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Solution of fractional third-order dispersive partial differential equations

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ABSTRACT

In this paper, we proposed fractional differential transform method(FDTM) and modified fractional differential transform method(MFDTM) for the solution of fractional third-order dispersive partial differential equations in one- and higher-dimensional spaces. The plotted graphs illustrate the behavior of the solution for different values of fractional orderα. The efficiency and accurateness of the proposed methods are examined by means of four numerical experiments.

1. Introduction

In recent past, the glorious developments have been envisaged in the field of fractional calculus and fractional differential equations. Differential equations involving fractional order derivatives are used to model a variety of systems, of which the important applications lie in field of viscoelasticity, electrode-electrolyte polarization, heat conduction, electromagnetic waves, diffusion equation and so on [1,2]. Due to its tremendous scope and applications in several disciplines, a considerable attention has been given to exact and numerical solutions of fractional differential equations. A great deal of researchers has shown the advantageous use of the fractional calculus in the modeling and control of many dynamical systems [3–10]. Other than modeling aspects of these differential equations, the solution techniques and their reliability are rather more important aspects. It is also equally important to handle critical points which cause sudden divergence convergence and bifurcation of the solutions of the model. In order to achieve the goal of highly accurate and reliable solutions, several methods have been proposed to solve the fractional order differential equations. Some of the recent analytical/numerical methods are Adomian decomposition method (ADM) [11–16], finite difference method [17], Variational iteration method (VIM) [18,19], Operational matrix

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method [20], Homotopy analysis method [21,22], generalized differential transform method [23,24], finite element method [25], fractional differential transform method [26,27] and references therein.

The main aim of this work is to apply FDTM and MFDTM to solve third-order dispersive partial differential equations [28–32]. The DTM obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor's series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. The use of DTM in electric circuit analysis was pioneered by Zhou [33]. Since then, DTM was successfully applied for large variety of problems such as partial differential equations [34,35], solitary wave solutions for the KdV and mKdV equations [36], linear and nonlinear Schrodinger equations [37], linear and nonlinear Klein-Gordon equations [38], nonlinear oscillators with fractional nonlinearities [39], fractional linear and nonlinear Schrodinger equation [41], nonlinear fractional Klein-Gordon Equation [42] and references therein. Recently, in [40] presented a novel technique to obtain the differential transform of nonlinearities by the Adomian polynomials. The proposed FDTM and MFDTM do not require linearization, discretization or perturbation unlike the method discussed in the literature. The main drawback of the ADM is to calculate Adomian polynomials for a nonlinear operator where the procedure is very complex. The difficulty in VIM has an inherent inaccuracy in identifying the Lagrange multiplier, correctional functional and stationary conditions for the fractional order. The disadvantage of the Homotopy perturbation method is to solve functional equation in each iteration, which is sometimes complicated and unattainable. Therefore, the proposed FDTM and MFDTM are much easier when compared with ADM, VIM and HPM. The outline of this paper is as follows. In section 2 the basic definitions of fractional calculus are discussed. The basic definitions of two-dimensional FDTM and MFDTM are presented in section three and four. Four clear cut test problems of fractional third-order dispersive partial differential equations are given to elucidate the proposed methods in section 5. At the end, we write the conclusions of the work in section 6.

2. Basic definitions of fractional calculus [2,10]

For convenience of the reader, we present a review of the basic definitions and properties of the fractional calculus theory.

Definition 1. A real function $f(x)$, $x>0$ in the spacec_μ, $\mu\in R$ if there exists a real number $p>\mu$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,\infty)$ and it is said to be in the space if $f^m\in C_\mu$, $m\in N$.

Definition 2. The left-sided Riemann-Liouville fractional integral operator of order $\alpha\geq 0$, of a function $f\in C_\mu$, $\mu\geq -1$ is defined as $I^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \alpha>0, x>0$ and $J^0 f(x)=f(x)$.

Definition 3. The fractional derivative of $f(x)$ in the Caputo [2] sense is defined as

$$D_*^\alpha f(x)=J^{m-\alpha}D^m f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}f^m(t)dt \text{ for } m-1<\alpha<m, \quad m\in N, x>0, f\in C_{-1}^n.$$

The unknown function $f=f(x,t)$ is assumed to be a casual function of fractional derivatives (i.e., vanishing for $\alpha<0$) taken in Caputo sense as follows.

Definition 4. For m as the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha>0$ is defined as

$$D_{*t}^\alpha f(x,t)=\frac{\partial^\alpha f(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m f(x,\tau)}{\partial \tau^m}, & m-1<\alpha<m \\ \frac{\partial^m f(x,t)}{\partial t^m}, & \alpha=m\in N \end{cases}.$$

3. Two-dimensional fractional differential transform method

Consider a function of two variables $u(x,t)$ and suppose that it can be represented as a product of two single variable functions i.e., $u(x,t)=f(x)g(t)$. Based on the properties of two-dimensional fractional differential transform, the function $u(x,t)$ can be represented as

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha,1}(k,h)(x-x_0)^k(t-t_0)^{h\alpha} \quad (1)$$

where $0<\alpha$, $U_{\alpha,1}(k,h)$ is called the spectrum of $u(x,t)$. The generalized two-dimensional fractional differential transform of the function $u(x,t)$ is given by

$$U_{\alpha,1}=\frac{1}{\Gamma(k+1)\Gamma(\alpha h+1)}\left[\left(D_{*x_0}^1\right)^k\left(D_{*t_0}^\alpha\right)^h u(x,t)\right]_{x_0,t_0} \quad (2)$$

where $(D_{*t_0}^\alpha)^h=\underbrace{D_{*t_0}^\alpha D_{*t_0}^\alpha \ldots D_{*t_0}^h}_{h}$. In real applications the function $u(x,t)$ is represented by a finite series of Eq. (1) can be written as

$$u(x,t)=\sum_{k=0}^{l}\sum_{h=0}^{n}U_{\alpha,1}(k,h)x^k t^{\alpha h}+R_{ln}(x,t) \quad (3)$$

and (1) implies that $R_{ln}(x,t)=\sum_{k=l+1}^{\infty}\sum_{h=n+1}^{\infty}U_{\alpha,1}(k,h)x^k t^{\alpha h}$ is negligibly small. Usually, the values of l and n are decided by convergence of the series solution. In case of $\alpha=1$, the generalized two-dimensional fractional differential transform method (1) reduces to classical two-dimensional differential transform [34–38]. The fundamental mathematical operations performed by two-dimensional fractional differential transform method are listed in Table 1.

Table 1 – The operations for the two-dimensional fractional differential transform method.

Original function	Transformed function
$w(x,t) = u(x,t) \pm v(x,t)$	$W_{\alpha,1}(k,h) = U_{\alpha,1}(k,h) \pm V_{\alpha,1}(k,h)$
$w(x,t) = \mu u(x,t)$	$W_{\alpha,1}(k,h) = \mu U_{\alpha,1}(k,h)$
$w(x,t) = \frac{\partial u(x,t)}{\partial x}$	$W_{\alpha,1}(k,h) = (k+1)U_{\alpha,1}(k+1,h)$
$w(x,t) = D^{\alpha}_{*t_0} u(x,t), 0<\alpha \le 1$	$W_{\alpha,1}(k,h) = \frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(k,h+1)$
$w(x,t) = (x-x_0)^m (t-t_0)^{n\alpha}$	$W_{\alpha,1}(k,h) = \delta(k-m, h\alpha - n) = \begin{cases} 1, k=m, h=n \\ 0, \text{otherwise} \end{cases}$
$w(x,t) = u^2(x,t)$	$W_{\alpha,1}(k,h) = \sum_{m=0}^{k} \sum_{n=0}^{h} U_{\alpha,1}(m,h-n) U_{\alpha,1}(k-m,n)$
$w(x,t) = u^3(x,t)$	$W_{\alpha,1}(k,h) = \sum_{r=0}^{k} \sum_{q=0}^{k-r} \sum_{s=0}^{h} \sum_{p=0}^{h-s} U_{\alpha,1}(r,h-s-p) U_{\alpha,1}(q,s) U_{\alpha,1}(k-r-q,p)$

4. Modified fractional differential transform method

However, there are difficulties in FDTM while handling the non-linear functions in two-dimension. Let us consider the differential transform for

$$u^3(x,t) = \sum_{r=0}^{k} \sum_{q=0}^{k-r} \sum_{s=0}^{h} \sum_{p=0}^{h-s} U_{\alpha,1}(r,h-s-p) U_{\alpha,1}(q,s) U_{\alpha,1}(k-r-q,p) \tag{4}$$

(4) involves four summations. Thus it is necessary to have a lot of computational work to calculate such differential transform $U_{\alpha,1}(k,h)$ for the large number of (k, h). As we know that, FDTM is based on the Taylor series for all variables. To avoid these difficulties, MFDTM is considered the Taylor's series of the function$u(x,t)$ with respect to the specific variable. Assume that the specific variable't' then, we have the Taylor's series expansion of the function $u(x,t)$ at$t = t_0$ as follows.

$$u(x,t) = \sum_{h=0}^{\infty} \frac{1}{\Gamma(\alpha h+1)} \left(\frac{\partial^{\alpha h} u(x,t)}{\partial t^{\alpha h}} \right) (t-t_0)^{\alpha h} \tag{5}$$

Definition 5. The modified fractional differential transform $U_{\alpha,1}(x,h)$ of $u(x,t)$ with respect to the variable t at t_0 is defined by

$$U_{\alpha,1}(x,h) = \frac{1}{\Gamma(\alpha h+1)} \left(\frac{\partial^{\alpha h} u(x,t)}{\partial t^{\alpha h}} \right)_{t=t_0} \tag{6}$$

Definition 6. The modified fractional differential inverse transform $U_{\alpha,1}(x,h)$ of $u(x,t)$ with respect to the variable t at t_0 is defined by

$$u(x,t) = \sum_{h=0}^{\infty} U_{\alpha,1}(x,h)(t-t_0)^{\alpha h} \tag{7}$$

Since the MFDTM results from the Taylor's series of the function with respect to the specific variable it is expected that the corresponding algebraic equation from the given problem is much simpler than the result obtained by the standard FDTM. The fundamental mathematical operations performed by modified fractional differential transform method are listed in Table 2.

5. Applications

In this section, four numerical examples are tested to authenticate the proposed FDTM and MFDTM.

Example 1. We consider the linear fractional dispersive KdV equation

$$u_t^{\alpha} + 2u_x + u_{xxx} = 0, t>0 \tag{8}$$

Subject to the initial condition

$$u(x,0) = \sin x \tag{9}$$

FDTM: The transformed version of (8) is

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(k,h+1) + 2(k+1)U_{\alpha,1}(k+1,h) + (k+1)(k+2)(k+3)U_{\alpha,1}(k+3,h) = 0 \tag{10}$$

The transformed version of (9) is

$$U_{\alpha,1}(k,0) = \begin{cases} 0, k=0,2,4,8,\ldots \\ \frac{1}{k!}, k=1,5,9,\ldots \\ \frac{-1}{k!}, k=3,7,11,\ldots \end{cases} \tag{11}$$

Substituting (11) in (10), yields the $U_{\alpha,1}(k,h)$ values,

$$U_{\alpha,1}(0,1) = \frac{-1}{\Gamma(\alpha+1)}, U_{\alpha,1}(1,1) = 0, U_{\alpha,1}(2,1) = \frac{1}{2\Gamma(\alpha+1)},$$
$$U_{\alpha,1}(3,1) = 0, U_{\alpha,1}(4,1) = \frac{-1}{24\Gamma\alpha+1}, \ldots$$

$$U_{\alpha,1}(0,2) = 0, U_{\alpha,1}(1,2) = \frac{-1}{\Gamma(2\alpha+1)}, U_{\alpha,1}(2,2) = 0,$$
$$U_{\alpha,1}(3,2) = \frac{1}{6\Gamma(2\alpha+1)}, U_{\alpha,1}(4,2) = 0, \ldots$$

Using $U_{\alpha,1}(k,h)$ values in (1), we obtained the series solution as

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,1}(k,h) x^k t^{\alpha h} \\ &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &- \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots \end{aligned} \tag{12}$$

When $l,n \le 5$, then the FDTM solution (3) takes the following form

Table 2 – The operations for the modified fractional differential transform method.

Original function	Transformed function
$w(x,t) = u(x,t) \pm v(x,t)$	$W_{\alpha,1}(x,h) = U_{\alpha,1}(x,h) \pm V_{\alpha,1}(x,h)$
$w(x,t) = \mu u(x,t)$	$W_{\alpha,1}(x,h) = \mu U_{\alpha,1}(x,h)$
$w(x,t) = \frac{\partial u(x,t)}{\partial x}$	$W_{\alpha,1}(x,h) = \frac{\partial U_{\alpha,1}(x,h)}{\partial x}$
$w(x,t) = D^{\alpha}_{*t_0} u(x,t), 0<\alpha\le 1$	$W_{\alpha,1}(x,h) = \frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(x,h+1)$
$w(x,t) = (x-x_0)^m (t-t_0)^{n\alpha}$	$W_{\alpha,1}(x,h) = (x-x_0)^m \delta(h\alpha - n)$
$w(x,t) = u^2(x,t)$	$W_{\alpha,1}(x,h) = \sum_{m=0}^{h} U_{\alpha,1}(x,m) U_{\alpha,1}(x,h-m)$
$w(x,t) = u^3(x,t)$	$W_{\alpha,1}(x,h) = \sum_{m=0}^{h}\sum_{l=0}^{m} U_{\alpha,1}(x,h-m) U_{\alpha,1}(x,l) U_{\alpha,1}(x,m-l)$

$$u(x,t) = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} \quad (13)$$

MFDTM: The transformed version of (8) with respect to‘t’ is

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(x,h+1) + 2\frac{\partial U_{\alpha,1}(x,h)}{\partial x} + \frac{\partial^3 U_{\alpha,1}(x,h)}{\partial x^3} = 0 \quad (14)$$

The transformed version of (9) is

$$U_{\alpha,1}(x,0) = \sin x \quad (15)$$

The MFDTM recurrence Eq. (14) yields the $U_{\alpha,1}(x,h)$ values

$$U_{\alpha,1}(x,1) = \frac{-\cos x}{\Gamma(\alpha+1)}, U_{\alpha,1}(x,1) = \frac{-\sin x}{\Gamma(2\alpha+1)},$$
$$U_{\alpha,1}(x,3) = \frac{\cos x}{\Gamma(3\alpha+1)}, \ldots$$

Substituting $U_{\alpha,1}(x,h)$'s into Eq. (7), we obtained solution in the following form

$$u(x,t) = \sin x - \frac{\cos x}{\Gamma(\alpha+1)} t^{\alpha} - \frac{\sin x}{\Gamma(2\alpha+1)} t^{2\alpha} + \ldots \quad (16)$$

When $\alpha \to 1$, the approximate solution Eqs. (12) and (16) takes the following form

$$u(x,t) = \sin x \cos t - \cos x \sin t = \sin(x-t)$$

which is exactly same as the solution obtained in [32]. In Figs. 1–2 we have shown the solution $u(x,t)$ obtained by five term FDTM, MFDTM and exact solution for the two different values of fractional orderα. It can be seen that the MFDTM results are far better approximations than the FDTM results.

Example 2. Next, consider the linear fractional dispersive KdV equation in two-dimensional space

$$u_t^{\alpha} + 2u_{xxx} + u_{yyy} = 0, t>0 \quad (17)$$

Subject to the initial condition

$$u(x,y,0) = \cos(x+y) \quad (18)$$

FDTM: The transformed version of (17) is

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(k,l,h+1) + (k+1)(k+2)(k+3)U_{\alpha,1}(k+3,l,h) + (l+1)(l+2)(l+3)U_{\alpha,1}(k,l+3,h) = 0 \quad (19)$$

The transformed version of (18) is

$$U_{\alpha,1}(k,l,0) = \frac{\cos\left(\frac{k\pi}{2}\right)}{k!}\frac{\cos\left(\frac{l\pi}{2}\right)}{l!} - \frac{\sin\left(\frac{k\pi}{2}\right)}{k!}\frac{\sin\left(\frac{l\pi}{2}\right)}{l!}, k,l = 0,1,2,\ldots \quad (20)$$

Substituting Eq. (20) in Eq. (19), yields the $U_{\alpha,1}(k,l,h)$ values

$$U_{\alpha,1}(0,0,1) = 0, U_{\alpha,1}(1,0,1) = \frac{-2}{\Gamma(1+\alpha)},$$
$$U_{\alpha,1}(2,0,1) = 0, U_{\alpha,1}(3,0,1) = \frac{1}{3\Gamma(1+\alpha)}, \ldots$$

$$U_{\alpha,1}(0,1,1) = \frac{-2}{\Gamma(1+\alpha)}, U_{\alpha,1}(1,1,1) = 0,$$
$$U_{\alpha,1}(2,1,1) = \frac{1}{\Gamma(1+\alpha)}, U_{\alpha,1}(3,1,1) = 0, \ldots$$

$$U_{\alpha,1}(0,2,1) = 0, U_{\alpha,1}(1,2,1) = \frac{1}{\Gamma(1+\alpha)},$$
$$U_{\alpha,1}(2,2,1) = 0, U_{\alpha,1}(3,2,1) = \frac{-1}{6\Gamma(1+\alpha)}, \ldots$$

Using $U_{\alpha,1}(k,l,h)$ values in Eq. (1), we obtained the series solution as

$$u(x,y,t) = 1 - \frac{x^2}{2} - xy - \frac{x^3 y}{6} - \frac{y^2}{2} + \frac{x^2 y^2}{4} + \frac{xy^3}{6} - \frac{x^3 y^3}{36} + \ldots + \left(-2x + \frac{x^3}{3} - 2y + x^2 y + xy^2 - \frac{x^3 y^2}{6} + \frac{y^3}{3} - \frac{x^2 y^3}{6} + \ldots\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \ldots \quad (21)$$

MFDTM: The transformed version of Eq. (17) is w. r. t ‘t’ is

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(x,y,h+1) + \frac{\partial^3 U_{\alpha,1}(x,y,h)}{\partial x^3} + \frac{\partial^3 U_{\alpha,1}(x,y,h)}{\partial y^3} = 0 \quad (22)$$

The transformed version of (18) is

$$U_{\alpha,1}(x,y,0) = \cos(x+y) \quad (23)$$

The MFDTM recurrence Eq. (22) yields the $U_{\alpha,1}(x,y,h)$ values

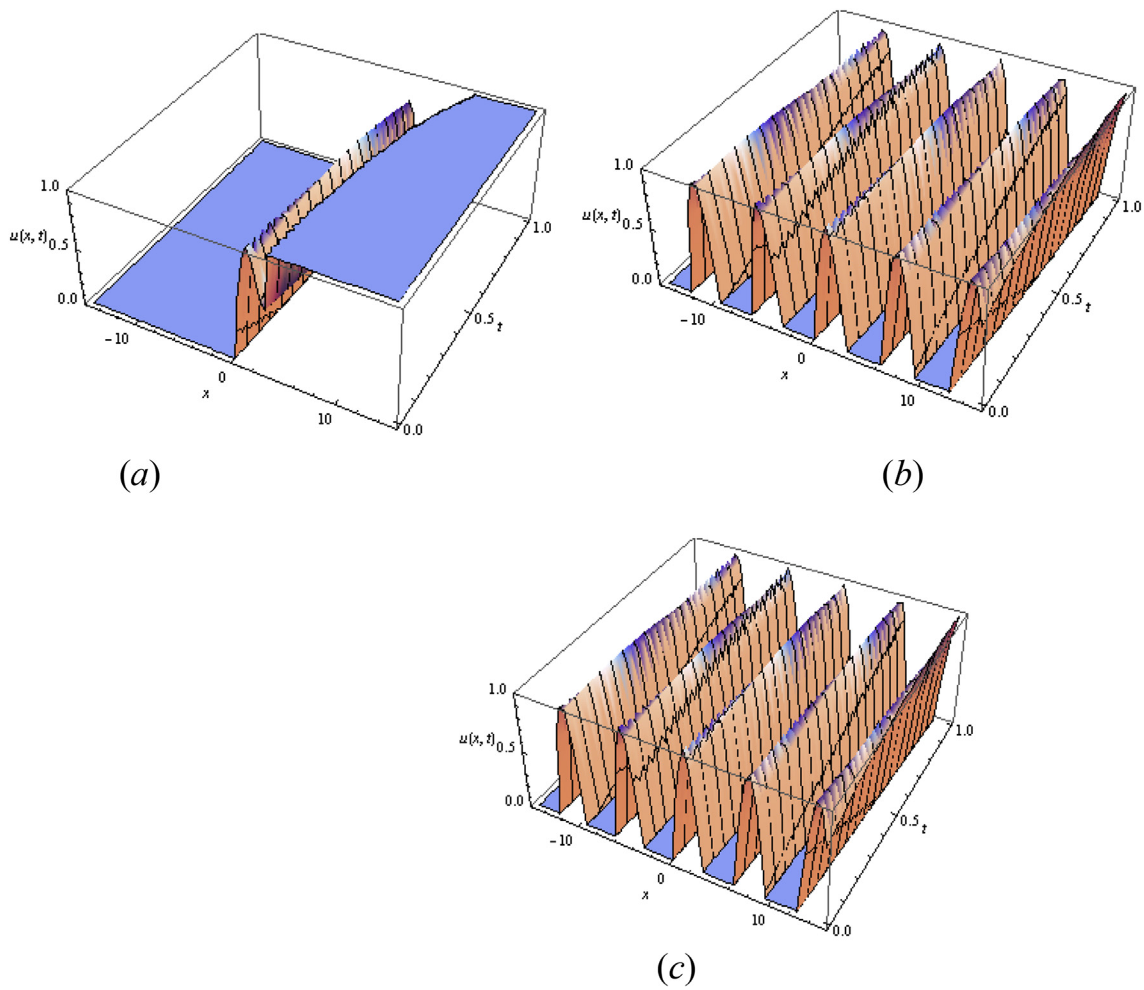

Fig. 1 – *u(x,t)* obtained by using five term *(a)* FDTM, (b) MFDTM and (c) Exact solution when $\alpha = 0.95$.

$$U_{\alpha,1}(x,y,1) = \frac{-2\sin(x+y)}{\Gamma(\alpha+1)}, U_{\alpha,1}(x,y,2) = \frac{-4\cos(x+y)}{\Gamma(2\alpha+1)},$$
$$U_{\alpha,1}(x,y,3) = \frac{8\sin(x+y)}{\Gamma(3\alpha+1)}, \ldots$$

Using the inverse MFDTM, we obtained the solution in the following form,

$$u(x,y,t) = \cos(x+y) - \frac{2\sin(x+y)}{\Gamma(\alpha+1)}t^{\alpha} - \frac{4\cos(x+y)}{\Gamma(2\alpha+1)}t^{2\alpha} + \ldots \quad (24)$$

When$\alpha \to 1$, the approximate solution (21) and (24) takes the following form

$$u(x,t) = \sin(x+y+2t)$$

which is exactly same as the solution obtained in [32].

Example 3. Consider the non-homogeneous fractional third-order dispersive partial differential equation

$$u_t^{\alpha} + u_{xxx} = -\sin\pi x\sin t - \pi^3\cos\pi x\cos t,\ 0<x<1,\ t>0 \quad (25)$$

Subject to the initial condition

$$u(x,0) = \sin\pi x \quad (26)$$

and time-dependent boundary conditions

$$u(0,t) = 0, u_x(0,t) = \pi\cos t, u_{xx}(0,t) = 0 \quad (27)$$

FDTM: The transformed version of (25) is

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{\alpha,1}(k,h+1) + (k+1)(k+2)(k+3)U_{\alpha,1}(k+3,h)$$
$$= -(\pi)^k\frac{\sin\left(\frac{k\pi}{2}\right)}{k!}\frac{\sin\left(\frac{h\pi}{2}\right)}{\Gamma(\alpha h+1)} - \pi^3(\pi)^k\frac{\cos\left(\frac{k\pi}{2}\right)}{k!}\frac{\cos\left(\frac{h\pi}{2}\right)}{\Gamma(\alpha h+1)} \quad (28)$$

The transformed version of (26) is

$$U_{\alpha,1}(k,0) = \begin{cases} 0, k = 0,2,4,6,\ldots \\ \frac{(\pi)^k}{k!}, k = 1,5,9,\ldots \\ -\frac{(\pi)^k}{k!}, k = 3,7,11,\ldots \end{cases} \quad (29)$$

MFDTM: The transformed version of (25) is w. r. t 't' is

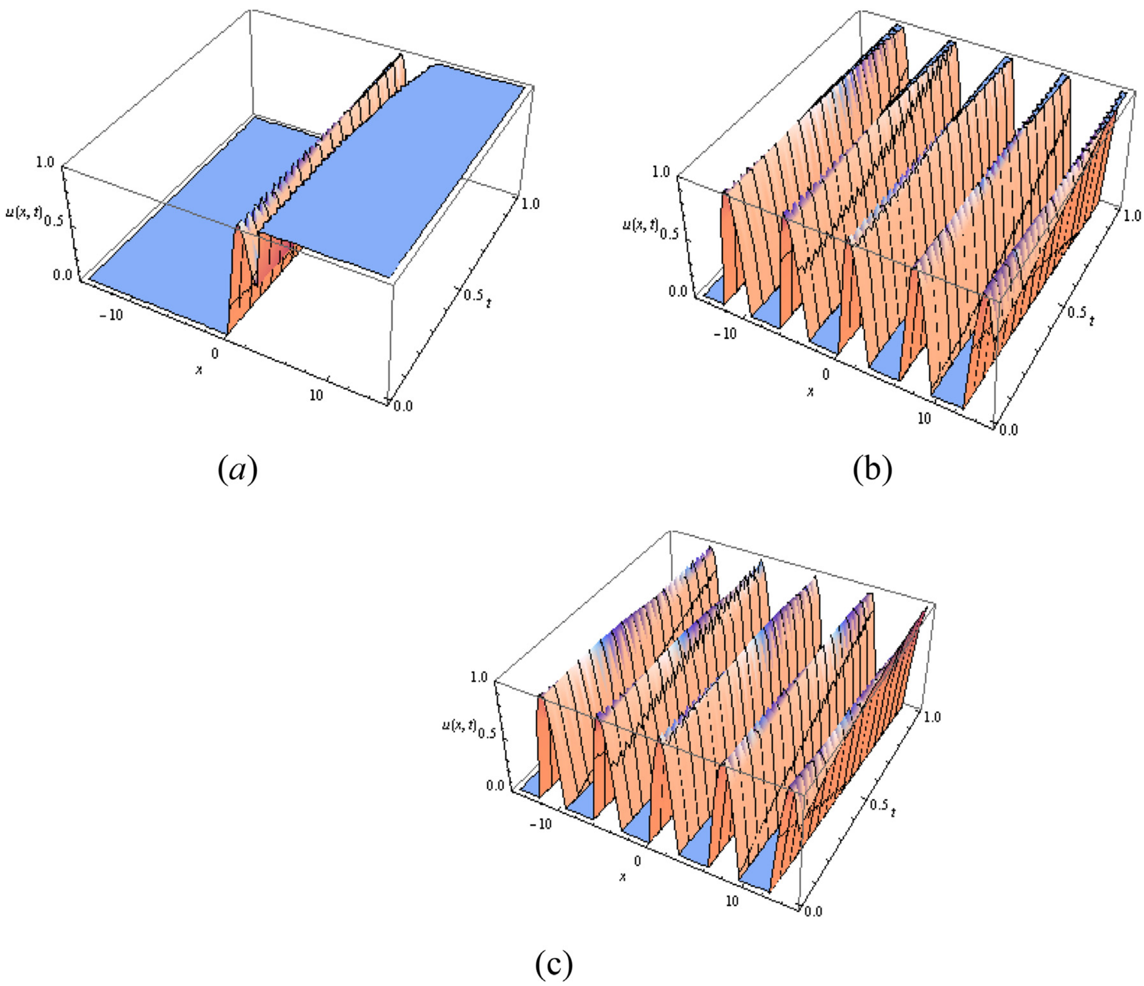

Fig. 2 – *u(x,t)* obtained by using five term (*a*) FDTM, (b) MFDTM and (c) Exact solution when $\alpha = 1.5$.

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{\alpha,1}(x,h+1)+\frac{\partial^3 U_{\alpha,1}(x,h)}{\partial x^3} = \sin\pi x\frac{\sin\left(\frac{\pi h}{2}\right)}{\Gamma(\alpha h+1)}-\pi^3\cos\pi x\frac{\cos\left(\frac{\pi h}{2}\right)}{\Gamma(\alpha h+1)} \quad (30)$$

The transformed version of (26) is

$$U_{\alpha,1}(x,0)=\sin\pi x \quad (31)$$

Following the same procedure in Example 1 and 2, we obtained the FDTM and MFDTM series solution

$$u(x,t)=\left(\pi x-\frac{(\pi x)^3}{3!}+\frac{(\pi x)^5}{5!}-\ldots\right)\left(1-\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}-\ldots\right) \quad (32)$$

$$u(x,t)=\sin\pi x\left(1-\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}-\ldots\right) \quad (33)$$

respectively. When $\alpha\to 1$, the approximate solution (32) and (33) takes the following form

$$u(x,t)=\sin\pi x\cos t \quad (34)$$

which is exactly same as the solution obtained in [32]. Five term solution of *u*(*x*,*t*) using FDTM, MFDTM corresponding to the values of α=0.5,1.95 and exact solution are plotted in Figs. 3–4 respectively.

Example 4. Finally, consider the non-homogeneous fractional third-order dispersive partial differential equation in three dimensional space

$$u_t^\alpha+u_{xxx}+\frac{1}{8}u_{yyy}+\frac{1}{27}u_{zzz}=-3\cos(x+2y+3z)\sin t + \sin(x+2y+3z)\cos t,\ t>0 \quad (35)$$

Subject to the initial condition

$$u(x,y,z,0)=0 \quad (36)$$

FDTM: The transformed version of (35) is

$$
\begin{aligned}
&\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{\alpha,1}(k,l,m,h+1)+(k+1)(k+2)(k+3)U_{\alpha,1}(k+3,l,m,h) \\
&+\frac{1}{8}(l+1)(l+2)(l+3)U_{\alpha,1}(k,l+3,m,h)+\frac{1}{27}(m+1)(m+2)(m+3)U_{\alpha,1}(k,l,m+3,h) \\
&=\left(-3\frac{\cos\left(\frac{k\pi}{2}\right)}{k!}\frac{(3)^m\cos\left(\frac{m\pi}{2}\right)}{m!}+3\frac{\sin\left(\frac{k\pi}{2}\right)}{k!}\frac{(3)^m\sin\left(\frac{m\pi}{2}\right)}{m!}\right)\frac{(2)^l\cos\left(\frac{l\pi}{2}\right)}{l!}\frac{\sin\left(\frac{h\pi}{2}\right)}{\Gamma(\alpha h+1)} \\
&+\left(3\frac{\sin\left(\frac{k\pi}{2}\right)}{k!}\frac{(3)^m\cos\left(\frac{m\pi}{2}\right)}{m!}+3\frac{\cos\left(\frac{k\pi}{2}\right)}{k!}\frac{(3)^m\sin\left(\frac{m\pi}{2}\right)}{m!}\right)\frac{(2)^l\sin\left(\frac{l\pi}{2}\right)}{l!}\frac{\sin\left(\frac{h\pi}{2}\right)}{\Gamma(\alpha h+1)} \\
&+\left(\frac{\sin\left(\frac{k\pi}{2}\right)}{k!}\frac{(3)^m\cos\left(\frac{m\pi}{2}\right)}{m!}-\frac{\cos\left(\frac{k\pi}{2}\right)}{k!}\frac{(3)^m\sin\left(\frac{m\pi}{2}\right)}{m!}\right)\frac{(2)^l\cos\left(\frac{l\pi}{2}\right)}{l!}\frac{\cos\left(\frac{h\pi}{2}\right)}{\Gamma(\alpha h+1)} \\
&-\left(\frac{\cos\left(\frac{k\pi}{2}\right)}{k!}\frac{(3)^m\cos\left(\frac{m\pi}{2}\right)}{m!}-\frac{\sin\left(\frac{k\pi}{2}\right)}{k!}\frac{(3)^m\sin\left(\frac{m\pi}{2}\right)}{m!}\right)\frac{(2)^l\sin\left(\frac{l\pi}{2}\right)}{l!}\frac{\cos\left(\frac{h\pi}{2}\right)}{\Gamma(\alpha h+1)}
\end{aligned} \tag{37}
$$

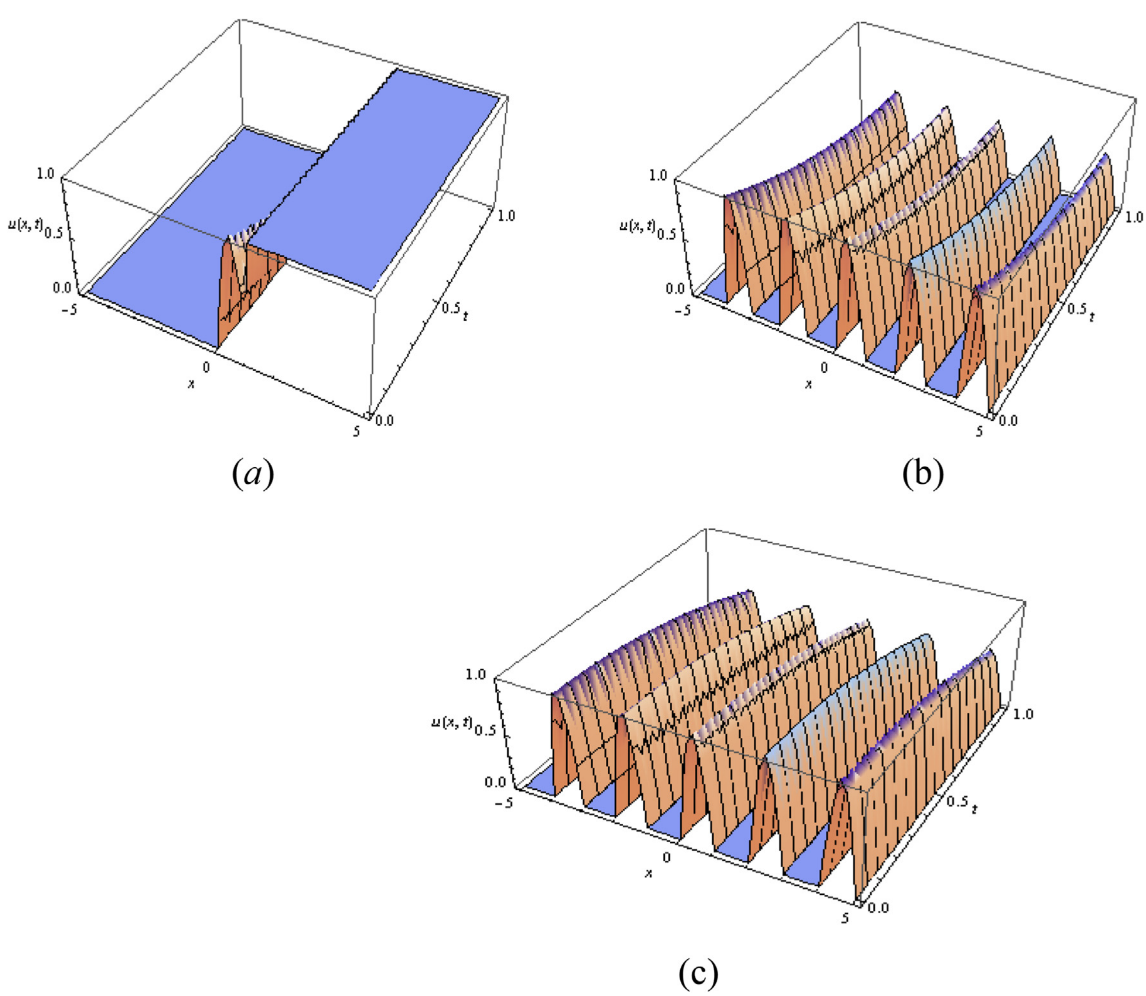

Fig. 3 – $u(x,t)$ obtained by using five term (*a*) FDTM, (*b*) MFDTM and (*c*) Exact solution when $\alpha = 0.5$.

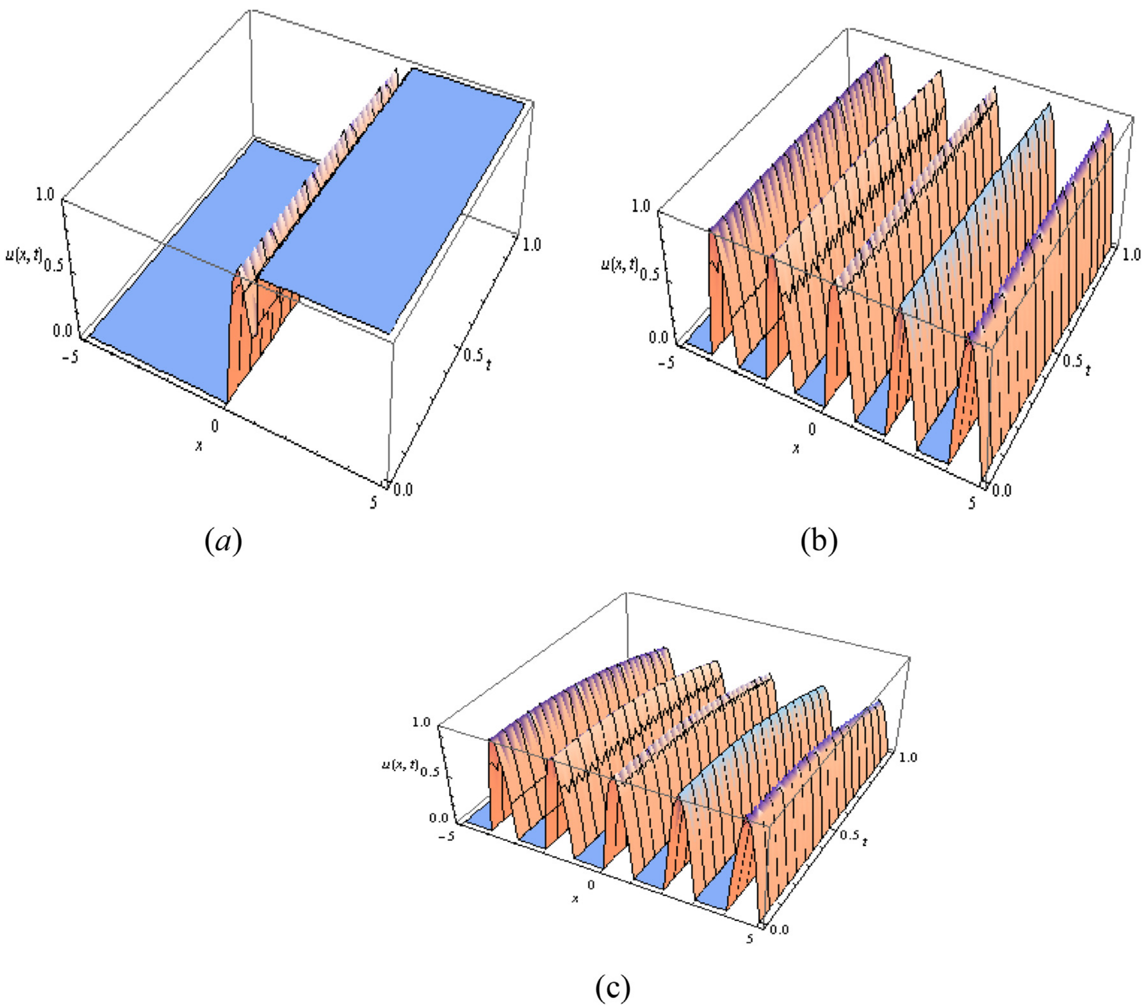

Fig. 4 – u(x,t) obtained by using five term (a) FDTM, (b) MFDTM and (c) Exact solution when $\alpha = 1.95$.

The transformed version of (36) is

$$U_{\alpha,1}(k,l,m,0) = 0, k,l,m = 0,1,2,\ldots \tag{38}$$

MFDTM: The transformed version of (35) is w. r. t 't' is

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{\alpha,1}(x,y,z,h+1) + \frac{\partial^3 U_{\alpha,1}(x,y,z,h)}{\partial x^3} + \frac{1}{8}\frac{\partial^3 U_{\alpha,1}(x,y,z,h)}{\partial y^3} + \frac{1}{27}\frac{\partial^3 U_{\alpha,1}(x,y,z,h)}{\partial z^3} = -3\cos(x+2y+3z)\frac{\sin\left(\frac{\pi h}{2}\right)}{\Gamma(\alpha h+1)} + \sin(x+2y+3z)\frac{\cos\left(\frac{\pi h}{2}\right)}{\Gamma(\alpha h+1)} \tag{39}$$

The transformed version of (36) is

$$U_{\alpha,1}(x,y,z,0) = 0 \tag{40}$$

Following the same procedure in Example 1 and 2, we obtained the FDTM and MFDTM series solution

$$u(x,t) = \left(x+2y+3z-\frac{x^3}{6}-\frac{4}{3}y^3-\frac{9}{2}z^3-x^2y-9z^2y-\frac{3}{2}zx^2 - 2y^2x-6zy2^2+\ldots\right)\frac{t^\alpha}{\Gamma(\alpha+1)}-\ldots \tag{41}$$

$$u(x,y,z,t) = \sin(x+2y+3z)\left(\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right) \tag{42}$$

respectively. When$\alpha \to 1$, the approximate solution (32) and (33) takes the following form

$$u(x,y,z,t) = \sin(x+2y+3z)\sin t \tag{43}$$

which is exactly same as the solution obtained in [32].

6. Conclusions

In this paper, we implemented the two-dimensional FDTM and MFDTM for solving fractional third-order dispersive partial differential equation. DTM is an attractive tool for solving linear and nonlinear partial differential equations and it does not require linearization, discretization or perturbation. But it also faces some difficulties while constructing recursive equation for the function of three or more variables and it requires an expensive computational cost to solve the algebraic recursive equation. The proposed MFDTM for the specific variable can obtain the simple recursive equation. Thus it is concluded that MFDTM enhances the effectiveness of the computational work when compared with the FDTM. The proposed methods are simpler in its principles and effective in solving linear and nonlinear differential equations of fractional order and promising tool for solving wider class of nonlinear fractional models in mathematical physics with high accuracy.

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