rangle)_w}^{\mathbf{H}^\circ.\langle {}^ba\rangle}(1)$. On the other hand, the fact that $\mathbf{T} := (\mathbf{T}_0.\langle a\rangle)_w$ is of type $w \in W^{\circ\langle a\rangle}$ in $\mathbf{H}^\circ\langle a\rangle$ amounts to the fact that there is an $(\mathbf{H}^\circ\langle a\rangle)^F$-conjugate $\mathbf{T}'$ of $\mathbf{T}$ which contains a and such that $(\mathbf{T}'^{\langle a\rangle})^\circ$ is of type w in $(\mathbf{H}^{\langle a\rangle})^\circ$, which in turn means that $(\mathbf{T}'^{\langle a\rangle})^\circ = ((\mathbf{T}_0{}^{\langle a\rangle})^\circ)^x$ for $x \in (\mathbf{H}^{\langle a\rangle})^\circ$ such that $x^{-1}F(x)(\mathbf{T}_0^{\langle a\rangle})^\circ = w$ (see Proposition 1.8). Then it is clear that ${}^b\mathbf{T}$ is $(\mathbf{H}^\circ\langle {}^ba\rangle)^F$-conjugate to ${}^b\mathbf{T}'$ containing ${}^ba$ and we have $({}^b\mathbf{T}'^{\langle {}^ba\rangle})^\circ = ((\mathbf{T}_0{}^{\langle {}^ba\rangle})^\circ)^{{}^bx}$ with $({}^bx)^{-1}F({}^bx)(\mathbf{T}_0^{\langle {}^ba\rangle})^\circ = {}^b(x^{-1}F(x)(\mathbf{T}_0^{\langle a\rangle})^\circ) = {}^bw$ since b is F-fixed. $\square$

Note that Definition 3.3 readily implies $\mathrm{Res}_{\mathbf{H}^{\circ F}}^{\mathbf{H}^{\circ F}.A_\eta^F}\widetilde{R}_\eta = R_\eta^\circ$.

The main result is as follows. The proof is in our last section.

Theorem 3.5. *Let $\eta \in \mathrm{Irr}(W^{\circ F})$, then $\epsilon_\eta\widetilde{R}_\eta \in \mathrm{Irr}(\mathbf{H}^{\circ F}.A_\eta^F)$.*

The consequence on parametrization of unipotent characters of $\mathbf{H}^F$ is as follows.

Theorem 3.6. *For $\eta \in \mathrm{Irr}(W^{\circ F})$, $\xi \in \mathrm{Irr}(A^F_\eta)$, thus defining $\eta * \xi \in \overline{\mathcal{I}}(W^{\circ F}, A^F)$ (see Definition 1.6), let*

$$R_{\eta*\xi} := \mathrm{Ind}^{\mathbf{H}^F}_{\mathbf{H}^{\circ F}.A^F_\eta}(\xi.\widetilde{R}_\eta) \in \mathcal{C}(\mathbf{H}^F).$$

Then one gets a bijection

$$\mathrm{Irr}\big(\mathrm{W}_{\mathbf{H}}(\mathbf{T}_0)^F\big) \to \mathcal{E}\big(\mathbf{H}^F, 1\big)$$

sending $\mathrm{Ind}^{W^{\circ F} \rtimes A^F}_{W^{\circ F} \rtimes A^F_\eta}(\xi.(\eta \rtimes A^F_\eta))$ *to* $\epsilon_\eta R_{\eta*\xi}$.

Proof. We have first $A^F_\eta = A^F_{R^\circ_\eta}$. Given the bijectivity of $\eta \mapsto R^\circ_\eta$ from Theorem 3.1.(ii), this is a consequence of the fact that for any $a \in A^F$, ${}^aR^\circ_\eta = R^\circ_{{}^a\eta}$, which is proved in the same fashion as the above Lemma 3.4.

One has $\mathrm{W}_{\mathbf{H}}(\mathbf{T}_0)^F \cong W^{\circ F} \rtimes A^F$, a wreath product, so Clifford theorem in the form of the above Proposition 1.7 shows that $\eta * \xi \mapsto \mathrm{Ind}^{W^{\circ F} \rtimes A^F}_{W^{\circ F} \rtimes A^F_\eta}(\xi.(\eta \rtimes A^F_\eta))$ gives a bijection $\overline{\mathcal{I}}(W^{\circ F}, A^F) \xrightarrow{\sim} \mathrm{Irr}(\mathrm{W}_{\mathbf{H}}(\mathbf{T}_0)^F)$.

For a given $\epsilon_\eta R^\circ_\eta \in \mathcal{E}(\mathbf{H}^{\circ F}, 1)$, $\epsilon_\eta \widetilde{R}_\eta \in \mathrm{Irr}(\mathbf{H}^{\circ F}.A^F_\eta)$ is an extension by Theorem 3.5. We have seen that $\mathbf{H}^{\circ F}.A^F_\eta$ is the stabilizer of $\epsilon_\eta R^\circ_\eta$ in $\mathbf{H}^F$. Therefore, by Clifford theorem [N98, 8.9], the irreducible characters of $\mathbf{H}^F$ lying over $\epsilon_\eta R^\circ_\eta$ are of the type $\epsilon_\eta R_{\eta*\xi}$ for some $\xi \in \mathrm{Irr}(A^F_\eta)$. This gives a bijection $\eta * \xi \mapsto \epsilon_\eta R_{\eta*\xi}$ between $\overline{\mathcal{I}}(W^{\circ F}, A^F)$ and $\mathcal{E}(\mathbf{H}^F, 1)$ by the definition of the latter. □

4. Twisted induction

We keep $(\mathbf{H}, F)$, $\mathbf{T}_0$, W° as in Section 3.

In this section, we assume that A^F is abelian and a p'-group (see however Remark 4.8 below). This suits the applications we have in mind, thanks to Proposition 2.5.(ii).

Here again we follow [B99, §7.6] to give the effect of Lusztig's twisted induction functor on the parametrization of Theorem 3.6. The first step is to apply a Mellin transform with regard to the group A^F to our irreducible characters $\epsilon_\eta R_{\eta*\xi}$. Recall the set $\overline{\mathcal{I}}^\wedge(W^{\circ F}, A^F)$ from Definition 1.6.

Definition 4.1. For $\eta * a \in \overline{\mathcal{I}}^\wedge(W^{\circ F}, A^F)$, let

$$\hat{R}_{\eta*a} = \sum_{\xi \in \mathrm{Irr}(A^F_\eta)} \xi\big(a^{-1}\big) R_{\eta*\xi}.$$

Using the commutativity of A^F, this is another orthogonal basis of $\mathbb{C}\mathcal{E}(\mathbf{H}^F, 1)$ with base change matrix $(\frac{\xi(a)}{|A^F_\chi|}\delta_{\chi,\chi'})_{\chi*a,\chi'*\xi}$ and for $h \in \mathbf{H}^{\circ F}$, $a, b \in A^F$

$$\hat{R}_{\eta*a}(hb) = \big|A^F_\eta\big| R_{\eta*1}(hb) \quad \text{if } a = b, \qquad 0 \quad \text{if } a \neq b \tag{4.2}$$

see [B99, 7.5.2].

Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{H}$. By [B99, 6.2.2], up to $\mathbf{H}^{\circ F}$-conjugacy, we may choose $\mathbf{L}$ such that $\mathbf{L}$ contains $A^F_{\mathbf{L}}$ where $A_{\mathbf{L}}$ is the image of $\mathbf{L}$ in $A \cong \mathbf{H}/\mathbf{H}^\circ$. Since with our definition $\mathbf{L}^\circ.A^F_{\mathbf{L}}$ is another Levi subgroup, we may replace $\mathbf{L}$ by another Levi subgroup with same $\mathbf{L}^F$ such that $\mathbf{L} = \mathbf{L}^\circ.\mathbf{L}^F = \mathbf{L}^\circ.A^F_{\mathbf{L}}$, so that

$$A^F_{\mathbf{L}} = A_{\mathbf{L}} = A \cap \mathbf{L}.$$

Note that this won't change the functor $\mathrm{R}^{\mathbf{H}}_{\mathbf{L}\subseteq\mathbf{P}}$ since the parabolic subgroup $\mathbf{P}$ will be changed into $\mathbf{P}^\circ.\mathbf{L}^F = \mathrm{R}_u(\mathbf{P})\mathbf{L}^\circ.\mathbf{L}^F$ with same unipotent radical hence same associated variety $\mathbf{Y}$, see Section 1.3. By Proposition 2.4.(ii), the parametrization of $\mathcal{E}(\mathbf{L}^F, 1)$ of Theorem 3.6 applies.

Let $\mathbf{T_L}$ a diagonal torus of $\mathbf{L}^\circ$ which can clearly be taken $A_\mathbf{L}$-stable.

Lemma 4.3. *One has* ${}^{g_1}\mathbf{T}_0 = \mathbf{T_L}$ *for some* $g_1 \in \mathbf{H}^{\circ A_\mathbf{L}}$ *such that* $w_1 := g_1^{-1}F(g_1) \in W^\circ \cap \mathbf{H}^{\circ A_\mathbf{L}}$. *Moreover* $A_\mathbf{L}^{w_1F} = A_\mathbf{L}^F = A_\mathbf{L} = A \cap \mathbf{L}$.

Proof. Since $\mathbf{T_L}$ is $A_\mathbf{L}$-stable, Proposition 2.4.(i) implies that ${}^{g}\mathbf{T}_0 = \mathbf{T_L}$ for some $g \in \mathbf{H}^{\circ A_\mathbf{L}}$. Then denote $w_1 = g^{-1}F(g) \in \mathrm{N}_\mathbf{H}(\mathbf{T}_0) \cap \mathbf{H}^{\circ A_\mathbf{L}}$. One may also assume that $w_1 \in W^\circ$ since $\mathrm{N}_{\mathbf{H}^\circ}(\mathbf{T}_0) = \mathbf{T}_0.W^\circ$ and if $g^{-1}F(g) = t.w_1$, with $t \in \mathbf{T}_0$, $w_1 \in W^\circ$, one may write $t = t'w_1F(t')^{-1}w_1^{-1}$ with $t' \in \mathbf{T}_0$ thanks to Lang theorem applied to w_1F in $\mathbf{T}_0$, then replace g with gt'.

For the last statement, note that, g_1 being $A_\mathbf{L}$-fixed and $A_\mathbf{L}$ being F-fixed, $w_1 = g_1^{-1}F(g_1)$ is also $A_\mathbf{L}$-fixed. Then $A_\mathbf{L} = A_\mathbf{L}^F = A_\mathbf{L}^{w_1F}$. $\square$

Note that $W_\mathbf{L}^\circ := \mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T_L})^{g_1} \subseteq \mathrm{W}_{\mathbf{H}^\circ}(\mathbf{T}_0)$ and the pair $(\mathbf{L}, F)$ is sent to $(\mathbf{L}^{g_1}, w_1F)$ by g_1-conjugacy. Theorem 3.6 for $\mathbf{L}$ gives a basis $\hat{R}^\mathbf{L}_{\lambda*a}$ (for $\lambda * a \in \overline{\mathcal{I}}^\wedge(\mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T_L})^F, A_\mathbf{L})$) of $\mathbb{C}\mathcal{E}(\mathbf{L}^F, 1)$. So determining the effect of the functor $\mathrm{R}^{\mathbf{H}}_{\mathbf{L}\subseteq\mathbf{P}}$ on $\mathcal{E}(\mathbf{L}^F, 1)$ amounts to give a formula for $\mathrm{R}^{\mathbf{H}}_{\mathbf{L}\subseteq\mathbf{P}}(\hat{R}^\mathbf{L}_{\lambda*a})$.

Note that the formula does not depend on the choice of $\mathbf{P}$. Indeed, by Proposition 1.9.(ii), one has independence for tori and, along with transitivity (Proposition 1.9.(i)) and our definition of $\hat{R}_{\lambda*a}$ as a combination of $\mathrm{R}^{\mathbf{H}}_{\mathbf{T}}$'s, parabolics do not matter. So we now omit $\mathbf{P}$ in our statements.

Let us fix $\lambda \in \mathrm{Irr}(\mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T_L})^F)$ and $a \in (A \cap \mathbf{L})_\lambda$, thus defining $\lambda * a \in \overline{\mathcal{I}}^\wedge(\mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T_L})^F, A \cap \mathbf{L})$. Recall $\lambda_a^F \in \mathrm{Irr}(\mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T_L})^{\langle a\rangle})^F$ associated with $\lambda \in \mathrm{Irr}(\mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T_L})^F)^{\langle a\rangle}$ thanks to (1.2). Let $\lambda'_a \in \mathrm{Irr}(W_\mathbf{L}^{\circ\langle a\rangle})^{w_1F}$ the image of λ_a^F by g_1-conjugacy $x \mapsto x^{g_1}$ (recall from Lemma 4.3 that g_1 commutes with a).

Recall the notations from Definition 1.3.

Theorem 4.4. *One has*

$$\mathrm{R}^{\mathbf{H}}_{\mathbf{L}}\left(\hat{R}^\mathbf{L}_{\lambda*a}\right) = \sum_{\eta \in \mathrm{Irr}(W^{\circ F})^{\langle a\rangle}} m_\eta \hat{R}^\mathbf{H}_{\eta*a}$$

where the scalars $m_\eta \in \mathbb{C}$ are defined by

$$\mathrm{Ind}^{W^{\circ\langle a\rangle}.\sigma}_{W_\mathbf{L}^{\circ\langle a\rangle}.w_1\sigma}\left(\lambda'_a \rtimes w_1\sigma\right) = \sum_{\eta \in \mathrm{Irr}(W^{\circ F})^{\langle a\rangle}} m_\eta(\eta_a \rtimes \sigma) \quad \textit{in } \mathcal{C}\left(W^{\circ\langle a\rangle}.\sigma\right).$$

Note that the m_η's are uniquely defined thanks to Proposition 1.5.

Proof of Theorem 4.4. We follow the main lines of the proof of [B99, 7.6.1].

First it is clear from (4.2) that $\hat{R}^\mathbf{L}_{\lambda*a}$ has support in $\mathbf{L}^{\circ F}.a$, and therefore also its image by $\mathrm{R}^\mathbf{H}_\mathbf{L}$ has support in $\mathbf{H}^{\circ F}.a$ thanks to the character formula in a case where $|\mathbf{H}/\mathbf{H}^\circ|$ is prime to p (see [DM94, 2.6, 2.7]). So it suffices to compare both sides of the claimed equality of Theorem 4.4 on restricting to $\mathbf{H}^{\circ F}.\langle a\rangle$.

For the right side, using the definition of $\hat{R}_{\eta*a}$ and $R_{\eta*\xi}$ (see Theorem 3.6), one has

$$\mathrm{Res}^{\mathbf{H}^F}_{\mathbf{H}^{\circ F}.\langle a\rangle}\hat{R}^\mathbf{H}_{\eta*a} = \mathrm{Res}^{\mathbf{H}^F}_{\mathbf{H}^{\circ F}.\langle a\rangle}\mathrm{Ind}^{\mathbf{H}^F}_{\mathbf{H}^{\circ F}.A_\eta^F}\left(\sum_{\xi \in \mathrm{Irr}(\mathbf{H}^{\circ F}.A_\eta^F/\mathbf{H}^{\circ F})} \xi\left(a^{-1}\right)\xi.\widetilde{R}_\eta\right).$$

Moreover, $\overline{\sum}_{\xi\in\mathrm{Irr}(\mathbf{H}^{\circ F}.A_\eta^F/\mathbf{H}^{\circ F})}\xi(a^{-1})\xi=\mathbf{1}_{\mathbf{H}^{\circ F}.a}$ (characteristic function of the coset $\mathbf{H}^{\circ F}.a\subseteq\mathbf{H}^{\circ F}.A_\eta^F$). Noting that $\mathbf{H}^F=\mathbf{H}^{\circ F}.A^F$ with abelian A^F, and using the Mackey formula for the ordinary restriction and induction from subgroups, one gets $\mathrm{Res}^{\mathbf{H}^F}_{\mathbf{H}^{\circ F}.\langle a\rangle}\hat{R}^{\mathbf{H}}_{\eta*a}=\sum_{b\in A^F}\mathrm{Res}^{\mathbf{H}^{\circ F}.A^F_\eta}_{\mathbf{H}^{\circ F}.\langle a\rangle}{}^b(\mathbf{1}_{\mathbf{H}^{\circ F}.a}.\widetilde{R}_\eta)$. Using now the definition of $\widetilde{R}_\eta$ (see Definition 3.3), one gets

$$\mathrm{Res}^{\mathbf{H}^F}_{\mathbf{H}^{\circ F}.\langle a\rangle}\hat{R}^{\mathbf{H}}_{\eta*a}=\sum_{b\in A^F}{}^b\Bigg(\mathbf{1}_{\mathbf{H}^{\circ F}.a}\overline{\sum_{w\in W^{\circ\langle a\rangle}}}\big(\eta_a^F\rtimes\sigma\big)(w\sigma)\mathrm{R}^{\mathbf{H}^\circ\langle a\rangle}_{(\mathbf{T}_0\langle a\rangle)_w}(1)\Bigg). \tag{4.5}$$

Concerning the left hand side of the claimed equality of Theorem 4.4, applying first Proposition 1.9.(iii), one has

$$\mathrm{Res}^{\mathbf{H}^F}_{\mathbf{H}^{\circ F}.\langle a\rangle}\mathrm{R}^{\mathbf{H}}_{\mathbf{L}}\big(\hat{R}^{\mathbf{L}}_{\lambda*a}\big)=|A\cap\mathbf{L}|^{-1}\sum_{b\in A^F}{}^b\big(\mathrm{R}^{\mathbf{H}^\circ.\langle a\rangle}_{\mathbf{L}^\circ\langle a\rangle}\big(\mathrm{Res}^{\mathbf{L}^F}_{\mathbf{L}^{\circ F}.\langle a\rangle}\hat{R}^{\mathbf{L}}_{\lambda*a}\big)\big).$$

On the other hand, arguments similar to the above allow to write

$$\mathrm{Res}^{\mathbf{L}^F}_{\mathbf{L}^{\circ F}.\langle a\rangle}\hat{R}^{\mathbf{L}}_{\lambda*a}=\sum_{c\in A\cap\mathbf{L}}{}^c\Bigg(\mathbf{1}_{\mathbf{L}^{\circ F}.a}\overline{\sum_{w\in \mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T}_\mathbf{L})^{\langle a\rangle}}}\big(\lambda_a^F\rtimes\sigma\big)(w\sigma)\mathrm{R}^{\mathbf{L}^\circ\langle a\rangle}_{(\mathbf{T}_\mathbf{L}\langle a\rangle)_w}(1)\Bigg)$$

and therefore

$$\mathrm{Res}^{\mathbf{H}^F}_{\mathbf{H}^{\circ F}.\langle a\rangle}\mathrm{R}^{\mathbf{H}}_{\mathbf{L}}\big(\hat{R}^{\mathbf{L}}_{\lambda*a}\big)=\sum_{b\in A^F}{}^b\Bigg(\mathrm{R}^{\mathbf{H}^\circ.\langle a\rangle}_{\mathbf{L}^\circ\langle a\rangle}\Bigg(\mathbf{1}_{\mathbf{L}^{\circ F}.a}\overline{\sum_{w\in \mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T}_\mathbf{L})^{\langle a\rangle}}}\big(\lambda_a^F\rtimes\sigma\big)(w\sigma)\mathrm{R}^{\mathbf{L}^\circ\langle a\rangle}_{(\mathbf{T}_\mathbf{L}\langle a\rangle)_w}(1)\Bigg)\Bigg).$$

By a classical application of the character formula (see [DM91, 12.6] for the connected case) and since A is a p'-group, one has $\mathrm{R}^{\mathbf{H}^\circ.\langle a\rangle}_{\mathbf{L}^\circ\langle a\rangle}(\mathbf{1}_{\mathbf{L}^{\circ F}a}.f)=\mathbf{1}_{\mathbf{H}^{\circ F}a}.\mathrm{R}^{\mathbf{H}^\circ.\langle a\rangle}_{\mathbf{L}^\circ\langle a\rangle}(f)$ for any $f\in\mathcal{C}(\mathbf{L}^{\circ F}\langle a\rangle)$. Applying also transitivity of $\mathrm{R}^{\mathbf{G}}_{\mathbf{L}}$ functors (Proposition 1.9.(i)), one gets

$$\mathrm{Res}^{\mathbf{H}^F}_{\mathbf{H}^{\circ F}.\langle a\rangle}\mathrm{R}^{\mathbf{H}}_{\mathbf{L}}\big(\hat{R}^{\mathbf{L}}_{\lambda*a}\big)=\sum_{b\in A^F}{}^b\Bigg(\mathbf{1}_{\mathbf{H}^{\circ F}.a}\overline{\sum_{w\in \mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T}_\mathbf{L})^{\langle a\rangle}}}\big(\lambda_a^F\rtimes\sigma\big)(w\sigma)\mathrm{R}^{\mathbf{H}^\circ\langle a\rangle}_{(\mathbf{T}_\mathbf{L}\langle a\rangle)_w}(1)\Bigg).$$

Using now conjugacy by $g_1\in(\mathbf{H}^\circ)^{A\cap\mathbf{L}}$ from Lemma 4.3, one sees that a maximal quasi-torus of $\mathbf{H}^\circ\langle a\rangle$ of type $w\in\mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T}_\mathbf{L})^{\langle a\rangle}$ with regard to $\mathbf{T}_\mathbf{L}\langle a\rangle$ will be of type $w^{g_1}.w_1\in W^\circ_\mathbf{L}w_1\subseteq W^\circ=\mathrm{W}_{\mathbf{H}^\circ}(\mathbf{T}_0)$ with regard to $\mathbf{T}_0\langle a\rangle$. Indeed if that torus is ${}^g\mathbf{T}_\mathbf{L}$ with $g^{-1}F(g).\mathbf{T}_\mathbf{L}=w$, then ${}^g\mathbf{T}_\mathbf{L}={}^{gg_1}\mathbf{T}_0$ with $(gg_1)^{-1}F(gg_1)=(g^{-1}F(g))^{g_1}.g_1^{-1}F(g_1)\in w^{g_1}w_1\mathbf{T}_0$. Conjugacy by g_1 sends $(\mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T}_\mathbf{L}),a,F)$ to $(W^\circ_\mathbf{L},a,w_1F)$, so one may rewrite now $\overline{\sum}_{w\in\mathrm{W}_{\mathbf{L}^\circ}(\mathbf{T}_\mathbf{L})^{\langle a\rangle}}(\lambda_a^F\rtimes\sigma)(w\sigma)\mathrm{R}^{\mathbf{H}^\circ\langle a\rangle}_{(\mathbf{T}_\mathbf{L}\langle a\rangle)_w}(1)=\overline{\sum}_{w'\in W_\mathbf{L}^{\circ\langle a\rangle}}(\lambda'_a\rtimes w_1\sigma)(w'w_1\sigma)\mathrm{R}^{\mathbf{H}^\circ\langle a\rangle}_{(\mathbf{T}_0\langle a\rangle)_{w'w_1}}(1)$, thus implying

$$\mathrm{Res}^{\mathbf{H}^F}_{\mathbf{H}^{\circ F}.\langle a\rangle}\mathrm{R}^{\mathbf{H}}_{\mathbf{L}}\big(\hat{R}^{\mathbf{L}}_{\lambda*a}\big)=\sum_{b\in A^F}{}^b\Bigg(\mathbf{1}_{\mathbf{H}^{\circ F}.a}\overline{\sum_{w'\in W_\mathbf{L}^{\circ\langle a\rangle}}}\big(\lambda'_a\rtimes w_1\sigma\big)\big(w'w_1\sigma\big)\mathrm{R}^{\mathbf{H}^\circ\langle a\rangle}_{(\mathbf{T}_0\langle a\rangle)_{w'w_1}}(1)\Bigg). \tag{4.6}$$

So in view of (4.6) and (4.5), the claim of our theorem is established once we can show that

$$\overline{\sum_{w'\in W_{\mathbf{L}}^{\circ\langle a\rangle}}}\big(\lambda'_a \rtimes w_1\sigma\big)\big(w'w_1\sigma\big)\mathrm{R}^{\mathbf{H}^\circ\langle a\rangle}_{(\mathbf{T}_0\langle a\rangle)_{w'w_1}}(1)$$
$$=\sum_{\eta\in\mathrm{Irr}(W^{\circ F})^{\langle a\rangle}} m_\eta \overline{\sum_{w\in W^{\circ\langle a\rangle}}}\big(\eta_a^F\rtimes\sigma\big)(w\sigma)\mathrm{R}^{\mathbf{H}^\circ\langle a\rangle}_{(\mathbf{T}_0\langle a\rangle)_w}(1). \tag{4.7}$$

This follows from a quite formal argument, applying Lemma 3.1.1 in [B99] with (in the notations of that lemma) $H = W^{\circ\langle a\rangle}$, $K = W_{\mathbf{L}}^{\circ\langle a\rangle}$, $x = w_1$, $E(H,\sigma) = \mathcal{C}(\mathbf{H}^{\circ F}\langle a\rangle)$, $\rho_w^H = \mathrm{R}^{\mathbf{H}^\circ\langle a\rangle}_{(\mathbf{T}_0\langle a\rangle)_w}(1)$ for $w\in H$, $E(K,x\sigma) = \mathcal{C}(\mathbf{L}^{\circ F}\langle a\rangle)$, $\rho_{w'}^K = \mathrm{R}^{\mathbf{L}^\circ\langle a\rangle}_{(\mathbf{T}_0\langle a\rangle)_{w'w_1}}(1)$ for $w'\in K$, and $R_K^H = \mathrm{R}^{\mathbf{H}^\circ\langle a\rangle}_{\mathbf{L}^\circ\langle a\rangle}$. □

Remark 4.8. C. Bonnafé has pointed to us that one may lift the hypothesis that A is a p'-group. This was used twice in the above proof to show that when $f\in\mathcal{C}(\mathbf{L}^F)$ has support in a single class $\mathbf{L}^{\circ F}.a$, then $\mathrm{R}^{\mathbf{H}}_{\mathbf{L}\subseteq\mathbf{P}}(f)$ has support included in $\mathbf{H}^{\circ F}.a$. This can be seen by noting first that the variety $\mathbf{Y}$ described in Section 1.3 above decomposes as $\coprod_{a\in A^F}\mathbf{Y}^\circ.a$ with $\mathbf{Y}^\circ := \mathbf{Y}\cap\mathbf{H}^\circ$ an open subvariety. Then if $(h,l)\in\mathbf{H}^F\times(\mathbf{L}^F)^{\mathrm{opp}}$ is such that $h\in\mathbf{H}^{\circ F}a$ and $l\in\mathbf{L}^{\circ F}b$ with $a\neq b$ in A^F, (h,l) will permute the summands above without stabilizing any, hence will have trace 0 on the corresponding cohomology groups $\mathrm{H}^i_c(\mathbf{Y},\overline{\mathbb{Q}}_\ell)$.

Let $\mathbf{G} := \mathrm{SL}_n(\mathbb{F})$ and $F:\mathbf{G}\to\mathbf{G}$ the raising of matrix entries to the q-th power composed with some power of the transpose-inverse automorphism. Let $\mathbf{G}^* = \mathrm{PGL}_n(\mathbb{F})$ endowed with the Frobenius endomorphism F^* defined in the same way. If $s\in\mathbf{G}^{*F^*}$ is a semi-simple element, the rational series $\mathcal{E}(\mathbf{G}^F,[s])\subseteq\mathrm{Irr}(\mathbf{G}^F)$ are defined in [B06, 11.A]. Following [B06, 32.1], the bijection $\mathcal{E}(\mathbf{G}^F,[s])\to\mathrm{Irr}(W(s)^{w_sF^*})$ [B06, 23.7] and the above Theorem 3.6 and Theorem 4.4 (generalizing [B99, 7.4.3 and 7.6.1], respectively) imply the following

Theorem 4.9. *Assume q is such that the conjecture $\mathfrak{G}$ of [B06, §14.E] is satisfied.*

For any semi-simple element $s\in\mathbf{G}^{*F^*}$, one has a bijection $\aleph_{\mathbf{G},s}:\mathcal{E}(\mathbf{G}^F,[s])\to\mathcal{E}(\mathrm{C}_{\mathbf{G}^*}(s)^{F^*},1)$, such that for any Levi $\mathbf{L}^*$ such that $\mathbf{L}^{*F^*}$ contains s we have a commutative square

$$\begin{array}{ccc}
\mathbb{Z}\mathcal{E}(\mathbf{L}^F,[s]) & \xrightarrow{\aleph_{\mathbf{L},s}} & \mathbb{Z}\mathcal{E}(\mathrm{C}_{\mathbf{L}^*}(s)^{F^*},1)\\
\Big\downarrow{\scriptstyle \epsilon_{\mathbf{G}}\epsilon_{\mathbf{L}}\mathrm{R}^{\mathbf{G}}_{\mathbf{L}}} & & \Big\downarrow{\scriptstyle \epsilon_{\mathrm{C}_{\mathbf{G}^*}(s)^\circ}\epsilon_{\mathrm{C}_{\mathbf{L}^*}(s)^\circ}\mathrm{R}^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathrm{C}_{\mathbf{L}^*}(s)}}\\
\mathbb{Z}\mathcal{E}(\mathbf{G}^F,[s]) & \xrightarrow{\aleph_{\mathbf{G},s}} & \mathbb{Z}\mathcal{E}(\mathrm{C}_{\mathbf{G}^*}(s)^{F^*},1)
\end{array}$$

(*where we have extended linearly the bijections* $\aleph$).

Proof. Thanks to Proposition 2.5.(ii), $\mathbf{H} := \mathrm{C}_{\mathbf{G}^*}(s)$ is of the type studied in the preceding section.

Let's recall why $\mathrm{C}_{\mathbf{L}^*}(s)$ is a Levi subgroup of $\mathbf{H}$ in the sense of our Section 1.3. Indeed, if $\mathbf{P}^* = \mathbf{U}^*.\mathbf{L}^*$ is a parabolic subgroup of $\mathbf{G}^*$ of which $\mathbf{L}^*$ is a Levi subgroup, the classical description of $\mathrm{C}_{\mathbf{G}^*}(s)$ in terms of roots (see [DM91, 2.3]) implies that $\mathrm{C}^\circ_{\mathbf{P}^*}(s)$ is a parabolic subgroup of $\mathrm{C}_{\mathbf{G}^*}(s)^\circ$ with Levi decomposition $\mathrm{C}_{\mathbf{U}^*}(s)\mathrm{C}_{\mathbf{L}^*}(s)^\circ$. Then $\mathrm{C}_{\mathbf{P}^*}(s)$ is a parabolic subgroup of $\mathrm{C}_{\mathbf{G}^*}(s)$ and $\mathrm{C}_{\mathbf{L}^*}(s) = \mathrm{N}_{\mathrm{C}_{\mathbf{P}^*}(s)}(\mathrm{C}_{\mathbf{P}^*}(s)^\circ,\mathrm{C}_{\mathbf{L}^*}(s)^\circ)$ since the intersection of the latter with $\mathrm{C}_{\mathbf{U}^*}(s)$ is trivial, due to $\mathrm{C}_{\mathbf{U}^*}(s) = \mathrm{C}_{\mathbf{U}^*}(s)^\circ$ (see [DM91, 2.5]) and the Levi decomposition of $\mathrm{C}_{\mathbf{P}^*}(s)^\circ$.

Note that the vertical arrows do not mention the parabolic subgroups. For the one on the right, this is due to the fact mentioned above before stating Theorem 4.4. Then one may choose matching parabolics on both sides and, once the theorem is proved, independence on the right side implies independence on the left. However, independence of $\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ with regard to $\mathbf{P}$ is known (see [B00, 6.1.1]).

Let $A_{\mathbf{G}^*}(s) := \mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}^\circ_{\mathbf{G}^*}(s)$, so that $\mathrm{C}_{\mathbf{G}^*}(s) = \mathrm{C}^\circ_{\mathbf{G}^*}(s) \rtimes A_{\mathbf{G}^*}(s)$, both factors being F^*-stable (see Proposition 2.5.(ii)) and denote $W(s) := \mathrm{W}_{\mathrm{C}_{\mathbf{G}^*}(s)}(\mathbf{T}^*) \supseteq W^\circ(s) := \mathrm{W}_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)}(\mathbf{T}^*)$ where $\mathbf{T}^*$ is any maximally split torus of $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$. As in [B06, §23], let $w_s \in W^\circ(s)^{A_{\mathbf{G}^*}(s)^{F^*}}$ such that $w_s F^*$ acts on $W^\circ(s)$ by permutation of irreducible components (see [B06, 23.1]).

One defines the bijection $\aleph_{\mathbf{G},s}$ as a composite of two bijections

$$\aleph_{\mathbf{G},s} : \mathcal{E}\big(\mathbf{G}^F, [s]\big) \leftarrow \mathrm{Irr}\big(W(s)^{w_s F^*}\big) \to \mathcal{E}\big(\mathrm{C}_{\mathbf{G}^*}(s)^{F^*}, 1\big).$$

For the left side, one has the isometry $R[s] : \mathbb{C}\,\mathrm{Irr}(W(s)^{w_s F^*}) \to \mathbb{C}\mathcal{E}(\mathbf{G}^F, [s])$ of [B06, 23.5] such that for any $\theta \in \mathrm{Irr}(W(s)^{w_s F^*})$, $R[s](\theta) \in \epsilon_\theta \mathcal{E}(\mathbf{G}^F, [s])$ for a sign $\epsilon_\theta = \epsilon_{\mathbf{G}} \epsilon_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)} \epsilon_\eta$ if θ lies above some $\eta \in \mathrm{Irr}(W^\circ(s)^{w_s F^*})$ (see [B06, 23.9], this is where the conjecture $\mathfrak{G}$ is assumed). The sign ϵ_η is the one denoted that way in the above Theorem 3.1 as can be seen by applying [B06, 23.13 and 23.15] with $s = 1$.

For the right hand side, note first that $W(s)^{w_s F^*} \cong \mathrm{W}_{\mathrm{C}_{\mathbf{G}^*}(s)}(\mathbf{T}^*_0)^{F^*}$ where $\mathbf{T}^*_0$ is a diagonal torus of $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$. This is because, by [B06, §23], the type of $\mathbf{T}^*_0$ with regard to $\mathbf{T}^*$ is the longest element of the Weyl group on each summand of unitary type (and trivial otherwise), which coincides with w_s except for summands of rank 1, but then the fixed point groups are the same. Now the bijection of Theorem 3.6 for $\mathbf{H} := \mathrm{C}_{\mathbf{G}^*}(s)$ gives a bijection $\mathrm{Irr}(W(s)^{w_s F^*}) \to \mathcal{E}(\mathrm{C}_{\mathbf{G}^*}(s)^{F^*}, 1)$.

The composite $\aleph_{\mathbf{G},s}$ we get is by

$$\epsilon_{\mathbf{G}} \epsilon_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)} \epsilon_\eta R[s](\theta) \hookleftarrow \theta := \eta * \xi \mapsto \epsilon_\eta R_\theta$$

(notations of Theorem 3.6).

In view of the formula of compatibility with $\mathrm{R}^{\mathbf{G}}_{\mathbf{L}}$ functors to prove, we can remove the signs and study the maps extended by linearity

$$\mathbb{C}\mathcal{E}\big(\mathbf{G}^F, [s]\big) \leftarrow \mathbb{C}\,\mathrm{Irr}\big(W(s)^{w_s F^*}\big) \to \mathbb{C}\mathcal{E}\big(\mathrm{C}_{\mathbf{G}^*}(s)^{F^*}, 1\big)$$

induced by

$$R[s](\theta) \hookleftarrow \theta := \eta * \xi \mapsto R_\theta.$$

Denote $A := A_{\mathbf{G}^*}(s)^{F^*}$. Each map is a direct sum of the maps

$$\mathbb{C}\mathcal{E}\big(\mathbf{G}^F, [s]\big) \leftarrow \mathcal{C}\big(W^\circ(s)^{w_s F^*} a\big)^A \to \mathbb{C}\mathcal{E}\big(\mathrm{C}_{\mathbf{G}^*}(s)^{F^*}, 1\big)$$

defined for each $a \in A_{\mathbf{G}^*}(s)^{F^*}$ as follows. On the right this is $\eta \rtimes a \mapsto \hat{R}_{\eta * a}$ from Definition 4.1. The compatibility we expect is by our Theorem 4.4 for this side and each $a \in A_{\mathbf{L}^*}(s)^F$.

On the left side the map is $R[s, a] : \mathcal{C}(W^\circ(s)^{w_s F^*} a)^A \to \mathbb{C}\mathcal{E}(\mathbf{G}^F, [s])$ from [B06, §23.C]. Through the identification $\mathcal{C}(W^\circ(s)^{w_s F^*} a) \cong \mathcal{C}(W^\circ(s)^{\langle a \rangle} w_s F^*) = \mathcal{C}(W^\circ(s)^{\langle a \rangle} F^*)$ from the wreath product structure, this is deduced from a map $\mathcal{R}[s, a] : \mathcal{C}(W^\circ(s)^{\langle a \rangle} F^*)^A \to \mathbb{C}\mathcal{E}(\mathbf{G}^F, [s])$ (see [B06, 17.18]). The compatibility with $\mathrm{R}^{\mathbf{G}}_{\mathbf{L}}$ is [B06, 17.24]. To sum up, the $\mathcal{R}[s, a]$ maps for $\mathbf{G}$ and $\mathbf{L}$ transform the $\mathrm{R}^{\mathbf{G}}_{\mathbf{L}}$ functor into an induction of central functions on right cosets in $W(s)$ matching the one satisfied by $\hat{R}$ seen above. This gives our claim. □

5. Proof of Theorem 3.5

We now give the proof of Theorem 3.5. It follows the same steps as in [B99, §8], and we summarize the main ideas. In order to conform to the notations in [B99], we switch back to denoting by $\mathbf{G}$ (instead of $\mathbf{H}$) a wreath product of general linear groups as in Notation 2.1. We keep the other notations n_i, d_i, A_i, $\mathbf{T}_0$, W°, etc.

Let us recall the construction of $\widetilde{R}_\eta$.

Let $\eta \in \mathrm{Irr}(W^{\circ F})$, and $\eta_a^F \in \mathrm{Irr}(W^{\circ\langle a\rangle})$ for $a \in A_\eta$ associated with η through the identification $\mathrm{Irr}(W^{\circ F})^{\langle a\rangle} = \mathrm{Irr}(W^{\circ\langle a,F\rangle}) = \mathrm{Irr}(W^{\circ\langle a\rangle})^F$ (see (1.2)). Let $\widetilde{R}_\eta^{\mathbf{G}} \in \mathcal{C}(\mathbf{G}^{\circ F}.A_\eta^F)$ be defined for $g \in \mathbf{G}^{\circ F}$, $a \in A_\eta^F$, by (see Definition 3.3)

$$\widetilde{R}_\eta^{\mathbf{G}}(ga) = \overline{\sum_{w\in W^{\circ\langle a\rangle}}} \big(\eta_a^F \rtimes \langle\sigma\rangle\big)(w\sigma)\mathrm{R}_{(\mathbf{T}_0.\langle a\rangle)_w}^{\mathbf{G}^\circ.\langle a\rangle}(1)(ga),$$

where $(\mathbf{T}_0.\langle a\rangle)_w$ denotes any maximal quasi-torus in $\mathbf{G}^\circ\langle a\rangle$ of type w with regard to $\mathbf{T}_0\langle a\rangle$ (see Proposition 1.8).

One must prove $\widetilde{R}_\eta^{\mathbf{G}} \in \pm\mathrm{Irr}(\mathbf{G}^{\circ F}.A_\eta^F)$.

Step 1. $\widetilde{R}_1^{\mathbf{G}} = 1$.

This is [DM94, Proposition 4.12].

Step 2. $\widetilde{R}_\eta$ has norm 1.

The proof of [B99, 8.1.2], via computation of scalar products $\langle \mathrm{R}_{(\mathbf{T}_0.\langle a\rangle)_v}^{\mathbf{G}^\circ.\langle a\rangle}(1), \mathrm{R}_{(\mathbf{T}_0.\langle a\rangle)_w}^{\mathbf{G}^\circ.\langle a\rangle}(1)\rangle_{\mathbf{G}^{\circ F}.a}$ (see [DM94, 4.8]), applies without change. $\square$

It now suffices to prove that

$$\widetilde{R}_\eta \in \mathbb{Z}\,\mathrm{Irr}\big(\mathbf{G}^{\circ F}.A_\eta^F\big)$$

to have our claim.

By direct product, one assumes that there is just one i, so that $\mathbf{G}^\circ = \mathrm{GL}_n(\mathbb{F})^d$. In view of the claim and the definition of $\widetilde{R}_\eta$, one may also assume that $A = A^F$, i.e. that $A \subseteq \mathfrak{S}_d$ centralizes σ. One may also assume $A = \mathrm{C}_{\mathfrak{S}_d}(\sigma)$ since proving our claim in this case will imply it in the case of any subgroup A of $\mathrm{C}_{\mathfrak{S}_d}(\sigma)$. Again by direct product, one assumes that all cycles of σ have same length a divisor e of $d = ek$.

Then

$$\mathbf{G} = \mathrm{GL}_n(\mathbb{F})^{ek} \rtimes A \quad \text{with } A \cong (\mathbb{Z}/e\mathbb{Z})^k \rtimes \mathfrak{S}_k$$

with A embedding in $\mathfrak{S}_{ek}$ as $\mathrm{C}_{\mathfrak{S}_{ek}}(\sigma)$ for $\sigma = (1,\dots,e)(e+1,\dots,2e)\dots(ek-e+1,\dots,ek)$.

Step 3. Assume moreover $k = 1$, that is $e = d$. Then $\widetilde{R}_\eta$ is a generalized character.

Note that $A = \langle\sigma\rangle$ now fixes any element hence any character of any subgroup of $W^{\circ F}$.

Let $\mathbf{L}_1 \subseteq \mathrm{GL}_n(\mathbb{F})$ be a product of linear groups associated with a composition of n (that is a standard Levi subgroup for the usual BN-pair). Then $\mathbf{L} := (\mathbf{L}_1)^e \rtimes A$ is an F-stable Levi subgroup of $\mathbf{G}$. Let $W_{\mathbf{L}}^\circ = W^\circ \cap \mathbf{L}$ and $\lambda \in \mathrm{Irr}(W_{\mathbf{L}}^{\circ F})$. This is consistent with the notation of Theorem 4.4 above. Then the above process defines a central function $\widetilde{R}_\lambda^{\mathbf{L}}$ on $\mathbf{L}^F$.

Lemma 5.1.

$$\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}\big(\widetilde{R}_\lambda^{\mathbf{L}}\big) = \sum_{\eta\in\mathrm{Irr}(W^{\circ F})} m_\eta \widetilde{R}_\eta^{\mathbf{G}}$$

where $m_\eta = \langle \eta, \mathrm{Ind}_{W_{\mathbf{L}}^{\circ F}}^{W^{\circ F}}(\lambda)\rangle_{W^{\circ F}}$.

Let us show why this lemma is enough to establish (E) in this case.

By a classical theorem about characters of symmetric groups (see for instance [JK81, 2.2.10]), the inverse of the matrix $(\langle \mathrm{Ind}_{W_{\mathbf{L}}^{\circ F}}^{W^{\circ F}}(1_{W_{\mathbf{L}}^{\circ F}}), \eta\rangle_{W^{\circ F}})_{\mathbf{L},\eta}$ for $\mathbf{L}_1$ ranging over Levi subgroups defined by partitions of n and $\eta \in \mathrm{Irr}(W^{\circ F})$ with $W^{\circ F} \cong \mathfrak{S}_n$ is integral. Applying Lemma 5.1 with each λ the trivial character of $W_{\mathbf{L}}^{\circ F}$, allows to express $\widetilde{R}_\eta$ as an integral combination of central functions of type $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\widetilde{R}_1^{\mathbf{L}})$. But Step 1 tells us that $\widetilde{R}_1^{\mathbf{L}} = 1_{\mathbf{L}^F}$. Hence our claim since $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}$ maps generalized characters to generalized characters.

Proof of Lemma 5.1. One checks the claimed equality on ga with $a = \sigma^i$ for $i \geqslant 0$.

The situation is a special case of the one described by Theorem 4.4 with $A = A_{\mathbf{L}}$, $w_1 = 1$ and one has basically to prove the same statement as (4.7) with our m_η's. Applying [B99, 3.1.1] again, one gets

$$\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\widetilde{R}_\lambda^{\mathbf{L}}) = \sum_{\eta \in \mathrm{Irr}(W^{\circ F})} \left\langle \eta_{\sigma^i} \rtimes \sigma, \mathrm{Ind}_{W_{\mathbf{L}}^{\circ\langle\sigma^i\rangle}.\sigma}^{W^{\circ\langle\sigma^i\rangle}.\sigma}(\lambda_{\sigma^i} \rtimes \sigma)\right\rangle_{W^{\circ\langle\sigma^i\rangle}.\sigma} \widetilde{R}_\eta^{\mathbf{G}}.$$

There remains to check that $\langle \eta_{\sigma^i} \rtimes \sigma, \mathrm{Ind}_{W_{\mathbf{L}}^{\circ\langle\sigma^i\rangle}.\sigma}^{W^{\circ\langle\sigma^i\rangle}.\sigma} \lambda_{\sigma^i} \rtimes \sigma\rangle_{W^{\circ\langle\sigma^i\rangle}.\sigma} = \langle \eta, \mathrm{Ind}_{W_{\mathbf{L}}^{\circ F}}^{W^{\circ F}}(\lambda)\rangle_{W^{\circ F}}$. This is a general property relating induction on right cosets and ordinary induction in wreath products, see [B99, 3.2.1]. □

Step 4. Assume arbitrary k. Then $\widetilde{R}_\eta$ is a generalized character.

We define a more general framework to which [B99, §8.4] applies.

Take $(\mathbf{G}_0, F_0)$ a connected reductive group defined over $\mathbb{F}_q$ having an F_0-stable maximal torus $\mathbf{S}_0$ such that F_0 acts trivially on $\mathrm{W}_{\mathbf{G}_0}(\mathbf{S}_0)$.

Let

$$\mathbf{G} = \mathbf{G}_0^{ek} \rtimes A \quad \text{with } A = (\mathbb{Z}/e\mathbb{Z})^k \rtimes C$$

for $C \subseteq \mathfrak{S}_k$ embedded as before in $\mathfrak{S}_{ek}$ as a subgroup of the centralizer $\cong \mathbb{Z}/e\mathbb{Z} \wr \mathfrak{S}_k$ of $\sigma = (1,\dots,e)(e+1,\dots,2e)\dots(e(k-1)+1,\dots,ek)$. Let F_0 extend to $\mathbf{G}$ by acting the same on each factor of $\mathbf{G}^\circ = \mathbf{G}_0^d$ and trivially on A, let $F = \sigma F_0$.

Denote $B \cong (\mathbb{Z}/e\mathbb{Z})^k$ so that $A \cong B \rtimes C$. Note that B fixes any element hence any character of $W^{\circ F}$.

Denote $\mathbf{H} := \mathbf{G}^\circ \rtimes B = (\mathbf{G}_0^e \rtimes \mathbb{Z}/e\mathbb{Z})^k$, so that $\mathbf{G} = \mathbf{H} \rtimes C$ with F acting trivially on the second term and the pair $(\mathbf{H}, F)$ being a product of k identical pairs $(\mathbf{H}_1, F)$ as in the above Step 3.

Denote $W^\circ = \mathrm{W}_{\mathbf{G}^\circ}(\mathbf{T}_0)$ where $\mathbf{T}_0 = (\mathbf{S}_0)^d$.

For $\eta \in \mathrm{Irr}(W^{\circ F})^C = \mathrm{Irr}(W^{\circ F})^A$, let $\widetilde{R}_\eta^{\mathbf{G}} \in \mathcal{C}(\mathbf{G}^F)$ defined by

$$\mathrm{Res}_{\mathbf{G}^{\circ F}.a}^{\mathbf{G}^F} \widetilde{R}_\eta^{\mathbf{G}} = \overline{\sum_{w \in W^{\circ\langle a\rangle}}} (\eta_a^F \rtimes a)(w\sigma).\mathrm{Res}_{\mathbf{G}^{\circ F}.a}^{\mathbf{G}^{\circ F}.\langle a\rangle} \mathrm{R}_{(\mathbf{T}_0.\langle a\rangle)_w}^{\mathbf{G}^\circ.\langle a\rangle}(1)$$

for any $a \in A = A_\eta^F$, and $\widetilde{R}_\eta^{\mathbf{H}} \in \mathcal{C}(\mathbf{H}^F)$ by $\mathrm{Res}_{\mathbf{G}^{\circ F}.b}^{\mathbf{H}^F} \widetilde{R}_\eta^{\mathbf{H}} = \overline{\sum}_{w \in W^{\circ\langle b\rangle}} (\eta_b^F \rtimes b)(w\sigma).\mathrm{Res}_{\mathbf{G}^{\circ F}.b}^{\mathbf{G}^{\circ F}.\langle b\rangle} \mathrm{R}_{(\mathbf{T}_0.\langle b\rangle)_w}^{\mathbf{G}^\circ.\langle b\rangle}(1)$ for any $b \in B = B_\eta^F$.

Note that the group $\mathbf{G}^F$ is also a wreath product $\mathbf{H}^F \rtimes C = (\mathbf{H}_1)^F \wr C$. With this description, $\widetilde{R}_\eta^{\mathbf{H}}$ is a tensor product of similarly defined representations $\widetilde{R}_{\eta_j}^{\mathbf{H}_j}$ $(j = 1,\dots,k)$ of summands $\cong (\mathbf{H}_1)^F$ with $\eta_j = \eta_{c.j}$ for each $c \in C$. Remark 1.4 then allows to define $\widetilde{R}_\eta^{\mathbf{H}} \rtimes C \in \mathcal{C}(\mathbf{H}^F \rtimes C)$.

Theorem 5.2. *On $\mathbf{G}^F = \mathbf{H}^F \rtimes C$, one has $\widetilde{R}_\eta^{\mathbf{G}} = \widetilde{R}_\eta^{\mathbf{H}} \rtimes C$.*

Let's say how this would complete our last step.

With $\mathbf{G}_0 = \mathrm{GL}_n(\mathbf{F})$, F_0 the raising of matrix entries to the q-th power possibly composed with the transpose-inverse automorphism, $\mathbf{S}_0$ the diagonal torus, and $C = (\mathfrak{S}_k)_\eta$ we cover our initial situation. Theorem 5.2 then implies that $\epsilon_\eta \widetilde{R}^{\mathbf{G}}_\eta = \epsilon_\eta \widetilde{R}^{\mathbf{H}}_\eta \rtimes C$ which in turn is an irreducible character thanks to Definition 1.3 and the fact that $\epsilon_\eta \widetilde{R}^{\mathbf{H}}_\eta$ is an irreducible character by Step 3 as said before.

Proof of Theorem 5.2. By direct product, one may assume that C is transitive. Given the definition of $\widetilde{R}^{\mathbf{G}}_\eta$ and the one in Remark 1.4, one may also assume that C is cyclic. Since it commutes with $\sigma = (1,\dots,e)(e+1,\dots,2e)\dots(e(k-1)+1,\dots,ek)$ in $\mathfrak{S}_{ek}$, one may then index elements in $\{1,\dots,e\}$, $\{e+1,\dots,2e\}\dots\{e(k-1)+1,\dots,ek\}$ so that C is generated by $c = (1,\dots,ke)^e = (1, 1+e,\dots,1+e(k-1))(2, 2+e,\dots,2+e(k-1))\dots$ and one evaluates the equality on $\mathbf{H}^F.c$.

Up to some adequate conjugacy inside $(\mathbb{Z}/e\mathbb{Z}) \wr C$ (see [B99, p. 95]), not involving the type of $\mathbf{G}_0$, one may content ourselves with evaluating our functions on $h^\circ.bc$ where $h^\circ = (h_1^\circ,\dots,h_k^\circ) \in \mathbf{G}^{\circ F} = ((\mathbf{G}_0^e)^F)^k \cong (\mathbf{G}_0^{Fe})^k$, $b = (\tau, 1,\dots,1)^m \in B$ for τ corresponding with the cycle $(1,\dots,e)$ and $m \geqslant 1$.

On the left side of our claim we have

$$\widetilde{R}^{\mathbf{G}}_\eta\big(h^\circ.bc\big) = \overline{\sum_{w\in W^{\circ\langle bc\rangle}}} \eta^F_{bc} \rtimes \sigma(w\sigma)\mathrm{R}^{\mathbf{G}^\circ\langle bc\rangle}_{(\mathbf{T}_0\langle bc\rangle)_w}(1)\big(h^\circ.bc\big). \tag{5.3}$$

For the right side one must compute $(\widetilde{R}^{\mathbf{H}}_\eta \rtimes C)((h^\circ.b)c) = (\widetilde{R}^{\mathbf{H}}_\eta)_c(\pi_c(h^\circ.b))$ (see Definition 1.3).

For $x \in \mathbf{H}_1 = \mathbf{G}_0^e \rtimes \mathbb{Z}/e\mathbb{Z}$, denote $x^{(k)} = (x,\dots,x)$ (k times), as element of $\mathbf{H}$. We have $(\mathbf{H}^F)^{\langle c\rangle} = \{g^{(k)} \mid g \in \mathbf{H}_1^F\}$ and $\pi_c(h_1,h_2,\dots,h_k) = (h_k\cdots h_1)^{(k)}$ for $h_1,h_2,\dots,h_k \in \mathbf{H}_1^F$. So $\pi_c(h^\circ.b) = (h_k^\circ\dots h_1^\circ\tau^m)^{(k)} = (h_k^\circ\dots h_1^\circ)^{(k)}\sigma^m$. We have $\widetilde{R}^{\mathbf{H}}_\eta = (\widetilde{R}^{\mathbf{H}_1}_{\eta_1})^{\otimes k}$, so $(\widetilde{R}^{\mathbf{H}}_\eta)_c$ may be identified with $\widetilde{R}^{\mathbf{H}_1}_{\eta_1}$ or $\widetilde{R}^{\mathbf{H}^{\langle c\rangle}}_{\eta^{\langle c\rangle}}$ where $\eta^{\langle c\rangle}(w^{(k)}) = \eta_1(w)$ for any $w \in W^\circ \cap \mathbf{H}_1^F$. So $(\widetilde{R}^{\mathbf{H}}_\eta \rtimes C)((h^\circ b)c) = \widetilde{R}^{\mathbf{H}^{\langle c\rangle}}_{\eta^{\langle c\rangle}}((h_k^\circ\dots h_1^\circ\tau^m)^{(k)}) = \overline{\sum}_{w\in W^{\circ\langle c,b\rangle}}((\eta^{\langle c\rangle})^F_{\sigma^m} \rtimes \sigma)(w\sigma)\mathrm{R}^{(\mathbf{G}^\circ\langle\sigma^m\rangle)^{\langle c\rangle}}_{\mathbf{T}'_w}(1)((h_k^\circ\dots h_1^\circ)^{(k)}\sigma^m)$ where $\mathbf{T}'_w$ denotes the maximal quasi-torus of $(\mathbf{G}^\circ\langle\sigma^m\rangle)^{\langle c\rangle}$ of type $w \in W^{\circ\langle c,b\rangle}$.

Comparing this with the formula (5.3) above, one notices that $W^{\circ\langle bc\rangle} = W^{\circ\langle c,b\rangle}$ (since $\langle bc\rangle$ and $\langle c,b\rangle$ have same orbits on $\{1,\dots,ek\}$), and similarly $\eta^F_{bc} \rtimes \sigma = (\eta^{\langle c\rangle})^F_{\sigma^m} \rtimes \sigma$, which leads to ask if

$$\mathrm{R}^{\mathbf{G}^\circ\langle bc\rangle}_{(\mathbf{T}_0\langle bc\rangle)_w}(1)\big(h^\circ.bc\big) = \mathrm{R}^{(\mathbf{G}^\circ\langle\sigma^m\rangle)^{\langle c\rangle}}_{\mathbf{T}'_w}(1)\big(\pi_c\big(h^\circ.b\big)\big). \tag{5.4}$$

As in [B99, 8.4.3], one first notes that $((\mathbf{T}_0\langle bc\rangle)_w)^{\langle c\rangle} = \mathbf{T}'_w$ from equalities in $B \rtimes C$ and then that (5.4) holds thanks to [DM94, 5.3] relating $\mathrm{R}^{\mathbf{G}}_{\mathbf{T}}(1)$'s with varieties of type $\mathbf{X}_w$ (see for instance [CE04, 7.13]) and [B99, 8.4.5] giving an isomorphism between varieties $\mathbf{X}_w$ for $\mathbf{G}^{\langle c\rangle}$ and some corresponding subvarieties for $\mathbf{G}$, both holding for a $\mathbf{G}_0$ of any type. This completes the proof of Theorem 5.2. □

References

[B99] C. Bonnafé, Produits en couronne de groupes linéaires, J. Algebra 211 (1999) 57–98.

[B00] C. Bonnafé, Mackey formula in type A, Proc. Lond. Math. Soc. (3) 80 (3) (2000) 545–574.

[B06] C. Bonnafé, Sur les caractères des groupes réductifs finis à centre non connexe: applications aux groupes spéciaux linéaires et unitaires, Astérisque 306 (2006).

[CE04] M. Cabanes, M. Enguehard, Representation Theory of Finite Reductive Groups, Cambridge University Press, Cambridge, 2004.

[Cr85] R. Carter, Finite Groups of Lie Type: Conjugacy Classes and Complex Characters, Wiley, New York, 1985.

[DM91] F. Digne, J. Michel, Representations of Finite Groups of Lie Type, Cambridge University Press, Cambridge, 1991.

[DM94] F. Digne, J. Michel, Groupes réductifs non connexes, Ann. Sci. Éc. Norm. Super. (4) 27 (1994) 345–406.

[JK81] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Addison–Wesley, Reading, 1981.

[L77] G. Lusztig, Irreducible representations of finite classical groups, Invent. Math. 43 (1977) 125–175.

[L84] G. Lusztig, Characters of Reductive Groups Over a Finite Field, Ann. of Math. Stud., vol. 107, Princeton University Press, Princeton, 1984.

[LS77] G. Lusztig, B. Srinivasan, The characters of the finite unitary groups, J. Algebra 49 (1977) 167–171.

[N98] G. Navarro, Characters and Blocks of Finite Groups, Cambridge University Press, Cambridge, 1998.