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Classification of quasifinite modules over the Lie algebras of Weyl type ☆

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Abstract

For a nondegenerate additive subgroup Γ of the n-dimensional vector space $\mathbb{F}^n$ over an algebraically closed field $\mathbb{F}$ of characteristic zero, there is an associative algebra and a Lie algebra of Weyl type $\mathcal{W}(\Gamma, n)$ spanned by all differential operators $uD_1^{m_1} \cdots D_n^{m_n}$ for $u \in \mathbb{F}[\Gamma]$ (the group algebra), and $m_1, \ldots, m_n \geqslant 0$, where $D_1, \ldots, D_n$ are degree operators. In this paper, it is proved that an irreducible quasifinite $\mathcal{W}(\mathbb{Z}, 1)$-module is either a highest or lowest weight module or else a module of the intermediate series; furthermore, a classification of uniformly bounded $\mathcal{W}(\mathbb{Z}, 1)$-modules is completely given. It is also proved that an irreducible quasifinite $\mathcal{W}(\Gamma, n)$-module is a module of the intermediate series and a complete classification of quasifinite $\mathcal{W}(\Gamma, n)$-modules is also given, if Γ is not isomorphic to $\mathbb{Z}$.

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1. Introduction

Let n be a positive integer. A (classical) *Weyl algebra of rank n* is the associative algebra $A_n^+ = \mathbb{C}[t_1, \ldots, t_n, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}]$ or $A_n = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}]$ of differential operators over the complex field $\mathbb{C}$. The Lie algebra with A_n as the underlined vector space and the commutator as the Lie bracket is called a *Lie algebra of Weyl type*, and denoted by $\mathcal{W}(n)$. The Lie algebra $\mathcal{W}(n)$ is a $\mathbb{Z}^n$-graded Lie algebra

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$\mathscr{W}(n) = \oplus_{\alpha\in\mathbb{Z}^n}\mathscr{W}(n)_\alpha$ with the grading space $\mathscr{W}(n)_\alpha$ spanned by

$$t^\alpha D^\mu = t_1^{\alpha_1}\cdots t_n^{\alpha_n}D_1^{\mu_1}\cdots D_n^{\mu_n} \quad \text{for } \mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}_+^n,$$

where $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n$ and $D_i = t_i\frac{\partial}{\partial t_i}$. It is known [8,14,19] that $\mathscr{W}(n)$ has a nontrivial universal central extension if and only if $n=1$. The universal central extension $\hat{\mathscr{W}}(1)$ of $\mathscr{W}(1)$ is the well-known Lie algebra $\mathscr{W}_{1+\infty}$ of the $\mathscr{W}$-infinity algebras, which arise naturally in various physical theories such as conformal field theory, the theory of the quantum Hall effect, etc. and which have received intensive studies in the literature (see for example, [1–6,8–14,16,19–21]).

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. Consider the vector space $\mathbb{F}^n$. An element in $\mathbb{F}^n$ is denoted by $\alpha=(\alpha_1,\dots,\alpha_n)$. Let Γ be an additive subgroup of $\mathbb{F}^n$ such that Γ is *nondegenerate*, i.e., it contains an $\mathbb{F}$-basis of $\mathbb{F}^n$. Let $\mathbb{F}[\Gamma]$ denote the group algebra of Γ spanned by $\{t^\alpha \mid \alpha\in\Gamma\}$ with the algebraic operation $t^\alpha\cdot t^\beta = t^{\alpha+\beta}$ for $\alpha,\beta\in\Gamma$. We define the *degree operators* D_i to be the derivations of $\mathbb{F}[\Gamma]$ determined by $D_i : t^\alpha \mapsto \alpha_i t^\alpha$ for $\alpha\in\Gamma$, where $i=1,\dots,n$. The *Lie algebra of generalized Weyl type of rank n* (or simply a *Lie algebra of Weyl type*, also called a *Lie algebra of generalized differential operators*) is a tensor product space $\mathscr{W}(\Gamma,n) = \mathbb{F}[\Gamma,D_1,\dots,D_n] = \mathbb{F}[\Gamma]\otimes\mathbb{F}[D_1,\dots,D_n]$ of the commutative associative algebra $\mathbb{F}[\Gamma]$ with the polynomial algebra $\mathbb{F}[D_1,\dots,D_n]$ of $D_1,\dots,D_n$, which is spanned by $\{t^\alpha D^\mu \mid \alpha\in\Gamma,\ \mu\in\mathbb{Z}_+^n\}$, where D^μ stands for $\prod_{i=1}^n D_i^{\mu_i}$, with the Lie bracket:

$$[t^\alpha D^\mu, t^\beta D^\nu] = (t^\alpha D^\mu)\cdot(t^\beta D^\nu) - (t^\beta D^\nu)\cdot(t^\alpha D^\mu) \tag{1.1}$$

and

$$(t^\alpha D^\mu)\cdot(t^\beta D^\nu) = \sum_{\lambda\in\mathbb{Z}_+^n}\binom{\mu}{\lambda}[\beta]^\lambda t^{\alpha+\beta}D^{\mu+\nu-\lambda}, \tag{1.2}$$

where

$$[\beta]^\lambda = \prod_{i=1}^n \beta_i^{\lambda_i}, \quad \binom{\mu}{\lambda} = \prod_{i=1}^n\binom{\mu_i}{\lambda_i}, \quad \binom{i}{j} = \begin{cases} \frac{i(i-1)\cdots(i-j+1)}{j!} & \text{if } j\geqslant 0,\\ 0 & \text{otherwise.}\end{cases}$$

The associative algebra with the underlined vector space $\mathscr{W}(\Gamma,n)$ and product (1.2) is called a *generalized Weyl algebra of rank n*, denoted by $\mathscr{A}(\Gamma,n)$. Then the classical Lie algebra $\mathscr{W}(n)$ of Weyl type is simply the Lie algebra $\mathscr{W}(\mathbb{Z}^n,n)$ by our definition.

Clearly $\mathscr{W}(\Gamma,n)$ is a Γ-graded Lie algebra $\mathscr{W}(\Gamma,n) = \oplus_{\alpha\in\Gamma}\mathscr{W}(\Gamma,n)_\alpha$ with the grading space

$$\mathscr{W}(\Gamma,n)_\alpha = \text{span}\{t^\alpha D^\mu \mid \mu\in\mathbb{Z}_+^n\} \quad \text{for } \alpha\in\Gamma. \tag{1.3}$$

It is proved in [19] that $\mathscr{W}(\Gamma,n)$ has a nontrivial universal central extension if and only if $n=1$. The universal central extension $\hat{\mathscr{W}}(\Gamma,1)$ of $\mathscr{W}(\Gamma,1)$ is defined as follows: The Lie bracket (1.1) is replaced by

$$[t^\alpha[D]_\mu, t^\beta[D]_\nu] = (t^\alpha[D]_\mu)\cdot(t^\beta[D]_\nu) - (t^\beta[D]_\nu)\cdot(t^\alpha[D]_\mu) + \delta_{\alpha,-\beta}(-1)^\mu \mu!\nu!\binom{\alpha+\mu}{\mu+\nu+1}C \tag{1.4}$$

for $\alpha,\beta\in\Gamma\subset\mathbb{F}, \mu,\nu\in\mathbb{Z}_+$, where $[D]_\mu = D(D-1)\cdots(D-\mu+1)$, and C is a central element of $\hat{\mathscr{W}}(\Gamma,1)$ (the 2-cocycle of $\mathscr{W}(\Gamma,1)$ corresponding to (1.4) seems to appear first in [10]).

A $\mathscr{W}(\Gamma,n)$-module (or an $\mathscr{A}(\Gamma,n)$-module) V is called a *quasifinite module* [1,6,8,11,13,16] if $V=\oplus_{\alpha\in\Gamma}V_\alpha$ is Γ-graded such that $\mathscr{W}(\Gamma,n)_\alpha V_\beta\subset V_{\alpha+\beta}$ for $\alpha,\beta\in\Gamma$ and such that each grading space V_α is finite-dimensional (this is equivalent to saying that a quasifinite module is a module having finite-dimensional generalized weight spaces with respect to the commutative subalgebra $\mathscr{W}_0$). A quasifinite module V is called a *uniformly bounded module* if the dimensions of grading spaces V_α are uniformly bounded, i.e., there exists a positive integer N such that $\dim V_\alpha\leqslant N$ for all $\alpha\in\Gamma$. A quasifinite module V is called a *trivial module* if $\mathscr{W}(\Gamma,n)$ acts trivially on V.

Clearly an $\mathscr{A}(\Gamma,n)$-module is a $\mathscr{W}(\Gamma,n)$-module, but not the converse. Thus it suffices to consider $\mathscr{W}(\Gamma,n)$-modules. The quasifinite highest weight modules over $\hat{\mathscr{W}}(1)=\hat{\mathscr{W}}(\mathbb{Z},1)$ were intensively studied in [1,6,9–11,13,16,21] (it is also worth mentioning that Block [3] studied arbitrary irreducible modules over the classical Weyl algebra A_1^+ of rank 1).

Let $p\geqslant 1$. Denote by $M_{p\times p}(\mathbb{F})$ the set of $p\times p$ matrices with entries in $\mathbb{F}$. Denote

$$M^n_{p\times p}(\mathbb{F}) = \{G=(G_1,\ldots,G_n)\mid G_i\in M_{p\times p}(\mathbb{F}), G_iG_j=G_jG_i \text{ for } i,j=1,\ldots,n\},$$

the set of n-tuples of commuting $p\times p$ matrices. Denote by $\mathbf{1}_p$ the $p\times p$ identity matrix. Denote $\mathbb{1}=(\mathbf{1}_p,\ldots,\mathbf{1}_p)\in M^n_{p\times p}(\mathbb{F})$. Let $G=(G_1,\ldots,G_n)\in M^n_{p\times p}(\mathbb{F})$. Then one can define a quasifinite $\mathscr{W}(\Gamma,n)$-module $A_{p,G}$ as follows: it has a basis $\{y^{(i)}_\alpha\mid \alpha\in\Gamma,\ i=1,\ldots,p\}$ such that

$$(t^\alpha D^\mu)Y_\beta = Y_{\alpha+\beta}[\beta\,\mathbb{1}+G]^\mu \quad \text{for } \alpha,\beta\in\Gamma, \mu\in\mathbb{Z}^n_+ \tag{1.5}$$

where

$$Y_\beta=(y^{(1)}_\beta,\ldots,y^{(p)}_\beta),\quad [\beta\,\mathbb{1}+G]^\mu=\prod_{i=1}^n(\beta_i\,\mathbf{1}_p+G_i)^{\mu_i}.$$

Here $\beta_i\cdot\mathbf{1}_p$ denotes the scalar multiplication of the identity matrix. Clearly $A_{p,G}$ is also an $\mathscr{A}(\Gamma,n)$-module. By Su and Zhao [20], there exists a Lie algebra isomorphism $\sigma:\mathscr{W}(\Gamma,n)\cong\mathscr{W}(\Gamma,n)$ such that

$$\sigma(t^\alpha D^\mu) = (-1)^{|\mu|+1}([D^\mu,t^\alpha]+t^\alpha D^\mu) \text{ for } \alpha\in\Gamma,\mu\in\mathbb{Z}_+, \tag{1.6}$$

where $|\mu| = \sum_{i=1}^{n} \mu_p$ is the *level* of $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_+^n$. This isomorphism is uniquely determined by $\sigma(t^\alpha) = -t^\alpha$, $\sigma(D_i^j) = (-1)^{j+1} D_i$ for $\alpha \in \Gamma$, $i = 1, \ldots, n$, $j \in \mathbb{Z}_+$. Using this isomorphism, we have another $\mathscr{W}(\Gamma, n)$-module $\bar{A}_{p,G}$, called the *twisted module of* $A_{p,G}$, for the pair (p, G), defined by

$$(t^\alpha D^\mu) Y_\beta = (-1)^{|\mu|+1} Y_{\alpha+\beta}[(\alpha+\beta)\cdot \mathbb{1} + G]^\mu \quad \text{for } \alpha, \beta \in \Gamma, \mu \in \mathbb{Z}_+^n. \tag{1.7}$$

Obviously, $\bar{A}_{p,G}$ is not an $\mathscr{A}(\Gamma, n)$-module. When $\Gamma = \mathbb{Z}$, $n = p = 1$, the above modules $A_{p,G}, \bar{A}_{p,G}$ were obtained in [21]. Clearly, $A_{p,G}$ or $\bar{A}_{p,G}$ is indecomposable if and only if at least one G_i is an indecomposable matrix (here a $p \times p$ matrix B is called *indecomposable* if there does not exist an invertible matrix P such that $P^{-1}BP = \mathrm{diag}(B_1, B_2)$ for some $p_i \times p_i$ matrices B_i with $p_i < p$, $i = 1, 2$) and $A_{p,G}$ or $\bar{A}_{p,G}$ is irreducible if and only if $p = 1$. When $p = 1$, we refer $A_G = A_{1,G}$ or $\bar{A}_G = \bar{A}_{1,G}$ to as a *module of the intermediate series* (a notion borrowed from that of modules over the Virasoro algebra, cf. [15]).

Since each grading space in (1.3) is infinite-dimensional, the classification of quasifinite modules is thus a nontrivial problem, as pointed in [11,13]. The aim of this paper is to prove the following theorem (the analogous results for the affine Lie algebras and the Virasoro algebra were obtained in [7,15]).

Theorem 1.1. (i) *A uniformly bounded module over* $\mathscr{W}(1) = \mathscr{W}(\mathbb{Z}, 1)$ *or over* $\hat{\mathscr{W}}(1) = \hat{\mathscr{W}}(\mathbb{Z}, 1)$ *is a direct sum of a trivial module, a module* $A_{p,G}$ *and a module* $\bar{A}_{p',G'}$ *for some positive integers* p, p' *and some* $G \in M_{p\times p}^n(\mathbb{F}), G' \in M_{p'\times p'}(\mathbb{F})$ (*in the central extension case, the central element* C *acts trivially on a uniformly bounded module*); *a quasifinite irreducible module is either a highest or lowest weight module or else a module of the intermediate series.*

(ii) *Suppose* Γ *is not isomorphic to* $\mathbb{Z}$. *A quasifinite module over* $\mathscr{W}(\Gamma, n)$ *or over* $\hat{\mathscr{W}}(\Gamma, 1)$ *is a direct sum of a trivial module and a uniformly bounded module*; *a uniformly bounded module is a direct sum of a trivial module, a module* $A_{p,G}$ *and a module* $\bar{A}_{p',G'}$ *for some positive integers* p, p' *and some* $G \in M_{p\times p}^n(\mathbb{F}), G' \in M_{p'\times p'}^n(\mathbb{F})$. *In particular*, *a nontrivial quasifinite irreducible module is a module of the intermediate series.*

Thus, we in particular obtain that an indecomposable uniformly bounded $\mathscr{W}(\Gamma, n)$-module is simply an $\mathscr{A}(\Gamma, n)$-module (if the central element $t^0 D^0$ acts by 1) or its twist (if $t^0 D^0$ acts by -1), and that there is an equivalence between the category of uniformly bounded $\mathscr{A}(\Gamma, n)$-modules without the trivial composition factor and the category of the finite-dimensional $\mathscr{W}_0$-modules obtained by restriction to any nonzero graded subspace.

A *composition series of a module* V is a finite or infinite series of submodules $V = V^{(0)} \supset V^{(1)} \supset V^{(2)} \supset \cdots \supset \{0\}$ such that each $V^{(i)}/V^{(i+1)}$, called a *composition factor*, is irreducible.

Remark 1.2. Note that the definition of quasifiniteness does not require that V is a *weight module* (i.e., the actions of $D_i,\ i=1,\dots,n$ on V are diagonalizable). If we require that V is a weight module, then each G_i in (1.5) is diagonalizable, and thus all uniformly bounded modules are completely reducible. Also note that in Theorem 1.1(ii), if a module have infinite number of the trivial composition factor, then it is not necessarily uniformly bounded since any Γ-graded vector space can be defined as a trivial module.

We shall prove Theorem 1.1(i) and (ii) in Sections 2 and 3, respectively.

2. Proof of Theorem 1.1(i)

For convenience, we shall only work on noncentral extension case since the proof of central extension case is exactly similar. In this section, we shall consider the Lie algebra $\mathcal{W}(\mathbb{Z},1)=\text{span}\{t^iD^j\,|\,(i,j)\in\mathbb{Z}\times\mathbb{Z}_+\}$, which is now denoted by $\mathcal{W}$. Then $\mathcal{W}$ is $\mathbb{Z}$-graded $\mathcal{W}=\oplus_{i\in\mathbb{Z}}\mathcal{W}_i$ with $\mathcal{W}_i=\text{span}\{t^iD^j\,|\,j\in\mathbb{Z}_+\}$, and it has a triangular decomposition $\mathcal{W}=\mathcal{W}_+\oplus\mathcal{W}_0\oplus\mathcal{W}_-$ with $\mathcal{W}_+=\oplus_{i>0}\mathcal{W}_i, \mathcal{W}_-=\oplus_{i<0}\mathcal{W}_i$. Observe that $\mathcal{W}_+$ is generated by the adjoint action of $L_{1,0}$ on $\mathcal{W}_0$ and that $\text{ad}_{L_{1,0}}$ is locally nilpotent on $\mathcal{W}$, where for convenience, we denote

$$L_{i,j}=t^i[D]_j \text{ for } (i,j)\in\mathbb{Z}\times\mathbb{Z}_+.$$

Note that in fact $L_{i,j}=t^{i+j}(\frac{d}{dt})^j$ (cf. notation in (1.4)). Suppose $V=\oplus_{i\in\mathbb{Z}}V_i$ is a quasifinite module over $\mathcal{W}$. For any $a\in\mathbb{F}$, we denote

$$V(a)=\bigoplus_{i\in\mathbb{Z}}V(a)_i \quad \text{where } V(a)_i=\{v\in V_i\,|\,L_{0,1}v=(a+i)v\}.$$

Since $[L_{0,1},L_{i,j}]=iL_{i,j}$, it is straightforward to verify that $V(a)$ is a submodule of V. Since V_i is finite-dimensional, there exists at least an eigenvector of $L_{0,1}$ in V_i, i.e., $V(a)\neq 0$ for some $a\in\mathbb{F}$. If V is irreducible, then $V=V(a)$ for some $a\in\mathbb{F}$, i.e., $L_{0,1}$ is diagonalizable on V and so V is a weight module. Since $L_{0,0}=1$ is a central element and $L_{0,0}|_{V_0}$ has at least an eigenvector, we must have

$$L_{0,0}|_V=c\cdot\mathbf{1}_V \quad \text{for some } c\in\mathbb{F} \text{ if } V \text{ is indecomposable.} \tag{2.1}$$

Proposition 2.1. *Suppose V is a quasifinite irreducible $\mathcal{W}$-module which is neither a highest nor a lowest weight module. Then $L_{1,0}:V_i\to V_{i+1}$ and $L_{-1,0}:V_i\to V_{i-1}$ are injective and thus bijective for all $i\in\mathbb{Z}$. In particular, V is uniformly bounded.*

Proof. Say $L_{1,0}v_0=0$ for some $0\neq v_0\in V_i$. By shifting the grading index if necessary, we can suppose $i=0$. Since $L_{0,0}|_{V_0}, L_{0,1}|_{V_0},\dots$ are linear transformations on the finite-dimensional vector space V_0, there exists $s\geqslant 2$ such that for all $k\geqslant s$, $L_{0,k}|_{V_0}$ are linear combinations of $L_{0,p}|_{V_0}$, $0\leqslant p<s$. This implies that $\mathcal{W}_0v_0=Sv_0$, where $S=$

$\mathrm{span}\{L_{0,p}\,|\,0\leqslant p<s\}$. Recall that $\mathrm{ad}_{L_{1,0}}$ is locally nilpotent such that $\mathscr{W}_k=\mathrm{ad}^k_{L_{1,0}}(\mathscr{W}_0)$ for $k>0$. Choose $m>0$ such that $\mathrm{ad}^m_{L_{1,0}}(S)=0$, then for $k\geqslant m$, one has

$$\mathscr{W}_k v_0=(\mathrm{ad}^k_{L_{1,0}}(\mathscr{W}_0))v_0=L^k_{1,0}\mathscr{W}_0v_0=L^k_{1,0}Sv_0=(\mathrm{ad}^k_{L_{1,0}}(S))v_0=0.$$

This means

$$\mathscr{W}_{[m,\infty)}v_0=0, \tag{2.2}$$

where in general, for any $\mathbb{Z}$-graded space N, we use notations N_+, N_-, N_0 and $N_{[p,q)}$ to denote the subspaces spanned by elements of degree k with $k>0, k<0, k=0$ and $p\leqslant k<q$ respectively. For any subspace M of $\mathscr{W}$, we use $U(M)$ to denote the subspace, which is the span of standard monomials with respect to a basis of M, of the universal enveloping algebra of $\mathscr{W}$. Since $\mathscr{W}=\mathscr{W}_{[1,m)}+\mathscr{W}_0+\mathscr{W}_-+\mathscr{W}_{[m,\infty)}$, using the PBW theorem and the irreducibility of V, we have

$$\begin{aligned}V=U(\mathscr{W})v_0&=U(\mathscr{W}_{[1,m)})U(\mathscr{W}_0+\mathscr{W}_-)U(\mathscr{W}_{[m,\infty)})v_0\\&=U(\mathscr{W}_{[1,m)})U(\mathscr{W}_0+\mathscr{W}_-)v_0.\end{aligned} \tag{2.3}$$

Note that V_+ is a $\mathscr{W}_+$-module. Let V'_+ be the $\mathscr{W}_+$-submodule of V_+ generated by $V_{[0,m)}$. We want to prove

$$V_+=V'_+. \tag{2.4}$$

Let $x\in V_+$ have degree k. If $0\leqslant k<m$, then by definition $x\in V'_+$. Suppose $k\geqslant m$. Using (2.3), x is a linear combination of the form u_1x_1 with $u_1\in\mathscr{W}_{[1,m)}, x_1\in V$. Thus the degree $\deg u_1$ of u_1 satisfies $1\leqslant\deg u_1<m$, so $0<\deg x_1=k-\deg u_1<k$. By inductive hypothesis, $x_1\in V'_+$, and thus $x\in V'_+$. This proves (2.4).

Eq. (2.4) means that V_+ is finitely generated as a $\mathscr{W}_+$-module. Choose a basis B of $V_{[0,m)}$, then for any $x\in B$, we have $x=u_xv_0$ for some $u_x\in U(\mathscr{W})$. Regarding u_x as a polynomial with respect to a basis of $\mathscr{W}$, by induction on the polynomial degree and using the formula $[u,w_1w_2]=[u,w_1]w_2+w_1[u,w_2]$ for $u\in\mathscr{W}, w_1,w_2\in U(\mathscr{W})$, we see that there exists a positive integer k_x large enough such that $k_x>m$ and $[\mathscr{W}_{[k_x,\infty)},u_x]\subset U(\mathscr{W})\mathscr{W}_{[m,\infty)}$. Then by (2.2), $\mathscr{W}_{[k_x,\infty)}x=[\mathscr{W}_{[k_x,\infty)},u_x]v_0+u_x\mathscr{W}_{[k_x,\infty)}v_0=0$. Take $k=\max\{k_x\,|\,x\in B\}$, then

$$\mathscr{W}_{[k,\infty)}V_+=\mathscr{W}_{[k,\infty)}U(\mathscr{W}_+)V_{[0,m)}=U(\mathscr{W}_+)\mathscr{W}_{[k,\infty)}V_{[0,m)}=0.$$

Since we have $\mathscr{W}_+\subset\mathscr{W}_{[k,\infty)}+[\mathscr{W}_{[-k',0)},\mathscr{W}_{[k,\infty)}]$ for some $k'>k$, we get $\mathscr{W}_+V_{[k',\infty)}=0$. Now if $x\in V_{[k'+m,\infty)}$, by (2.3), it is a sum of elements of the form u_1x_1 such that $u_1\in\mathscr{W}_{[1,m)}$ and so $x_1\in V_{[k',\infty)}$, and thus $u_1x_1=0$. This proves that V has no degree $\geqslant k'+m$.

Now let p be maximal integer such that $V_p \neq 0$. Since $\mathcal{W}_0$ is commutative, there exists a common eigenvector v_0' of $\mathcal{W}_0$ in V_p. Then v_0' is a highest weight vector of $\mathcal{W}$, this contradicts the assumption of the proposition. $\square$

Proposition 2.2. *Suppose V is an indecomposable uniformly bounded $\mathcal{W}$-module without the trivial composition factor. Then V is a module of the form $A_{p,G}$ or $\bar{A}_{p,G}$.*

Proof. First we claim that $L_{i,0}$ acts nondegenerately on V for all $i \neq 0$. Say $L_{i_0,0}v_0 = 0$ for some $i_0 > 0$ and some $0 \neq v_0 \in V_0$. Denote $\mathcal{W}' = \mathcal{W}(\mathbb{Z}i_0, 1) = \mathrm{span}\{L_{i_0 j,k} \mid j,k \in \mathbb{Z}, k \geqslant 0\}$, a subalgebra of $\mathcal{W}$, which is clearly isomorphic to $\mathcal{W}$. Let V' be the $\mathcal{W}'$-submodule of V generated by v_0. Then by replacing $\mathcal{W}$ and V by $\mathcal{W}'$ and V', respectively, in the proof of Proposition 2.1, we see that V' has a highest weight vector v' with respect to $\mathcal{W}'$ (cf. (2.3), the left-hand side of (2.3) is now V'). Then v' generates a highest weight $\mathcal{W}'$-submodule V''. Since a nontrivial highest weight $\mathcal{W}'$-module is not uniformly bounded (see for example [11]), V'' must be a trivial $\mathcal{W}'$-module $V'' = \mathbb{F}v'$. Denote $\mathrm{Vir} = \mathrm{span}\{L_{j,1} \mid j \in \mathbb{Z}\}$, a subalgebra of $\mathcal{W}$, which is isomorphic to the centerless Virasoro algebra, then v' generates a uniformly bounded weight Vir-submodule U of V (note that in general we do not assume V is a weight module over $\mathcal{W}$). Since $L_{i_0 j,1}v' = 0$ for all $j \in \mathbb{Z}$, by a well-known result of a uniformly bounded weight Vir-module (see for example, [15,18]), we have $L_{i,1}v' = 0$ for all $i \in \mathbb{Z}$. But $\mathcal{W}$ is generated by $\mathcal{W}'$ and Vir, we obtain that $\mathbb{F}v'$ is a trivial $\mathcal{W}$-module. This is a contradiction with the assumption of the proposition. This proves the claim.

So, there exists $p \geqslant 1$ such that $\dim V_k = p$ for all $k \in \mathbb{Z}$, and one can choose a basis $Y_0 = (y_0^{(1)}, \ldots, y_0^{(p)})$ of V_0 and define a basis $Y_k = (y_k^{(1)}, \ldots, y_k^{(p)})$ of V_k by induction on $|k|$ such that

$$L_{1,0} Y_k = Y_{k+1} \quad \text{for } k \in \mathbb{Z}. \tag{2.5}$$

Furthermore, by induction on p, we see that V has a finite number of composition factors.

First note that

$$\left[t^i \left(\frac{d}{dt}\right)^j, t^k \left(\frac{d}{dt}\right)^\ell \right] = \sum_s \left(\binom{j}{s} [k]_s - \binom{\ell}{s} [i]_s \right) t^{i+k-s} \left(\frac{d}{dt}\right)^{j+\ell-s}, \tag{2.6}$$

where $[k]_j = k(k-1)\cdots(k-j+1)$ is a similar notation to $[D]_j$. Now assume that $L_{i-j,j} Y_n = (t^i (\frac{d}{dt})^j) Y_n = Y_{n+i-j} P_{i,j,n}$ for some $P_{i,j,n} \in M_{p \times p}$. In the following discussion, we remind the reader that $t^i (\frac{d}{dt})^j$ is in the grading space $\mathcal{W}_{i-j}$, not in $\mathcal{W}_i$. Applying $[t^i (\frac{d}{dt})^j, t] = j t^i (\frac{d}{dt})^{j-1}$ to Y_n, we obtain $P_{i,j,n+1} - P_{i,j,n} = j P_{i,j-1,n}$. Thus induction on j

gives

$$P_{i,0,n} = P_i,\ \ P_{i,1,n} = \bar{n}P_i + Q_i,$$
$$P_{i,2,n} = [\bar{n}]_2 P_i + 2\bar{n}Q_i + R_i,\ \ P_{i,3,n} = [\bar{n}]_3 P_i + 3[\bar{n}]_2 Q_i + 3\bar{n}R_i + S_i \tag{2.7}$$

for some $P_i, Q_i, R_i, S_i \in M_{p\times p}$, where $\bar{n} = n + G$ for some fixed $G \in M_{p\times p}$ (here and below, when the context is clear, we identify a scalar with the corresponding $p\times p$ scalar matrix), $[\bar{n}]_j$ is again a similar notation to $[D]_j$, and $Q_1 = 0$ (we use notation $\bar{n} = n + G$ in order to have $Q_1 = 0$; note from $[L_{0,1}, L_{i,j}] = iL_{i,j}$ that G commutes with all other matrices involved in the following discussion). Then by (2.1) and (2.5), $P_0 = c,\ P_1 = 1$ and P_i is invertible for $i \neq 0$ by the proof above. Applying $[\frac{d}{dt}, t^2] = 2t$ to Y_n and comparing the coefficients of $\bar{n}^0$, we obtain $2cP_2 + [Q_0, P_2] = 2$ (where $[P_2, Q_0] = P_2Q_0 - Q_0P_2$ denotes the usual commutator of matrices). Comparing the traces of matrices in this equation shows $c \neq 0$. Thus all P_i are invertible.

Now we encounter the difficulty that though P_i, Q_i, R_i, S_i satisfy lots of relations, most nontrivial relations are too complicated to be used; since the products of matrices are not commutative nor the cancellation law holds in general, there is still a problem in finding the solutions for P_i, Q_i, R_i, S_i. Our strategy is first to find some relatively simple nontrivial relations among P_i (cf. (2.9)).

First from $[t^i, t^j] = 0$, we obtain that $[P_i, P_j] = 0$. Applying $[\frac{d}{dt}, t^i] = it^{i-1}$ to Y_n and comparing the coefficients of $\bar{n}^0$, we obtain that $[Q_0, P_i] = -ic\,P_i + iP_{i-1}$. By induction on $i \geqslant 0$, we obtain an important fact that $P'_i = \sum_{s=0}^{i}\binom{i}{s}(-c)^s P_s$, if not zero, is an eigenvector for ad_{Q_0} with eigenvalue $-ic$. Since the operator ad_{Q_0} acting on the finite-dimensional vector space $M_{p\times p}$ has only a finite number of eigenvalues, we have $P'_i = 0$ for $i \gg 0$. Let $q \geqslant 0$ be the least number such that

$$P'_i = \sum_{s=0}^{i}\binom{i}{s}(-c)^s P_s = 0 \tag{2.8}$$

for $i > q$. Note that for any $j \geqslant 0$, we have

$$\begin{aligned}
P''_{i,j} &:= \sum_{k=0}^{j}(-1)^k\binom{i}{k}P'_k = \sum_{k=0}^{j}\sum_{s=0}^{k}(-1)^k\binom{i}{k}\binom{k}{s}(-c)^s P_s\\
&= \sum_{s=0}^{j} c^s\binom{i}{s}\left(\sum_{k=s}^{j}\binom{i-s}{k-s}(-1)^{k-s}\right)P_s\\
&= \sum_{s=0}^{j} c^s\binom{i}{s}\binom{i-1-s}{j-s}(-1)^{j-s}P_s.
\end{aligned}$$

In particular, letting $j=i$, we obtain that $P''_{i,i}=c^iP_i$ and letting $j=q$, by (2.8), we obtain

$$P_i=c^{-i}P''_{i,i}=c^{-i}P''_{i,q}=c^{-i}\sum_{s=0}^{q}\binom{i}{s}\binom{i-1-s}{q-s}(-1)^{q-s}c^sP_s \tag{2.9}$$

for $i>q$. If $0\leqslant i\leqslant q$, (2.9) holds trivially. Observe that $\binom{i}{s}\binom{i-1-s}{q-s}$ is a polynomial on i of degree q with the coefficient of i^q being $\frac{1}{s!(q-s)!}=\frac{1}{q!}\binom{q}{s}$. Thus $(-1)^qc^iP_i$ is a polynomial on i (with coefficients in $M_{p\times p}$) of degree q with the coefficient of i^q being $\frac{1}{q!}\sum_{s=0}^{q}\binom{q}{s}(-c)^sP_s$.

By (2.6), we have

$$3[t(\tfrac{d}{dt})^2,[t(\tfrac{d}{dt})^2,t^i]]-2(2i-1)[t(\tfrac{d}{dt})^3,t^i]+[i+1]_4t^{i-2}=0.$$

Applying this to Y_n and comparing the coefficients of $\bar{n}^0$, we obtain

$$\begin{aligned}f(i):=&3(([i-1]_2+R_1)([i]_2+R_1)P_i-2([i-1]_2+R_1)P_iR_1+P_i(2+R_1)R_1)\\&-2(2i-1)(([i]_3+3iR_1+S_1)P_i-P_iS_1)+[i+1]_4P_{i-2}=0.\end{aligned} \tag{2.10}$$

Using (2.9) in (2.10), we obtain that $(-1)^qc^if(i)$ is a polynomial on i of degree at most $q+4$. By comparing the coefficients of i^{q+4}, we obtain

$$\frac{1}{q!}(c^2-1)\sum_{s=0}^{q}\binom{q}{s}(-c)^sP_s=0. \tag{2.11}$$

If necessary, by using the isomorphism in (1.6) (which interchanges $A_{p,G}$ with $\bar{A}_{p,G}$), we can always suppose $c\neq-1$.

Assume that $c\neq1$. Then (2.11) shows that (2.8) also holds for $i=q$. Thus the minimality of q implies that $q=0$ and then (2.9) implies

$$P_i=c^{1-i} \tag{2.12}$$

for $i\geqslant0$. Assume that $c=1$. Using (2.9) in (2.10) and comparing the coefficients of i^{q+3}, we again obtain that (2.8) holds for $i=q$. Thus in any case, $q=0$ and we have (2.12).

For any $i\in\mathbb{Z}$, choose $j>0$ such that $i+j-1>0$. Applying $[t^i\frac{d}{dt},t^j]=jt^{i+j-1}$ to Y_n, we obtain $-j(P_jP_i-P_{i+j-1})=[Q_i,P_j]=0$. This implies that (2.12) holds for all $i\in\mathbb{Z}$.

Now using (2.12) and applying $[(\frac{d}{dt})^2,t^{i+1}]=2(i+1)t^i\frac{d}{dt}+[i+1]_2t^{i-1}$ to Y_n, by comparing the coefficients of $\bar{n}^0$, we obtain

$$(i+1)((iP_0+2Q_0)P_{i+1}-(2Q_i+iP_{i-1}))=[P_{i+1},R_0]=0.$$

Thus $Q_i = \frac{1}{2}c^{1-i}(1-c)i + c^{-i}Q_0$ for $i \neq -1$. Letting $i = 1$, since $Q_1 = 0$, we obtain that $Q_0 = -\frac{1}{2}c(1-c)$. Hence,

$$Q_i = \tfrac{1}{2}c^{1-i}(1-c)(i-1) \text{ for } i \neq -1. \tag{2.13}$$

Applying $[(\frac{d}{dt})^3, t^{i+1}] = 3(i+1)t^i(\frac{d}{dt})^2 + 3[i+1]_2 t^{i-1}\frac{d}{dt} + [i+1]_3 t^{i-2}$ to Y_n, we obtain

$$(i+1)(([i]_2P_0 + 3iQ_0 + 3R_0)P_{i+1} - (3R_i + 3iQ_{i-1} + [i]_2P_{i-2})) = [P_{i+1}, S_0] = 0.$$

Thus,

$$R_i = \tfrac{1}{6}c^{1-i}(c-1)(5-4c+(c-2)i)i + c^{-i}R_0 \quad \text{for } i \neq -1. \tag{2.14}$$

Using $[(\frac{d}{dt})^2, t^{i+1}\frac{d}{dt}] = 2(i+1)t^i(\frac{d}{dt})^2 + [i+1]_2 t^{i-1}\frac{d}{dt}$, we obtain

$$([i]_2P_0 + 2iQ_0)Q_i + 2P_{i+1}R_0 - (2(i+1)R_i + [i+1]_2Q_{i-1}) = [Q_{i+1}, R_0] = 0. \tag{2.15}$$

Using (2.12)–(2.14) in (2.15), by comparing the coefficients of i^3, i^2, i, we obtain $c = 1, R_0 = 0$. Then (2.12)–(2.15) show that $P_i = 1, Q_i = R_i = 0$.

Thus (2.7) shows that $(t^i(\frac{d}{dt})^j)Y_n = Y_{n+i-j}[n+G]_j$ for $j \leqslant 2$. Equivalently, $(t^iD^j)Y_n = Y_{n+i}(n+G)^j$ for $j \leqslant 2$. Since $\mathscr{W}$ is generated by $\{t^iD^j \mid i \in \mathbb{Z}, 0 \leqslant j \leqslant 2\}$, we obtain that V is the module $A_{p,G}$ defined in (1.5) (if we have used the isomorphism in (1.6) in the above proof, then V is the module $\bar{A}_{p,G}$). $\square$

Proof of Theorem 1.1(i). Assume that V is a nontrivial indecomposable uniformly bounded module over $\mathscr{W}$. Then V has at least a nontrivial composition factor, so Proposition 2.2 means that $L_{0,0}$ must acts as a nonzero scalar on V. Thus, V cannot contain a trivial composition factor. Thus, Theorem 1.1(i) follows from Propositions 2.1 and 2.2. $\square$

3. Proof of Theorem 1.2(ii)

In this section, we set $\mathscr{W} = \mathscr{W}(\Gamma, n)$, where Γ is not isomorphic to $\mathbb{Z}$. Denote $Witt = \text{span}\{t^\alpha D_i \mid \alpha \in \Gamma, i = 1, \ldots, n\}$. Then $Witt$ is a Witt algebra of rank n. In particular, if $n = 1$, $Witt$ is a (generalized centerless) Virasoro algebra.

Denote $\mathscr{D} = \text{span}\{D_i \mid i = 1, \ldots, n\}$. We can define an inner product on $\Gamma \times \mathscr{D}$ by

$$\langle \alpha, d \rangle = \sum_{i=1}^{n} \alpha_i d_i \quad \text{for } \alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma, \quad d = \sum_{i=1}^{n} d_i D_i \in \mathscr{D}. \tag{3.1}$$

Then $\langle \cdot, \cdot \rangle$ is *nondegenerate* in the sense that if $\langle \alpha, \mathscr{D} \rangle = 0$ for some $\alpha \in \Gamma$ then $\alpha = 0$ and if $\langle \Gamma, d \rangle = 0$ for some $d \in \mathscr{D}$ then $d = 0$.

Proposition 3.1. *Any quasifinite $\mathscr{W}$-module V' with a finite number of the trivial composition factor is a uniformly bounded module.*

Proof. Let V be a nontrivial composition factor of V'. Then V is a weight module over $\mathscr{W}$ (i.e., D_i act diagonalizable on V for $i = 1, \ldots, n$). Regarding V as a module over *Witt*, then V is a quasifinite weight module over *Witt*. If there exists some group embedding $\mathbb{Z} \times \mathbb{Z} \to \Gamma$, then by Su [18], V is uniformly bounded. If there does not exist a group embedding $\mathbb{Z} \times \mathbb{Z} \to \Gamma$, then $n = 1$ and Γ is a rank one group with infinite generators (since $\Gamma \not\cong \mathbb{Z}$). By choosing a total ordering on Γ compatible with its group structure one can prove (as in the proof of Proposition 2.1 or using similar arguments as in [17] since in this case the group Γ behaves just like the additive group $\mathbb{Q}$ of the rational numbers) that V is uniformly bounded. Hence in any case, V is uniformly bounded. Thus, V' has only a finite number of nontrivial composition factors and so it is uniformly bounded. $\square$

For any pair (α, d) with $\alpha \in \Gamma,\ d \in \mathscr{D}$ such that $\langle \alpha, d \rangle \neq 0$, we have a Lie algebra of Weyl type, denoted by $\mathscr{W}(\alpha, d)$, spanned by $\{t^{i\alpha} d^j \mid i \in \mathbb{Z}, j \in \mathbb{Z}_+\}$, which is isomorphic to $\mathscr{W}(\mathbb{Z}, 1)$.

Proposition 3.2. *Let V be a uniformly bounded $\mathscr{W}$-module without the trivial composition factor. Then $t^\alpha \cdot v \neq 0$ for all $\alpha \in \Gamma \setminus \{0\},\ v \in V \setminus \{0\}$.*

Proof. Suppose $t^\beta \cdot v_0 = 0$ for some $\beta \neq 0, v_0 \neq 0$. By shifting the grading index if necessary, we can suppose v_0 has degree 0. Let $V_0' = \{v \in V_0 \mid t^\beta \cdot v = 0\}$. Then V_0' is invariant under the action of $\mathscr{D}$ since $[\mathscr{D}, t^\beta] \subset \mathbb{F} t^\beta$. Thus we can find a common eigenvector (i.e., a weight vector), denoted again by v_0, of $\mathscr{D}$ in V_0'. Thus $t^\beta \cdot v_0 = 0$ and $\mathscr{D} \cdot v_0 \subset \mathbb{F} v_0$. For any $d' \in \mathscr{D}$ with $\langle \beta, d' \rangle \neq 0$, considering the $\mathscr{W}(\beta, d')$-submodule V'' of V generated by v_0, by Theorem 1.1(i), and by (1.5) and (1.7), V'' must be the trivial submodule $\mathbb{F} v_0$. In particular, $d' \cdot v_0 = 0$ for all d' with $\langle \beta, d' \rangle \neq 0$. Since such d' span $\mathscr{D}$, we obtain $\mathscr{D} \cdot v_0 = 0$. If $t^\alpha \cdot v_0 = 0$ for all $\alpha \in \Gamma \setminus \{0\}$, then by Theorem 1.1(i), $\mathbb{F} v_0$ is a trivial submodule over $\mathscr{W}(\alpha, d)$ for all pairs (α, d) with $\langle \alpha, d \rangle \neq 0$, and so it must be a trivial submodule over $\mathscr{W}$. Thus $t^\gamma \cdot v_0 \neq 0$ for some $\gamma \in \Gamma \setminus \{0\}$. Choose $d \in \mathscr{D}$ such that $\langle \beta, d \rangle \neq 0 \neq \langle \gamma, d \rangle$. Then by Theorem 1.1(i) and (1.5), v_0 generates a submodule of the intermediate series over $\mathscr{W}(\gamma, d)$ such that $(t^{i\gamma} d^j) \cdot v_0 = 0$ for $j \neq 0$ and all $i \in \mathbb{Z}$ (note that the coefficient of the right-hand side of (1.5) does not depend on α). So $t^{i\beta + j\gamma} \cdot v_0 = \langle i\beta, d \rangle^{-1} [t^{j\gamma} d, t^{i\beta}] \cdot v_0 = 0$ if $i \neq 0$. Similarly, $(t^{i\beta + j\gamma} d) \cdot v_0 = 0$ if $i \neq 0$. Choose some i, j with $i \neq 0$ such that $a = \langle i\beta + j\gamma, d \rangle \neq 0$, then $t^\gamma \cdot v_0 = a^{-1} [t^{-i\beta + (1-j)\gamma} d, t^{i\beta + j\gamma}] \cdot v_0 = 0$, a contradiction. $\square$

Proof of Theorem 1.1(ii). Suppose V is an indecomposable uniformly bounded module without the trivial composition factor. Choose a basis $Y_0 = (y_0^{(1)}, \ldots, y_0^{(p)})$ of V_0 and by Proposition 3.2, we can define basis $Y_\alpha = t^a \cdot Y_0$ of V_α for all $\alpha \in \Gamma \setminus \{0\}$.

For any $\beta \in \Gamma \setminus \{0\}$ with $\langle \beta, D_i \rangle \neq 0$ for $i = 1, \ldots, n$, set $V[\beta] = \oplus_{k \in \mathbb{Z}} V_{k\beta}$, then $V[\beta]$ is a submodule of V over $\mathscr{W}(\beta, D_i)$ of type A_{p,G_i} for some $G_i \in M_{p \times p}$, $i = 1, \ldots, n$. Since $[\mathscr{D}, \mathscr{D}] = 0$, we have $G_i G_j = G_j G_i$ for $i,j = 1, \ldots, n$. Since $\{\beta \in \Gamma \mid \langle \beta, D_i \rangle \neq 0,\ i = 1, \ldots, n\}$ generates Γ, it is straightforward to see that V is a $\mathscr{W}$-module of type $A_{p,G}$ or $\bar{A}_{p,G}$ with $G = (G_1, \ldots, G_n) \in M^n_{p \times p}(\mathbb{F})$. If V contains a trivial composition factor, then V must be a trivial module as in the proof of Theorem 1.1(i). Thus we obtain Theorem 1.1(ii). $\square$

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