JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 135, 591–610 (1988)

On the Asymptotic Equivalence of Systems with Impulse Effect

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Submitted by V. Lakshmikantham

Received February 10, 1987

In the present paper sufficient conditions for asymptotic equivalence and restricted asymptotic equivalence of a linear system of differential equations with impulse effect and its respective perturbed non-linear system are found. © 1988 Academic Press, Inc.

1. INTRODUCTION

The papers of many authors [1–10] are devoted to the finding of conditions for asymptotic equivalence of systems of ordinary differential equations. The present paper considers the problem of the asymptotic equivalence of the systems with impulse effect

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(t, x), && t \neq \tau_k, \\ \Delta x \,|_{t=\tau_k} &= Bx(\tau_k) + b(x(\tau_k)), && k = 1, 2, ..., \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{dy}{dt} &= Ay, && t \neq \tau_k, \\ \Delta y \,|_{t=\tau_k} &= By(\tau_k), && k = 1, 2, ..., \end{aligned} \tag{2}$$

where $x, y: I \to R^n$; $f: I \times R^n \to R^n$; $b_k: R^n \to R^n$; $I = [0, \infty)$; R^n is the n-dimensional Euclidean space with a norm $|\cdot|$; A and B are constant $n \times n$ matrices; the moments $\{\tau_k\}$ form an increasing sequence $0 \equiv \tau_0 < \tau_1 < \cdots < \tau_k < \cdots$, $\lim_{k \to \infty} \tau_k = \infty$.

For the systems with impulse effect of the kind (1) it is characteristic that at the moments $t = \tau_k$ the mapping point (t, x) undergoing a short period effect (a hit, an impulse) moves from position $(\tau_k, x(\tau_k))$ to position $(\tau_k, x(\tau_k) + \Delta x(\tau_k))$ simultaneously.

0022-247X/88 $3.00

We assume that at the moments of impulse effect $t=\tau_k$ the solutions of system (1) (or (2)) are left continuous, i.e., $x(\tau_k-0)=x(\tau_k)$, $\Delta x(\tau_k)=x(\tau_k+0)-x(\tau_k)$.

We shall use the following definitions for asymptotic equivalence.

DEFINITION 1. We say that the systems with impulse effect (1) and (2) are asymptotically equivalent, if to every solution $x(t)$ of (1) there corresponds a solution $y(t)$ of (2) such that

$$\lim_{t\to\infty} |x(t)-y(t)|=0 \tag{3}$$

and conversely, to every solution $y(t)$ of (2) there corresponds a solution $x(t)$ of (1) such that (3) holds.

DEFINITION 2. We shall say that between the systems with impulse effect (1) and (2) there exists a restricted asymptotic equivalence if relation (3) is fulfilled only between some subsets of solutions of (1) and (2).

We shall note that the first results, connected with establishing asymptotic equivalence of systems of differential equations with impulse effect, are obtained in [11].

2. PRELIMINARY NOTES

Let $t_0\in I$, $x_0\in R^n$. We denote with $x(t; t_0, x_0)$ the solution of system (1), for which $x(t_0+0; t_0, x_0)=x_0$, and with $\mathscr{T}^+(t_0, x_0)$ the maximum interval of the kind (t_0, ω) on which this solution is defined.

Further on we shall use the following notations: $i(s, t)$ is the number of the points τ_k lying in the interval (s, t); $\|A\|=\sup_{|x|=1}|Ax|$ is the norm of the $n\times n$ matrix A; E is the unit $n\times n$ matrix; 0_m is the zero $m\times m$ matrix; and $\operatorname{diag}(A_1, A_2)$ is the quasi-diagonal $n\times n$ matrix with blocks A_1 and A_2.

We say that the conditions (A) hold if the following conditions are fulfilled:

(A1) The function $f(t, x)$ is continuous and locally Lipschitzian in x on $I\times R^n$.

(A2) The functions $b_k(x)$, $k=1, 2, \ldots$, are continuous in R^n.

(A3) The matrices A and B commute and $\det(E+B)\neq 0$.

(A4) There exist constants $Q>0$ and $p\geqslant 0$ such that

$$|i(t_0, t)-p(t-t_0)|\leqslant Q \qquad \text{for} \quad 0\leqslant t_0\leqslant t<\infty.$$

(A5) For $(t, x) \in I \times R^n$ and $k = 1, 2, \ldots$, the inequalities

$$|f(t, x)| \leqslant H(t, |x|), \qquad |b_k(x)| \leqslant \beta_k(|x|)$$

hold, where the functions $H(t, u)$ and $\beta_k(u)$, $k = 1, 2, \ldots$, are continuous and non-negative for $t \geqslant 0$, $u \geqslant 0$ and non-decreasing in $u \geqslant 0$.

For proving the main results we shall use the following lemmas:

LEMMA 1 [5]. *Let the function $g(t)$ be non-negative and continuous in I and such that either $\int_0^\infty g(t)\,dt < \infty$ or $\lim_{t \to \infty} g(t) = 0$. Then for every $\sigma > 0$*

$$\lim_{t \to \infty} \int_0^t e^{-\sigma(t-s)} g(s)\,ds = 0.$$

LEMMA 2. *Let $q \in [0, 1)$, $\gamma_k \geqslant 0$, and $\lim_{k \to \infty} \gamma_k = 0$. Then*

$$\lim_{k \to \infty} \sum_{i=1}^{k} q^{k-i} \gamma_i = 0.$$

Proof. Let $\omega = \sup_k \gamma_k$ and the number $\varepsilon > 0$ be given. We choose the number N_1 such that for $k > N_1$ the inequality $\gamma_k < \varepsilon(1-q)/2$ is fulfilled and N_2 is such that $2\omega q^{N_2} < \varepsilon(1-q)$. Then for $k > N_1 + N_2$ we have

$$\begin{aligned} 0 \leqslant \sum_{i=1}^{k} q^{k-i}\gamma_i &= \sum_{i=1}^{N_1} q^{k-i}\gamma_i + \sum_{i=N_1+1}^{k} q^{k-i}\gamma_i \\ &\leqslant \omega q^{k-N_1} \frac{1-q^{N_1}}{1-q} + \frac{\varepsilon(1-q)}{2} \cdot \frac{1-q^{k-N_1}}{1-q} \\ &\leqslant \frac{\omega q^{N_2}}{1-q} q^{k-N_1-N_2} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

LEMMA 3. *Let $\sigma > 0$ and the sequences $\{\gamma_k\}$ and $\{\tau_k\}$ be such that $\gamma_k \geqslant 0$, $\lim_{k \to \infty} \gamma_k = 0$, $\tau_k - \tau_{k-1} \geqslant \theta > 0$, $\tau_1 > 0$. Then*

$$\lim_{t \to \infty} \sum_{0 < \tau_k < t} e^{-\sigma(t-\tau_k)} \gamma_k = 0. \tag{4}$$

Proof. Let $\tau_n < t \leqslant t_{n+1}$. Then $\tau_n - \tau_i \geqslant (n-i)\,\theta$ and

$$0 \leqslant \sum_{0 < \tau_i < t} e^{-\sigma(t-\tau_i)} \gamma_i \leqslant \sum_{i=1}^{n} e^{-\sigma(\tau_n - \tau_i)} \gamma_i \leqslant \sum_{i=1}^{n} e^{-\sigma\theta(n-i)} \gamma_i.$$

We apply Lemma 2 with $q = e^{-\sigma\theta}$ and get (4).

3. Main Results

First we shall investigate the question of the asymptotic equivalence of system (2) and the system

$$\begin{aligned} \frac{dz}{dt} &= Az + g(t), \qquad && t \neq \tau_k, \\ \Delta z \,|_{t=\tau_k} &= Bz(\tau_k) + b_k, \qquad && k = 1, 2, ..., \end{aligned} \tag{5}$$

where $b_k \in R^n$, $k = 1, 2, ...$, and the function $g: I \to R^n$ is continuous for $t \in I$, $t \neq \tau_k$, at the points $t = \tau_k$. $g(t)$ may have a first kind discontinuity and is left continuous.

Since the general solution Z of the linear non-homogeneous system (5) is in the form

$$Z = Y + \eta,$$

where Y is the general solution of the linear homogeneous system (2) and η is a partial solution of (5), then the following theorem holds.

THEOREM 1. *Let* $\det(E + B) \neq 0$. *Then the systems* (2) *and* (5) *are asymptotically equivalent if and only if system* (5) *has at least one solution* $z_0(t)$ *such that*

$$\lim_{t \to \infty} z_0(t) = 0.$$

Before stating the following results we shall do some preliminary reasoning and introduce new notations and assume everywhere that the conditions (A3) and (A4) are fulfilled.

Let $t_0 \in I$, $y_0 \in R^n$. Since the matrices A and B commute, then the solution $y(t; t_0, y_0)$ of system (2) is in the form

$$y(t; t_0, y_0) = (E + B)^{i(t_0, t)} e^{A(t - t_0)} y_0, \qquad t > t_0 \geqslant 0,$$

i.e., the Cauchy matrix $Y(t, t_0)$ of system (2) is

$$Y(t, t_0) = (E + B)^{i(t_0, t)} e^{A(t - t_0)} = (E + B)^{i(t_0, t) - p(t - t_0)} e^{\Lambda(t - t_0)},$$

where $\Lambda = A + p \ln(E + B)$.

The matrix Λ can be represented in the form

$$\Lambda = S^{-1} \operatorname{diag}(\Lambda_-, \Lambda_0)\, S,$$

where Λ_- is a Jordan $q \times q$ matrix with eigenvalues $\lambda_i(\Lambda_-)$, which have negative real parts, $\operatorname{Re} \lambda_i(\Lambda_-) \leqslant \max_{1 \leqslant i \leqslant q} \operatorname{Re} \lambda_i(\Lambda_-) = -\alpha < 0$, and Λ_0 is a Jordan $r \times r$ matrix with eigenvalues $\lambda_i(\Lambda_0)$, for which $0 \leqslant \lambda_i(\Lambda_0) \leqslant \max_{q+1 \leqslant i \leqslant n} \lambda_i(\Lambda_0) = \lambda$; $q + r = n$; $\det S \neq 0$.

If the matrix Λ has eigenvalues with a real part equal to μ then we denote with $d(\mu)$ the maximum order of these blocks of the matrix $\operatorname{diag}(\Lambda_-, \Lambda_0)$ which correspond to the eigenvalues with real parts μ.

We assume $m = d(\lambda)$, $w = d(-\alpha)$, $v = d(0)$ if Λ_0 has eigenvalues with real parts, equal to zero or $v = 1$, in the opposite case.

We introduce the matrix functions

$$G(t, s) = \begin{cases} (E+B)^{i(s,t) - p(t-s)} S^{-1} \operatorname{diag}(e^{\Lambda_-(t-s)}, e^{\Lambda_0(t-s)}) S, & t > s, \\ (E+B)^{-i(t,s) + p(s-t)} S^{-1} \operatorname{diag}(e^{\Lambda_-(t-s)}, e^{\Lambda_0(t-s)}) S, & t \leqslant s, \end{cases}$$

$$G_-(t, s) = \begin{cases} (E+B)^{i(s,t) - p(t-s)} S^{-1} \operatorname{diag}(e^{\Lambda_-(t-s}, O_r) S, & t > s, \\ (E+B)^{-i(t,s) + p(s-t)} S^{-1} \operatorname{diag}(e^{\Lambda_-(t-s)}, O_r) S, & t \leqslant s, \end{cases}$$

$$G_0(t, s) = \begin{cases} (E+B)^{i(s,t) - p(t-s)} S^{-1} \operatorname{diag}(O_q, e^{\Lambda_0(t-s)}) S, & t > s, \\ (E+B)^{-i(t,s) + p(s-t)} S^{-1} \operatorname{diag}(O_q, e^{\Lambda_0(t-s)}) S, & t \leqslant s. \end{cases}$$

The immediate check shows that

$$G(t, s) = G_-(t, s) + G_0(t, s), \tag{6}$$

$$G(t, t) = E, \qquad t \in I, \tag{7}$$

$$G(\tau_k + 0, \tau_k) = E, \qquad k = 1, 2, ..., \tag{8}$$

$$G(\tau_k + 0, s) = (E + B)\, G(\tau_k, s), \qquad s < \tau_k, \tag{9}$$

$$\frac{\partial U}{\partial t} = AU, \qquad t \neq \tau_k, \tag{10}$$

where U is one of the matrices G, G_-, or G_0.

$$G_0(t, s) = G(t, t_0)\, G_0(t_0, s), \qquad t > t_0, \quad s > t_0, \quad t \neq \tau_k, \quad s \neq \tau_k, \tag{11}$$

$$G_0(t, \tau_k) = G(t, t_0)\, F(t_0, \tau_k), \qquad t_0 \in I, \quad k = 1, 2, ..., \tag{12}$$

where $F(t_0, \tau_k) = (E + B)^{\omega} G_0(t_0, \tau_k)$ and the number ω is equal to 1, -1, or 0 depending on the mutual position of t_0, t, and τ_k. Then from (7)–(10) it follows that for the solutions $y(t; t_0, y_0)$, $z(t; t_0, z_0)$, and $x(t) = x(t; t_0, x_0)$ of systems (2), (5), and (1) for $t > t_0$ the relations

$$y(t; t_0, y_0) = G(t, t_0)\, y_0, \tag{13}$$

$$z(t; t_0, z_0) = G(t, t_0)\, z_0 + \int_{t_0}^{t} G(t, s)\, g(s)\, ds + \sum_{t_0 < \tau_k < t} G(t, \tau_k)\, b_k, \tag{14}$$

$$x(t) = G(t, t_0)x_0 + \int_{t_0}^{t} G(t, s) f(s, x(s))\, ds + \sum_{t_0 < \tau_k < t} G(t, \tau_k) b_k(x(\tau_k))$$

hold.

Having in mind the structure of the matrix A and condition (A4) we obtain that for the matrices G, G_-, G_0, and F the estimates of the kind

$$\|G_-(t, s)\| \leqslant a e^{-\lambda(t-s)} \chi_w(t-s), \qquad 0 \leqslant s \leqslant t < \infty, \tag{15}$$

$$\|G_0(t, s)\| \leqslant a \chi_v(s-t), \qquad 0 \leqslant t \leqslant s < \infty, \tag{16}$$

$$\|F(t_0, \tau_k)\| \leqslant a \chi_v(\tau_k - t_0), \qquad 0 \leqslant t_0 \leqslant \tau_k, \tag{17}$$

$$\|G(t, s)\| \leqslant c_0 e^{\lambda(t-s)} \chi_m(t-s), \qquad 0 \leqslant s \leqslant t < \infty, \tag{18}$$

hold, where $a \geqslant 0$ and $c_0 \geqslant 0$ are constants and

$$\chi_k(t) = \begin{cases} t^{k-1} & \text{for } t \geqslant 1 \\ 1 & \text{for } 0 \leqslant t \leqslant 1. \end{cases}$$

THEOREM 2. *Let the conditions* (A3) *and* (A4) *hold and*

$$\int_0^\infty t^{v-1}\, |g(t)|\, dt + \sum_{k=1}^{\infty} \tau_k^{v-1}\, |b_k| < \infty. \tag{19}$$

Then system (5) *has at least one solution* $z_0(t)$ *for which* $\lim_{t \to \infty} z_0(t) = 0$.

Proof. Let $z(t) = z(t; 0, z_0)$ be a solution of (5). From the estimates (16) and (17) and condition (19) it follows that

$$\left| \int_0^\infty G_0(t, s)\, g(s)\, ds \right| + \left| \sum_{k=1}^{\infty} F(t, \tau_k)\, b_k \right| < \infty$$

for every fixed $t \in I$. Then having in mind the formulae (11), (12), (14), and (6) we can write

$$\begin{aligned} z(t) = {} & G(t, 0) \left[z_0 + \int_0^\infty G_0(0, s)\, g(s)\, ds + \sum_{k=1}^{\infty} F(0, \tau_k)\, b_k \right] \\ & + \int_0^t G_-(t, s)\, g(s)\, ds + \sum_{\tau_k < t} G_-(t, \tau_k)\, b_k \\ & - \int_t^\infty G_0(t, s)\, g(s)\, ds + \sum_{t \leqslant \tau_k} G_0(t, \tau_k)\, b_k. \end{aligned}$$

We choose $z_0 = -\int_0^\infty G_0(0, s)\, g(s)\, ds - \sum_{k=1}^\infty F(0, \tau_k) b_k$. Then the solution $z_0(t)$ has the form

$$z_0(t) = \int_0^t G_-(t, s)\, g(s)\, ds + \sum_{0<\tau_k<t} G_-(t, \tau_k) b_k$$
$$- \int_t^\infty G_0(t, s)\, g(s)\, ds - \sum_{t \leqslant \tau_k} G_0(t, \tau_k) b_k.$$

Let $\sup_{u \geqslant 0} e^{-(\alpha u/2)} \chi_w(u) = M$. Then from estimate (15), condition (19), Lemma 1, and Lemma 3 it follows that

$$\begin{aligned}
\mathscr{T}_1 &= \left| \int_0^t G_-(t, s)\, g(s)\, ds + \sum_{0<\tau_k<t} G_-(t, \tau_k)\, b_k \right| \\
&\leqslant \int_0^t a e^{-\alpha(t-s)} \chi_w(t-s)\, |g(s)|\, ds + \sum_{0<\tau_k<t} a e^{-\alpha(t-\tau_k)} \chi_w(t-\tau_k)\, |b_k| \\
&\leqslant \int_0^t a e^{-(\alpha/2)(t-s)} \chi_w(t-s)\, e^{-(\alpha/2)(t-s)} |g(s)|\, ds \\
&\quad + \sum_{0<\tau_k<t} a e^{-(\alpha/2)(t-\tau_k)} \chi_w(t-\tau_k)\, e^{-(\alpha/2)(t-\tau_k)}\, |b_k| \\
&\leqslant \left(aM \int_0^t e^{-(\alpha/2)(t-s)}\, |g(s)|\, ds + aM \sum_{0<\tau_k<t} e^{-(\alpha/2)(t-\tau_k)}\, |b_k| \right) \to 0, \\
&\qquad \text{as} \quad t \to \infty.
\end{aligned}$$

From the estimate (16) and condition (19) it follows that for $t \geqslant 1$,

$$\begin{aligned}
\mathscr{T}_2 &= \left| \int_t^\infty G_0(t, s)\, g(s)\, ds + \sum_{t \leqslant \tau_k} G_0(t, \tau_k) b_k \right| \\
&\leqslant a \int_t^\infty \chi_v(s-t)\, |g(s)|\, ds + a \sum_{t \leqslant \tau_k} \chi_v(\tau_k - t)\, |b_k| \\
&\leqslant a \int_t^{t+1} |g(s)|\, ds + a \sum_{t \leqslant \tau_k < t+1} |b_k| \\
&\quad + a \int_{t+1}^\infty s^{v-1}\, |g(s)|\, ds + a \sum_{t+1 \leqslant \tau_k} \tau_k^{v-1}\, |b_k| \\
&\leqslant 2a \left(\int_t^\infty s^{v-1}\, |g(s)|\, ds + \sum_{t \leqslant \tau_k} \tau_k^{v-1}\, |b_k| \right) \to 0, \\
&\qquad \text{as} \quad t \to \infty.
\end{aligned}$$

Therefore $\lim_{t \to \infty} z_0(t) = 0$. Thus Theorem 2 is proved.

THEOREM 3. *Let the conditions* (A) *hold and*

$$\int_0^\infty t^{\nu-1} H(t, c)\, dt + \sum_{k=1}^\infty \tau_k^{\nu-1} \beta_k(c) < \infty \tag{20}$$

for every $c \geq 0$.

Let system (1) *have solutions which are defined and bounded on intervals of the kind* $[t_0, \infty)$.

Then for every such solution $x(t)$ *there exists a solution* $y(t)$ *of system* (2) *so that* (3) *is fulfilled.*

Proof. Let $x(t)$ be a bounded solution of system (1) defined on $[t_0, \infty) \subset I$ and $c = \sup_{t > t_0} |x(t)|$.

If $y(t)$ is an arbitrary solution (2) then $z(t) = x(t) - y(t)$ is a solution of the system

$$\begin{aligned} \frac{dz}{dt} &= Az + f(t, x(t)), \qquad & t \neq \tau_k, \\ \Delta z\,|_{t=\tau_k} &= Bz(\tau_k) + b_k(x(\tau_k)), \qquad & k = 1, 2, ..., \end{aligned} \tag{21}$$

and if $z(t)$ is a solution of (21) then $y(t) = x(t) - z(t)$ is a solution of (2).

Using (20) and condition (A5), we get

$$\begin{aligned} &\int_{t_0}^\infty t^{\nu-1} |f(t, x(t))|\, dt + \sum_{t_0 \leq \tau_k} \tau_k^{\nu-1} |b_k(x(\tau_k))| \\ &\leq \int_{t_0}^\infty t^{\nu-1} H(t, c)\, dt + \sum_{t_0 \leq \tau_k} \tau_k^{\nu-1} \beta_k(c) < \infty. \end{aligned}$$

Therefore, according to Theorem 2, system (21) has a solution $z_0(t)$ such that $\lim_{t\to\infty} z_0(t) = 0$. Then for the solution $y_0(t) = x(t) - z_0(t)$ of system (2) we have

$$\lim_{t\to\infty} |x(t) - y_0(t)| = 0.$$

THEOREM 4. *Let the following conditions be fulfilled*:

(1) *The conditions* (A) *hold.*

(2) $x(t)$ *is a solution of system* (1) *such that*

$$|x(t)| \leq c e^{\mu t} t^h, \qquad t \geq t_0 \geq 0,$$

where $c \geq 0$, $\mu \geq 0$, *and* $h \geq 0$ *are constants.*

(3) $$\int_0^\infty t^{\nu-1} H(t, c e^{\mu t} t^h)\, dt + \sum_{k=1}^\infty \tau_k^{\nu-1} \beta_k(c e^{\mu \tau_k} \tau_k^h) < \infty.$$

Then there exists a solution $y(t)$ of system (2) *for which* $\lim_{t\to\infty} |x(t)-y(t)|=0$.

The proof of Theorem 4 is analogous to the proof of Theorem 3.

If the conditions (A3) and (A4) are fulfilled, then from (13) and (18) it follows that for the solution $y(t)$ of system (2) an estimate of the kind

$$|y(t)| \leqslant c_0\, |y(0)|\, e^{\lambda t}\chi_m(t), \qquad t \geqslant 0, \tag{22}$$

holds.

If the systems (1) and (2) are asymptotically equivalent then for the solutions $x(t)=x(t;t_0,x_0)$ of system (1), estimates of the kind

$$|x(t)| \leqslant De^{\lambda(t-t_0)}\chi_m(t-t_0), \qquad t>t_0, \tag{23}$$

hold, where $D=D(t_0,x_0)$.

The following Theorem gives sufficient conditions under which for the solutions of system (1) the estimate (23) holds.

THEOREM 5. *Let the following conditions be fulfilled:*

(1) *The conditions* (A) *hold.*

(2) *For every* $c \geqslant 0$

$$\int_0^\infty e^{-\lambda t}H(t, ce^{\lambda t}\chi_m(t))\,dt + \sum_{k=1}^{\infty} e^{-\lambda\tau_k}\beta_k(ce^{\lambda\tau_k}\chi_m(\tau_k)) < \infty. \tag{24}$$

(3) *There exists* $t_0 \geqslant 0$ *such that*

$$\sup_{c\geqslant 1}\left[\frac{1}{c}\int_{t_0}^\infty e^{-\lambda t}H(t, ce^{\lambda t}\chi_m(t))\,dt \right.$$
$$\left. + \frac{1}{c}\sum_{t_0<\tau_k} e^{-\lambda\tau_k}\beta_k(ce^{\lambda\tau_k}\chi_m(\tau_k))\right] = S < \frac{1}{c_0}, \tag{25}$$

where c_0 is from (22). *Then every solution $x(t)=x(t;t_0,x_0)$ of system* (1) *is defined for $t>t_0$ and the estimate* (23) *holds.*

Proof. Let $x(t)=x(t;t_0,x_0)$ be a solution of (1). Then for $t\in\mathscr{T}^+(t_0,x_0)$,

$$x(t) = G(t,t_0)\,x_0 + \int_{t_0}^t G(t,s)f(s,x(s))\,ds$$
$$+ \sum_{t_0<\tau_k<t} G(t,\tau_k)b_k(x(\tau_k)). \tag{26}$$

First we assume that $\mathscr{T}^+(t_0, x_0) = (t_0, \infty)$. Then from (26), (18), and condition (A5) it follows that

$$|x(t)| \leqslant c_0 e^{\lambda(t-t_0)} \chi_m(t-t_0)\, |x_0|$$
$$+ \int_{t_0}^{t} c_0 e^{\lambda(t-s)} \chi_m(t-s)\, H(s, |x(s)|)\, ds$$
$$\sum_{t_0 < \tau_k < t} c_0 e^{\lambda(t-\tau_k)} \chi_m(t-\tau_k)\, \beta_k(|x(\tau_k)|).$$

Since $t_0 < s \leqslant t$, then $t-s < t-t_0$, $\chi_m(t-s) \leqslant \chi_m(t-t_0)$, and

$$|x(t)| \leqslant c_0 e^{\lambda t} \chi_m(t-t_0) \left[|x_0|\, e^{-\lambda t_0} + \int_{t_0}^{t} e^{-\lambda s} H(s, |x(s)|)\, ds \right.$$
$$\left. + \sum_{t_0 < \tau_k < t} e^{-\lambda \tau_k} \beta_k(|x(\tau_k)|) \right].$$

We set

$$A_T = |x_0|\, e^{-\lambda t_0} + \int_{t_0}^{T} e^{-\lambda s} H(s, |x(s)|)\, ds$$
$$+ \sum_{t_0 < \tau_k < T} e^{-\lambda \tau_k} \beta_k(|x(\tau_k)|), \qquad T > t_0. \tag{27}$$

Then

$$|x(t)| \leqslant c_0 e^{\lambda t} \chi_m(t-t_0) \qquad \text{for} \quad t_0 < t \leqslant T. \tag{28}$$

We shall consider two cases:

(a) $A_T < 1/c_0$ for $T > t_0$. Then from (28) it follows that

$$|x(t)| \leqslant e^{\lambda t_0} e^{\lambda(t-t_0)} \chi_m(t-t_0) \qquad \text{for} \quad t > t_0.$$

(b) There exists $T_0 > t_0$ such that $c_0 A_{\tau_0} \geqslant 1$. Then $c_0 A_T \geqslant 1$ for $T \geqslant T_0$ since A_T is non-decreasing in T and from (25) it follows that

$$\int_{t_0}^{\infty} e^{-\lambda s} H(s, c_0 A_T e^{\lambda s} \chi_m(s))\, ds$$
$$+ \sum_{t_0 < \tau_k} e^{-\lambda \tau_k} \beta_k(c_0 A_T e^{\lambda \tau_k} \chi_m(\tau_k)) \leqslant S c_0 A_T. \tag{29}$$

From (27)–(29) and the monotonicity of $H(t, u)$ and $\beta_k(u)$ in u we obtain

$$\begin{aligned} A_T &\leqslant |x_0|\, e^{-\lambda t_0} + \int_{t_0}^{T} e^{-\lambda s} H(s, c_0 A_T e^{\lambda s} \chi_m(s - t_0))\, ds \\ &\quad + \sum_{t_0 < \tau_k < T} e^{-\lambda \tau_k} \beta_k(c_0 A_T e^{\lambda \tau_k} \chi_m(\tau_k - t_0)) \\ &\leqslant |x_0|\, e^{-\lambda t_0} + \int_{t_0}^{\infty} e^{-\lambda s} H(s, c_0 A_T e^{\lambda s} \chi_m(s))\, ds \\ &\quad + \sum_{t_0 < \tau_k} e^{-\lambda \tau_k} \beta_k(c_0 A_T e^{\lambda \tau_k} \chi_m(\tau_k)) \\ &\leqslant |x_0|\, e^{-\lambda t_0} + S c_0 A_T \end{aligned}$$

or

$$A_T \leqslant \frac{|x_0|\, e^{-\lambda t_0}}{1 - S c_0}.$$

We substitute in (28) and get

$$|x(t)| \leqslant \frac{c_0\, |x_0|}{1 - S c_0}\, e^{\lambda(t - t_0)} \chi_m(t - t_0), \qquad t_0 < t \leqslant T. \tag{30}$$

But the right-hand side of the inequality (30) does not depend on T. Therefore (30) is fulfilled for every $t > t_0$ and an estimate (23) holds in case (b).

Now let us assume that $\mathscr{T}^+(t_0, x_0) = (t_0, \omega)$, $\omega < \infty$. Then $|x(t)|$ should be unbounded for $t \in (t_0, \omega)$. By the same reasoning as above for $T \in (t_0, \omega)$, we obtain that for $|x(t)|$ that the estimate (23) holds for $t \in (t_0, \omega)$. But this contradicts the unboundedness of $|x(t)|$. Therefore, $\mathscr{T}^+(t_0, x_0) = (t_0, \infty)$ and (23) holds for $t > t_0$.

Theorem 5 is proved.

Remark 1. Condition 3 of Theorem 5 can be replaced by the condition

$$\lim_{t_0 \to \infty} \frac{1}{c} \left[\int_{t_0}^{\infty} e^{-\lambda t} H(t, c e^{\lambda t} \chi_m(t))\, dt + \sum_{t_0 < \tau_k} e^{-\lambda \tau_k} \beta_k(c e^{\lambda \tau_k} \chi_m(\tau_k)) \right] = 0 \tag{31}$$

uniformly in $c \geqslant 1$.

Before formulating the following theorems we shall make some preliminary investigations. Let $D \subset I$ and the function $\rho\colon D \to I$ be positive and continuous. We denote with $S(D, R^n, \rho)$ the space of functions $f\colon D \to R^n$, which satisfy the following conditions:

(1) The function $f(t)$ is continuous for $t \in D$, $t \neq \tau_k$ and

$$\sup_{t \in D} \frac{|f(t)|}{\rho(t)} < \infty. \tag{32}$$

(2) There exist the limits

$$f(\tau_k - 0) = \lim_{t \to \tau_k - 0} f(t) \qquad \text{if} \quad \tau_k \in (\alpha, \beta],$$

$$f(\tau_k + 0) = \lim_{t \to \tau_k + 0} f(t) \qquad \text{if} \quad \tau_k \in [\alpha, \beta),$$

and

$$f(\tau_k - 0) = f(\tau_k + 0) \qquad \text{if} \quad \tau_k \in (\alpha, \beta] \cap D, \quad k = 1, 2, \ldots.$$

We denote with $M(D, R^n, \rho)$ the space of functions $f\colon D \to R^n$, for which the equality of the kind (32) is fulfilled.

Clearly, $S(D, R^n, \rho) \subset M(D, R^m, \rho)$.

In the spaces $S(D, R^n, \rho)$ and $M(D, R^m, \rho)$ we introduce the norm

$$\|f\|_\rho = \sup_{t \in D} \frac{|f(t)|}{\rho(t)} \qquad (f \in M(D, R^m, \rho)).$$

If $f_n, f \in M(D, R^n, \rho)$ and $\|f_n - f\|_\rho \to 0$ as $n \to \infty$, then $\lim_{n \to \infty} f_n(t) = f(t)$ uniformly in t on every finite subinterval of D.

Then following the proof of [12, Theorem 1.4.8], it is easily proved that $S(D, R^n, \rho)$ is closed in $M(D, R^n, \rho)$ and $M(D, R^n, \rho)$ is complete and according to [12, Lemma 1.4.7], $S(D, R^n, \rho)$ is complete, i.e., $S(D, R^n, \rho)$ is a Banach space.

Let $D_1 \subset D$ and $\mathscr{F} \subset S(D, R^n, \rho)$.

Definition 3. The set $\mathscr{F}$ is quasi-equicontinuous on D_1, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $f \in \mathscr{F}$; $k = 1, 2, \ldots$; $t_1, t_2 \in (\tau_{k-1}, \tau_k] \cap D_1$ and $|t_1 - t_2| < \delta$ then

$$|f(t_1) - f(t_2)| < \varepsilon,$$

Lemma 4. *The set $\mathscr{F} \subset S((0, \tau_k], R^n, \rho)$ is relatively compact if and only if*:

(1) *$\mathscr{F}$ is bounded, i.e., $\|f\|_\rho \leqslant c$ for every $f \in \mathscr{F}$ and some $c > 0$.*

(2) *$\mathscr{F}$ is quasi-equicontinuous on $(0, \tau_k]$.*

Proof. It is clear that $\mathscr{F} = \mathscr{F}_1 \times \cdots \times \mathscr{F}_k \subset S_1 \times \cdots \times S_k = S$, where $S_i = S((\tau_{i-1}, \tau_i], R^n, \rho)$, $i = 1, 2, \ldots, k$, and $\mathscr{F}_i$, $i = 1, 2, \ldots, k$, is the set of functions $f_i\colon (\tau_{i-1}, \tau_i] \to R^m$, for which

$$f_i(t) = f(t), \qquad t \in (\tau_{i-1}, \tau_i], \quad f \in \mathscr{F}.$$

The set $\mathscr{F}$ is relatively compact in S if and only if every $\mathscr{F}_i$ is relatively compact in S_i. But according to the Arzelà–Ascoli theorem, this is fulfilled if and only if $\mathscr{F}_i$ is bounded and equicontinuous, i.e., when conditions (1) and (2) of Lemma 4 are fulfilled.

Remark 2. Lemma 4 remains true if in its formulating we replace the interval $(0, \tau_k]$ by $[0, \tau_k]$.

LEMMA 5. *Let $\mathscr{F} \subset S(I, R^n, \rho)$ and the following conditions be fulfilled:*

(1) *There exists $\varphi \in S(I, R^n, \rho)$ and a number $d > 0$ such that*

$$\|f-\varphi\|_\rho \leqslant d \qquad \text{for} \quad f \in \mathscr{F}.$$

(2) *For every $\varepsilon > 0$ there exists $T > 0$ such that $|f(t)-\varphi(t)| < \varepsilon\rho(t)$ for $t > T$ and $f \in \mathscr{F}$ and $\mathscr{F}$ is quasi-equicontinuous in $[0, T]$.*

Then $\mathscr{F}$ is relatively compact.

Proof. Without loss of generality we assume that $\varphi = 0$. In the opposite case, the proof of Lemma 5 is done for the set $\mathscr{H} = \{h \in S(I, R^n, \rho)\colon h = f - \varphi, f \in \mathscr{F}\}$.

Let $\varepsilon > 0$ be given. We choose $T = \tau_k$ such that

$$|f(t)| \leqslant \varepsilon\rho(t) \qquad \text{for} \quad t > T. \tag{33}$$

But from the conditions of Lemma 5 it follows that the set

$$\mathscr{F}_T = \{g \in S([0, T], R^n, \rho)\colon g(t) = f(t),\ t \in [0, T], f \in \mathscr{F}\}$$

satisfies the conditions of Lemma 4. Therefore, $\mathscr{F}_T$ is relatively compact in $S([0, T], R^n, \rho)$.

Then there exist $f_k \in \mathscr{F}$, $k = 1, 2, ..., l$, such that for every $f \in \mathscr{F}$ and some $k = 1, 2, ..., l$ we have

$$|f(t) - f_k(t)| \leqslant \varepsilon\rho(t) \qquad \text{for} \quad t \in [0, T]. \tag{34}$$

From (33) and (34) it follows that the functions

$$g_k(t) = \begin{cases} f_k(t), & t \in [0, T], \\ 0, & t > T, \quad k = 1, 2, ..., l, \end{cases}$$

are such that for every $f \in \mathscr{F}$ there exists $k = 1, 2, ..., l$ such that

$$|f(t) - g_k(t)| \leqslant \varepsilon\rho(t) \qquad \text{for} \quad t \in I.$$

Therefore, $g_1, ..., g_l$ form a finite ε-net and $\mathscr{F}$ is relatively compact in $S(I, R^n, \rho)$.

THEOREM 6. *Let the following conditions be fulfilled*:

(1) *The conditions* (A) *and the condition* (31) *hold.*

(2) For $c \geqslant 0$ *and* $\lambda \geqslant 0$ *the inequality*

$$\int_0^\infty t^{\nu-1} H(t, ce^{\lambda t}\chi_m(t))\,dt + \sum_{k=1}^\infty \tau_k^{\nu-1} \beta_k(ce^{\lambda \tau_k}\chi_m(\tau_k)) < \infty \tag{35}$$

holds.

(3) *For* $c \geqslant 0$ *and* $\lambda < 0$ *the inequality*

$$\int_0^\infty e^{-\lambda t} H(t, ce^{\lambda t}\chi_m(t))\,dt + \sum_{k=1}^\infty e^{-\lambda \tau_k} \beta_k(ce^{\lambda \tau_k}\chi_m(\tau_k)) < \infty$$

is fulfilled.

Then the systems (1) *and* (2) *are asymptotically equivalent.*

Proof. From conditions (2) and (3) of Theorem 6 it follows that condition (24) is fulfilled. Therefore, Theorem 5 holds.

Let $\lambda < 0$. Then from Theorem 5 and estimates (22) and (23) it follows that $\lim_{t\to\infty} x(t; t_0, x_0) = \lim_{t\to\infty} y(t; t_0, x_0) = 0$ and the asymptotic equivalence of (1) and (2) is evident.

Now, let $\lambda \geqslant 0$ and $x(t) = x(t; t_0, x_0)$ be a solution of (1). Let us consider the system

$$\begin{aligned} \frac{dz}{dt} &= Az + f(t, x(t)), && t \neq \tau_k, \\ \Delta z\,|_{t=\tau_k} &= Bz(\tau_k) + b_k(x(\tau_k)), && k = 1, 2, \ldots. \end{aligned} \tag{36}$$

From Theorem 5 and condition (2) of Theorem 6 it follows that

$$\begin{aligned} t^{\nu-1}\,|f(t, x(t))| &\leqslant t^{\nu-1} H(t, ce^{\lambda t}\chi_m(t)), && t > t_0, \\ \tau_k^{\nu-1}\,|b_k(x(\tau_k))| &\leqslant \tau_k^{\nu-1} \beta_k(ce^{\lambda\tau_k}\chi_m(\tau_k)), && \tau_k > t_0. \end{aligned}$$

Then

$$\int_{t_0}^\infty t^{\nu-1} |f(t, x(t))|\,dt + \sum_{t_0 < \tau_k} \tau_k^{\nu-1}\,|b_k(x(\tau_k))| < \infty$$

and according to Theorem 2, system (36) has a solution $z_0(t)$, for which $\lim_{t\to\infty} z_0(t) = 0$. Then $y(t) = x(t) - z_0(t)$ is a solution of (2) and $\lim_{t\to\infty} |x(t) - y(t)| = 0$.

Let $y(t) = y(t; 0, y_0)$ be a solution of (2).

Let $\eta > 0$ be a constant and $\rho(t) = e^{\lambda t}\chi_m(t)$.

According to (35) we can choose $t_0 \geqslant 1$, $t_0 \neq \tau_k$ such that

$$R(t_0) \equiv (a + aM)\left[\int_{t_0}^{\infty} s^{\nu-1} H(s, c_1 e^{\lambda s}\chi_m(s))\, ds + \sum_{t_0 \leqslant \tau_k} \tau_k^{\nu-1} \beta_k(c_1 e^{\lambda \tau_k}\chi_m(\tau_k))\right] < \eta,$$

where $M = \sup_{u \geqslant 0} e^{-(\alpha u/2)}\chi_w(u)$, $c_1 = c_0\,|y_0| + \eta$, and the constants a, α, and c_0 are from the estimates (15) and (18).

We choose such a t_0. Then for the operator

$$V\varphi(t) = y(t) + \int_{t_0}^{t} G_-(t, s) f(s, \varphi(s))\, ds + \sum_{t_0 < \tau_k < t} G_-(t, \tau_k)\, b_k(\varphi(\tau_k)) - \int_{t}^{\infty} G_0(t, s) f(s, \varphi(s))\, ds - \sum_{t \leqslant \tau_k} G_0(t, \tau_k)\, b_k(\varphi(\tau_k)) \tag{37}$$

we shall prove the following:

(I) V is defined on $B_\eta = \{\varphi \in S([t_0, \infty), R^n, \rho) \colon \|\varphi - y\|_\rho \leqslant \eta\}$.
Indeed, let $\varphi \in B_\eta$, i.e.,

$$|\varphi(t) - y(t)| \leqslant \eta e^{\lambda t}\chi_m(t), \qquad t > t_0.$$

Then from (22) it follows that

$$|\varphi(t)| \leqslant (c_0\,|y_0| + \eta)\, e^{\lambda t}\chi_m(t) = c_1 e^{\lambda t}\chi_m(t), \qquad t > t_0, \tag{38}$$

and successively we get the estimates

$$|G_0(t, s) f(s, \varphi(s))| \leqslant a\chi_m(s - t)\, H(s, |\varphi(s)|) \leqslant a s^{\nu-1} H(s, c_1 e^{\lambda s}\chi_m(s)), \qquad s \geqslant t > t_0 \geqslant 1, \tag{39}$$

$$|G_0(t, \tau_k)\, b_k(\varphi(\tau_k))| \leqslant a\tau_k^{\nu-1} \beta_k(c_1 e^{\lambda \tau_k}\chi_m(\tau_k)), \qquad \tau_k \geqslant t > t_0 \geqslant 1. \tag{40}$$

From (39), (40), and (35) it follows that the improper integral and the series in (37) are convergent, i.e., the operator V is defined on B_η.

(II) $VB_\eta \subset B_\eta$. This follows from the estimate

$$|V\varphi(t) - y(t)| \leqslant \int_{t_0}^{t} |G_0(t, s)|\; |f(s, \varphi(s))|\, ds + \sum_{t_0 < \tau_k < t} |G_0(t, \tau_k)|\; |b_k(\varphi(\tau_k))| + \int_{t}^{\infty} |G_-(t, s)|\; |f(s, \varphi(s))|\, ds + \sum_{t \leqslant \tau_k} |G_-(t, s)|\; |b_k(\varphi(\tau_k))|$$

$$\leqslant a\int_{t_0}^{t} e^{-\alpha(t-s)}\chi_w(t-s)\, H(s, c_1 e^{\lambda s}\chi_m(s))\, ds$$

$$+a \sum_{t_0<\tau_k<t} e^{-\alpha(t-s)}\chi_w(t-\tau_k)\, \beta_k(c_1 e^{\lambda\tau_k}\chi_m(\tau_k))$$

$$+a\int_{t}^{\infty} \chi_v(s-t)\, H(s, c_1 e^{\lambda s}\chi_m(s))\, ds$$

$$+a\sum_{t\leqslant\tau_k} \chi_v(\tau_k-t)\, \beta_k(c_1 e^{\lambda\tau_k}\chi_m(\tau_k))$$

$$\leqslant aM\int_{t_0}^{t} H(s, c_1 e^{\lambda s}\chi_m(s))\, ds$$

$$+aM \sum_{t_0<\tau_k<t} \beta_k(c_1 e^{\lambda\tau_k}\chi_m(\tau_k))$$

$$+\; a\int_{t}^{\infty} s^{\nu-1} H(s, c_1 e^{\lambda s}\chi_m(s))\, ds$$

$$+a\sum_{t\leqslant\tau_k} \tau_k^{\nu-1}\beta_k(c_1 e^{\lambda_k}\chi_m(\tau_k))$$

$$\leqslant R(t_0)<\eta\leqslant\eta e^{\lambda t}\chi_m(t).$$

(III) V is continuous on B_η.

Let $\varepsilon>0$ be given, φ_n, $\varphi\in B_\eta$, and $\|\varphi_n-\varphi\|_\rho\to 0$ as $n\to\infty$. We choose $t_1>t_0$ such that $4R(t_1)<\varepsilon$. For $t>t_0$ we have

$$|V\varphi_n(t)-V\varphi(t)|\leqslant\int_{t_0}^{t} |G_0(t,s)|\, |f(s,\varphi_n(s))-f(s,\varphi(s))|\, ds$$

$$+\sum_{t_0<\tau_k<t} |G_0(t,\tau_k)|\, |b_k(\varphi_n(\tau_k))-b_k(\varphi(\tau_k))|$$

$$+\int_{t}^{\infty} |G_-(t,s)|\, |f(s,\varphi_n(s))-f(s,\varphi(s))|\, ds$$

$$+\sum_{t\leqslant\tau_k} |G_-(t,s)|\, |b_k(\varphi_n(\tau_k))-b_k(\varphi(\tau_k))|.$$

Using the estimates (15), (16), condition (A5), and considering cases $t_0<t<t_1$ and $t_0<t_1\leqslant t$, we get that for $t>t_0$ the inequality

$$|V\varphi_n(t)-V\varphi(t)|$$

$$\leqslant\int_{t_0}^{t_1} (aM+as^{\nu-1})\, |f(s,\varphi_n(s))-f(s,\varphi(s))|\, ds$$

$$+\sum_{t_0<\tau_k<t_1} (aM+a\tau_k^{\nu-1})\, |b_k(\varphi_n(\tau_k))-b_k(\varphi(\tau_k))|+2R(t_1)$$

$$\equiv\mathscr{T}_n+2R(t_1)$$

is fulfilled.

From the continuity of $f(t, x)$ and $b_k(x)$ and from the fact that $\varphi_n(s) \to \varphi(s)$, as $n \to \infty$, uniformly with respect to $s \in (t_0, t_1]$, it follows that there exists $n_0 > 0$ such that if $n > n_0$ then $\mathscr{T}_n < \varepsilon/2$.

Then $|V\varphi_n(t) - V\varphi(t)| < \varepsilon \leqslant \varepsilon e^{\lambda t}\chi_m(t)$, or $\|V\varphi_n - V\varphi\|_\rho = \varepsilon$, i.e., V is continuous.

(IV) $\lim_{t\to\infty} |V\varphi(t) - y(t)| = 0$ uniformly in $\varphi \in B_\eta$.

Indeed, for $t \geqslant 1$, $|V\varphi(t) - y(t)| \leqslant \mathscr{T}_1 + \mathscr{T}_2$, where

$$\begin{aligned}
\mathscr{T}_1 &= \left| \int_{t_0}^{t} G_-(t, s) f(s, \varphi(s))\, ds + \sum_{t_0 < \tau_k < t} G_-(t_1 \tau_k)\, b_k(\varphi(\tau_k)) \right| \\
&\leqslant a \int_{t_0}^{t} e^{-\alpha(t-s)} \chi_w(t-s)\, H(s, c_1 e^{\lambda s} \chi_m(s))\, ds \\
&\quad + a \sum_{t_0 < \tau_k < t} e^{-\alpha(t-s)} \chi_w(t - \tau_k)\, \beta_k(c_1 e^{\lambda \tau_k} \chi_m(\tau_k)) \\
&\leqslant aM \int_{t_0}^{t} e^{-(\alpha/2)(t-s)} H(s, c_1 e^{\lambda s} \chi_m(s))\, ds \\
&\quad + aM \sum_{t_0 < \tau_k < t} e^{-(\alpha/2)(t-s)} \beta_k(c_1 e^{\lambda \tau_k} \chi_m(\tau_k)),
\end{aligned} \tag{41}$$

$$\begin{aligned}
\mathscr{T}_2 &= \left| \int_{t}^{\infty} G_0(t, s) f(s, \varphi(s))\, ds + \sum_{t \leqslant \tau_k} G_0(t, \tau_k)\, b_k(\varphi(\tau_k)) \right| \\
&\leqslant a \int_{t}^{\infty} \chi_v(s-t)\, H(s, s_1 e^{\lambda s} \chi_m(s))\, ds \\
&\quad + a \sum_{t \leqslant \tau_k} \chi_v(\tau_k - t)\, \beta_k(c_1 e^{\lambda \tau_k} \chi_m(\tau_k)) \\
&\leqslant 2a \left(\int_{t}^{\infty} s^{v-1} H(s, c_1 e^{\lambda s} \chi_m(s))\, ds + \sum_{t \leqslant \tau_k} \tau_k^{v-1} \beta_k(c_1 e^{\lambda \tau_k} \chi_m(\tau_k)) \right).
\end{aligned} \tag{42}$$

From (41), condition (35), and Lemma 1 it follows that $\mathscr{T}_1 \to 0$ as $t \to \infty$. and from (42) and condition (35) it follows that $\mathscr{T}_2 \to 0$ as $t \to \infty$.

(V) The set VB_η is relatively compact.

We shall prove that $\mathscr{F} = VB_\eta$ satisfies the conditions of Lemma 5.

Let $\varepsilon > 0$ be given. We choose $T > t_0$ such that

$$|V\varphi(t) - y(t)| < \varepsilon \qquad \text{for} \quad t \leqslant T.$$

This is possible according to assertion (IV).

It remains to be proved that $\mathscr{F}$ is quasi-equicontinuous on $(t_0, T]$.

Let $T_1 \geqslant T$ be chosen such that $4R(T_1) < \varepsilon$. Using (15), (38), (39), and (40), we obtain that for $t_1, t_2 \in (\tau_{k-1}, \tau_k] \cap (t_0, T]$, $t_1 \leqslant t_2$, the estimate

$$
\begin{aligned}
&|V\varphi(t_2) - V\varphi(t_1)| \\
&\quad \leqslant |y(t_2) - y(t_1)| + \int_{t_0}^{t_1} |G_-(t_2, s) - G_-(t_1, s)|\, |f(s, \varphi(s))|\, ds \\
&\qquad + \sum_{t_0 < \tau_k < t_1} |G_-(t_2, \tau_k) - G_-(t_1, \tau_k)|\, |b_k(\varphi(\tau_k))| \\
&\qquad + \int_{t_2}^{T_1} |G_0(t_2, s) - G_0(t_1, s)|\, |f(s, \varphi(s))|\, ds \\
&\qquad + \sum_{t_2 \leqslant \tau_k < T_1} |G_0(t_2, \tau_k) - G_0(t_1, \tau_k)|\, |b_k(\varphi(\tau_k))| \\
&\qquad + \int_{t_1}^{t_2} |G_-(t_2, s)|\, |f(s, \varphi(s))|\, ds \\
&\qquad + \int_{t_1}^{t_2} |G_0(t_1, s)|\, |f(s, \varphi(s))|\, ds \\
&\qquad + \int_{T_1}^{\infty} |G_0(t_2, s) - G_0(t_1, s)|\, |f(s, \varphi(s))|\, ds \\
&\qquad + \sum_{T_1 \leqslant \tau_k} |G_0(t_2, \tau_k) - G_0(t_1, \tau_k)|\, |b_k(\varphi(\tau_k))| \\
&\quad \leqslant |y(t_2) - y(t_1)| + \int_{t_0}^{T} |G_-(t_2, s) - G_-(t_1, s)|\, H(s, c_1 e^{\lambda T} \chi_m(T))\, ds \\
&\qquad + \sum_{t_0 < \tau_k \leqslant T} |G_-(t_2, \tau_k) - G_-(t_1, \tau_k)|\, \beta_k(c_1 e^{\lambda T} \chi_m(T)) \\
&\qquad + \int_{t_2}^{T_1} |G_0(t_2, s) - G_0(t_1, s)|\, H(s, c_1 e^{\lambda T_1} \chi_m(T_1))\, ds \\
&\qquad + \sum_{t_2 \leqslant \tau_k < T_1} |G_0(t_2, \tau_k) - G_0(t_1, \tau_k)|\, \beta_k(c_1 e^{\lambda T_1} \chi_m(T_1)) \\
&\qquad + (aM + a)\, T^{\nu - 1} H(T, c_1 e^{\lambda T} \chi_m(T))(t_2 - t_1) + 2R(T_1) \\
&\quad \equiv |y(t_2) - y(t_1)| + \mathscr{T}(t_1, t_2) + 2R(T_1)
\end{aligned}
$$

is fulfilled.

From the last estimate and from the properties of the functions $G_0(t, s)$, $G_-(t, s)$, $y(t)$, and $\varphi(t)$ it follows that there exists $\delta > 0$ such that if $|t_2 - t_1| < \delta$ then

$$|y(t_2) - y(t_1)| + \mathscr{T}(t_1, t_2) < \varepsilon/2.$$

With this choice of δ we have that for every $k=1, 2, ..., \varphi \in B\eta$, and $t_1, t_2 \in (\tau_{k-1}, \tau_k] \cap (t_0, T]$, $|t_1 - t_2| < \delta$ the inequality

$$|V\varphi(t_2) - V\varphi(t_1)| < \varepsilon$$

is fulfilled, i.e., $\mathscr{F} = VB_\eta$ is quasi-equicontinuous.

The assertions (I), (II), (III), (IV), and (V) provide for the application of Schauder's theorem, according to which the operator V has a fixed point $x \in B_\eta$, i.e., $x(t) = Vx(t)$ for $t > t_0$. The immediate check shows that $x(t)$ is a solution of system (1) and from assertion (IV) it follows that $\lim_{t\to\infty} |x(t) - y(t)| = \lim_{t\to\infty} |Vx(t) - y(t)| = 0$.

THEOREM 7. *Let the conditions of Theorem* 3 *hold. Then there exists restricted asymptotic equivalence between the set of all bounded solutions of system* (1) *and the set of all bounded solutions of system* (2).

Proof. Let $x(t)$ be a bounded solution of system (1). Then, according to Theorem 3 there exists a solution $y(t)$ of system (2) which satisfies relation (3). But from (3) it follows that $y(t)$ is also bounded. Now, let $y(t) = y(t; 0, y_0)$ be a bounded solution of (2). The proof that there exists a solution $x(t)$ of system (1) which satisfies relation (3) is done in the same way as in Theorem 6. The only difference is that the operator V: $B_\eta \to B_\eta$ is considered on the set

$$B_\eta = \{\varphi \in S([t_0, \infty), R^n, \rho_1): \|\varphi - y\|_{\rho_1} \leqslant \eta\},$$

where $\rho_1(t) \equiv 1$, and the point $t_0 \geqslant 1$, $t_0 \neq \tau_k$ is chosen such that

$$R_1(t_0) \equiv (a + aM)\left(\int_{t_0}^{\infty} S^{\nu-1} H(s, c_1)\, ds + \sum_{t_0 \leqslant \tau_k} \tau_k^{\nu-1} \beta_k(c_1)\right) < \eta,$$

where $M = \sup_{u \geqslant 0} e^{-(\alpha u/2)} \chi_w(u)$, $c_1 = c_0 |y_0| + \eta$. Condition (20) is considerably used in the proof.

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