JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 76, 571–599 (1980)

Fuzzy Topology. I. Neighborhood Structure of a Fuzzy Point and Moore–Smith Convergence*

PU PAO-MING AND LIU YING-MING

Department of Mathematics, Sichuan University, Sichuan, China

Submitted by L. A. Zadeh

The fundamental concept of a fuzzy set, introduced by Zadeh in 1965 [1], provides a natural foundation for treating mathematically the fuzzy phenomena which exist pervasively in our real world and for building new branches of fuzzy mathematics. In the area of fuzzy topology, much research has been carried out [2–8] since 1968. We should like to mention here that in 1975 Chou Hao-xuan did some significant work concerning the relationship between fuzzy topological spaces and ordinary topological spaces (unpublished). But there are still two fundamental problems which remain to be solved. The first one concerns the concept of a fuzzy point and its neighborhood structure. A definition of a fuzzy point was given in [8] in such a way that a crisp singleton, equivalently, an ordinary point, was not a special case of a fuzzy point. Moreover, the research was carried on along with the same idea as that of the neighborhood system in general topology so that the results thus obtained could not reflect the features of neighborhood structure in fuzzy topological spaces. The properties derived therefrom are often pathologies and demonstrate departures from general topology, like those the author pointed out in [8]. The purpose of the present paper is to remedy these drawbacks so that we can develop the theory of fuzzy topology in a satisfactory way. We redefine a fuzzy point in such a way that it takes a crisp singleton, equivalently, an ordinary point, as a special case. As for the neighborhood structure of such a fuzzy point, in addition to the relation "ϵ" between fuzzy points and fuzzy sets and the corresponding neighborhood systems, we shall introduce another important relation "$\underset{\sim}{Q}$" between fuzzy points and fuzzy sets, called the $\underset{\sim}{Q}$-relation, and the corresponding neighborhood structure, called the $\underset{\sim}{Q}$-neighborhood system. In an ordinary topological space, as a special case of a fuzzy topological space, these concepts, neighborhood system and $\underset{\sim}{Q}$-neighborhood system, ϵ-relation and $\underset{\sim}{Q}$-relation coincide respectively.

The second problem concerns the theory of convergence. Since the concept

* This paper was completed in 1976 and published in 1977 in an internal publication, Sichuan Daxue Xuebao 1 (1977), 31–50 [in Chinese].

0022-247X/80/080571-29$02.00/0

of convergence occurring in [2] and [6] is limited to sequences of fuzzy sets and the neighborhood structure used to define convergence has the same character as the traditional neighborhood system, the conclusions thus obtained are unsatisfactory, to say nothing of generalizing Moore–Smith convergence of nets to fuzzy topological spaces. As pointed out in [6], new concepts of convergence and clustering are needed in order to develop the theory further in this direction. With the present treatment of these two problems, all the theorems concerning the neighborhood structure of a point and the theory of convergence in Chapters I and II of the celebrated book on general topology [10] are generalized to fuzzy topological spaces, with the exception of at most two less important ones. This means that these two problems in fuzzy topology have been solved to almost the same degree as the corresponding problems in general topology. In the framework of the present paper, there are still many properties of fuzzy topological spaces which can be investigated. Further results will be given in future papers.

1. Preliminaries

Since many concepts and statements have not yet taken their final forms, to begin with, let us recall some concepts occurring in the papers [2–9], which will be needed in the sequel. In the present paper X always denotes a non-empty (ordinary) set.

Definition 1.1. *A function A from X to the unit interval* $[0, 1]$ *is called a fuzzy set in X. For every $x \in A$, $A(x)$ is called* the grade of membership *of x in A. X is called the carrier of the fuzzy set A. The set $\{x \in X \mid A(x) > 0\}$ is called the support of A and is denoted by* Supp A *or* A_0. *If A takes only the values* 0, 1, *A is called a crisp set in X. From now on, we shall not differentiate between a crisp set A in X and* Supp A. *Particularly, the crisp set which always takes the value* 1 *on X is denoted by X, and the crisp set which always takes the value* 0 *on X is denoted by* $\varnothing$.

Definition 1.2. Let I be an indexed set, and let $\mathscr{A} = \{A_\alpha \mid \alpha \in I\}$ be a family of fuzzy sets in X. Then the union $\bigcup \{A_\alpha \mid \alpha \in I\}$ or $(\bigcup \mathscr{A})$ and the intersection $\bigcap \{A_\alpha \mid \alpha \in I\}$ (or $\bigcap \mathscr{A}$) are defined, respectively, by the following formulae (and hence are also fuzzy sets in X):

$$(\cup \mathscr{A})(x) = \sup\{A_\alpha(x) \mid \alpha \in I\}, \qquad x \in X,$$

$$(\cap \mathscr{A})(x) = \inf\{A_\alpha(x) \mid \alpha \in I\}, \qquad x \in X.$$

Definition 1.4. The complement of A, denoted by A', is defined by the formula: $A'(x) = 1 - A(x)$, $x \in X$.

By means of the properties of lub and glb of real numbers, it is easy to verify the following De Morgan's law:

$$(\cup\{A_\alpha \mid \alpha \in I\})' = \cap\{A'_\alpha \mid \alpha \in I\}.$$

DEFINITION 1.5. A family $\mathscr{T}$ of fuzzy sets in X is called a fuzzy topology for X iff (1) $\varnothing$, $X \in \mathscr{T}$, (2) $A \cap B \in \mathscr{T}$ whenever A, $B \in \mathscr{T}$ and (3) $\bigcup\{A_\alpha \mid \alpha \in I\} \in \mathscr{T}$ whenever each $A_\alpha \in \mathscr{T}$ $(\alpha \in I)$. Moreover, the pair $(X, \mathscr{T})$ is called a fuzzy topological space or fts, for short. Every member of $\mathscr{T}$ is called a $\mathscr{T}$-open fuzzy set (or simply open fuzzy set). The complement of a $\mathscr{T}$-open fuzzy set is called a $\mathscr{T}$-closed fuzzy set (or simply closed fuzzy set).

Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two fuzzy topologies for X. If the inclusion relation $\mathscr{T}_1 \subset \mathscr{T}_2$ holds, we say that $\mathscr{T}_2$ is finer than $\mathscr{T}_1$ and $\mathscr{T}_1$ is coarser than $\mathscr{T}_2$.

DEFINITION 1.6. Let $(X, \mathscr{T})$ be a fts. A subfamily $\mathscr{B}$ of $\mathscr{T}$ is called a base for $\mathscr{T}$ iff, for each $A \in \mathscr{T}$, there exists $\mathscr{B}_A \subset \mathscr{B}$ such that $A = \bigcup \mathscr{B}_A$; a subfamily $\mathscr{S}$ of $\mathscr{T}$ is called a subbase for $\mathscr{T}$ iff the family $\mathscr{B} = \{\bigcap \mathscr{F} \mid \mathscr{F}$ is a finite subset of $\mathscr{S}\}$ is a base for $\mathscr{T}$. $(X, \mathscr{T})$ is said to satisfy the second axiom of countability or is said to be a C_{II} space iff $\mathscr{T}$ has a countable base.

DEFINITION 1.7. Let A and A_α $(\alpha \in I)$ be fuzzy sets in X. $\{A_\alpha \mid \alpha \in I\}$ is called a cover of A iff $\bigcup\{A_\alpha \mid \alpha \in I\} \supset A$. If there exists a subset I_1 of I such that $\bigcup\{A_\alpha \mid \alpha \in I_1\} \supset A$, $\{A_\alpha \mid \alpha \in I\}$ is called a subcover.

$(X, \mathscr{T})$ will always denote a fuzzy topological space in this paper. We remark that all of the following definitions and conclusions take the corresponding definitions and conclusions take the corresponding definitions and conclusions in general topology as special cases, respectively. In general, we shall not repeat this remark any more.

2. CONCEPT OF A FUZZY POINT AND ITS NEIGHBORHOOD STRUCTURE

DEFINITION 2.1.[1] A fuzzy set in X is called a fuzzy point iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is λ $(0 < \lambda \leqslant 1)$ we denote this fuzzy point by x_λ, where the point x is called its support.

DEFINITION 2.2. The fuzzy point x_λ is said to be contained in a fuzzy set A, or to belong to A, denoted by $x_\lambda \in A$, iff $\lambda \leqslant A(x)$. Evidently, every fuzzy set A can be expressed as the union of all the fuzzy points which belong to A.

[1] The concept of a fuzzy point is actually the so-called crisp singleton given in section 4 of [9]. But the very important neighborhood construction was not mentioned at all in [9].

Definition 2.3. Two fuzzy sets A, B in X are said to be *intersecting* iff there exists a point $x \in X$ such that $(A \cap B)(x) \neq 0$. For such a case, we say that A and B intersect at x.

Definition 2.4. A fuzzy set A in $(X, \mathscr{T})$ is called a neighborhood of fuzzy point x_λ iff there exists a $B \in \mathscr{T}$ such that $x_\lambda \in B \subset A$; a neighborhood A is said to be open iff A is open. The family consisting of all the neighborhoods of x_λ is called the system of neighborhoods of x_λ.

Corresponding to the above definitions, we introduce the following important concepts.

Definition 2.2′. A fuzzy point x_λ is said to be quasi-coincident with A, denoted by $x_\lambda q A$, iff $\lambda > A'(x)$, or $\lambda + A(x) > 1$.

Definition 2.3′. A is said to be quasi-coincident with B, denoted by AqB, iff there exists $x \in X$ such that $A(x) > B'(x)$, or $A(x) + B(x) > 1$. If this is true, we also say that A and B are quasi-coincident (with each other) at x. It is clear that if A and B are quasi-coincident at x, both $A(x)$ and $B(x)$ are not zero and hence A and B intersect at x.

Definition 2.4′. A fuzzy set A in $(X, \mathscr{T})$ is called a Q-neighborhood of x_λ iff there exists a $B \in \mathscr{T}$ such that $x_\lambda q B \subset A$. The family consisting of all the Q-neighborhoods of x_λ is called the system of Q-neighborhoods of x_λ.

Note. A Q-neighborhood of a fuzzy point generally does not contain the point itself. The neighborhood structure of a point which does not contain the point itself was already studied in general topology by Fréchet in 1916 [cf. M. Fréchet, "Les espaces abstraits," Paris, p. 172] formed the foundation upon which the Fréchet (V)-space theory has been built [cf. W. Sierpinski," General Topology, Chap. I, Toronto, 1952]. But the fact that a set A and its complement A' should not intersect, which is true in the theory of (V)-spaces, is no longer true generally in the theory of fuzzy topological spaces. Hence our investigation of the Q-neighborhood structure differs from that of the Fréchet (V)-space theory.

The substitute for the fact that A and A' do not intersect in general topology is the fact that A and A' are not quasi-coincident in fuzzy topology. More generally, we have the following proposition.

Proposition 2.1. *$A \subset B$ iff A and B' are not quasi-coincident; particularly, $x_\lambda \in A$ iff x_λ is not quasi-coincident with A'.*

Proof. This follows from the fact:

$$A(x) \leqslant B(x) \quad \text{iff} \quad A(x) + B'(x) = A(x) + 1 - B(x) \leqslant 1.$$

PROPOSITION 2.2. *Let $\mathscr{U}_e$ be the family of Q-neighborhoods (resp. neighborhoods) of a fuzzy point e in $(X, \mathscr{T})$. Then we have*

(1) *If $U \in \mathscr{U}_e$, then e is quasi-coincident with (resp. belongs to) U.*

(2) *If $U, V \in \mathscr{U}_e$, then $U \cap V \in \mathscr{U}_e$.*

(3) *If $U \in \mathscr{U}_e$ and $U \subset V$, then $V \in \mathscr{U}_e$.*

(4) *If $U \in \mathscr{U}_e$, then there exists $V \in \mathscr{U}_e$ such that $V \subset U$ and $V \in \mathscr{U}_d$ for every fuzzy point d which is quasi-coincident with (resp. belongs to) V.*

Conversely, for each fuzzy point e in X. $\mathscr{U}_e$ is a family of fuzzy sets in X satisfying the above conditions (1)–(3), then the family $\mathscr{T}$ of all the fuzzy sets U, such that $U \in \mathscr{U}_e$ whenever eqU (resp. $e \in U$) is a fuzzy topology for X. If, in addition, $\mathscr{U}_e$ satisfies the condition (4), mentioned above, then $\mathscr{U}_e$ is exactly the Q-neighborhood (resp. neighborhood) system of e relative to the fuzzy topology $\mathscr{T}$.

The proof is straightforward (cf. [10 problem 1.B]).

PROPOSITION 2.3. *Let $\{\mathscr{A}_\alpha\}$ be a family of fuzzy sets in X. Then a fuzzy point e is quasi-coincident with $U\mathscr{A}$ iff there exists some $A_\alpha \in \mathscr{A}$ such that eqA_α.*

Proof. If eqA_α, it is evident that $eqUA_\alpha$. That the condition is necessary is easily proved by means of the properties of lub and the concept of being "quasi-coincident with."

PROPOSITION 2.4. *A subfamily $\mathscr{B}$ of a fuzzy topology $\mathscr{T}$ for X is a base for $\mathscr{T}$ iff for each fuzzy point e in $(X, \mathscr{T})$ and for each open Q-neighborhood U of e, there exists a member $B \in \mathscr{B}$ such that $eqB \subset A$.*

Proof. The necessity of the condition follows directly from the definition of a base and the necessary condition of Proposition 2.3. We shall now show its sufficiency. If $\mathscr{B}$ is not a base for $\mathscr{T}$, then there exists a member, $A \in \mathscr{T}$, such that $G = \{B \in \mathscr{B} \mid B \subset A\} \neq A$, and hence there is an x such that $G(x) < A(x)$. Let $\lambda = 1 - G(x)$, which is obviously positive; we obtain a fuzzy point $e = x_\lambda$. Since $A(x) + \lambda > G(x) + \lambda = 1$, eqA. But since any member $B \in \mathscr{B}$ which is contained in A is contained in G, we have $B(x) + \lambda \leqslant G(x) + \lambda = 1$; that is, $e = x_\lambda$ is not quasi-coincident with B. This contradicts the assumption.

3. LOCAL BASE. A COUNTEREXAMPLE

DEFINITION 3.1. Let $\mathscr{U}_{eQ}$ (resp. U_e) be a Q-neighborhood system (resp. neighrohood system) of a fuzzy point e in $(X, \mathscr{T})$. A subfamily $\mathscr{B}_{eQ}$ (resp. $\mathscr{B}_e$) of $\mathscr{U}_{eQ}$ (resp. $\mathscr{U}_e$) is called a Q-neighborhood base (resp. neighborhood base) of $\mathscr{U}_{eQ}$ (resp. $\mathscr{U}_e$) iff for each $A \in \mathscr{U}_{eQ}$ (resp. $A \in \mathscr{U}_e$) there exists a member $B \in \mathscr{B}_e\mathscr{U}_Q$ (resp. $B \in \mathscr{B}_e$) such that $B \subset A$. A fts $(X, \mathscr{T})$ is said to satisfy the Q-first axiom of countability (resp. first axiom of countability) or to be Q-C_1) iff every fuzzy point in $(X, \mathscr{T})$ has a countable Q-neighborhood base (resp. neighborhood base).

PROPOSITION 3.1. *If $(X, \mathscr{T})$ is a C_{I}-space, then it is a Q-C_{I}-space.*

Proof. Let $e = x_\lambda$ be an arbitrary fuzzy point. Consider a sequence $\{\mu_n\}_{n \in N}$ in $(1 - \lambda, 1]$ converging to $1 - \lambda$ and let $x_{\mu_n} = e_n$. For each $n \in N$, there exists a countable open neighborhood base $\mathscr{B}_n$ of e_n (there is evidently no loss of generality in assuming the openness of each member of $\mathscr{B}_n$). Each member B of B_n satisfies $B(x) \geqslant \mu_n > 1 - \lambda$ and hence is a Q-neighborhood of e. The collection $\mathscr{B}$ consisting of all the members of all $\mathscr{B}_n$ is a family of open Q-neighborhoods of e. Let A be an arbitrary Q-neighborhood of e, hence $A(x) > 1 - \lambda$. Since μ_n $1 - \lambda$, there exists $m \in N$ such that $A(x) \geqslant \mu_m > 1 - \lambda$, i.e., $e_m \in A$ and A is an open neighborhood of e_m. Therefore there exists a member $B \in \mathscr{B}_n \subset \mathscr{B}$ such that $B \subset A$, $B(x) \geqslant \mu_m > 1 - \lambda$. This shows that $\mathscr{B}$ is a countable Q-neighborhood base of e.

PROPOSITION 3.2. *If $(X, \mathscr{T})$ is C_{II}, then it is also Q-C_{I}.*

Proof. Let $\mathscr{B}$ be a countable base for $\mathscr{T}$. Let e be a fuzzy point in $(X, \mathscr{T})$. For any $U \in \mathscr{T}$ such that eqU, by Proposition 2.4, there exists $B \in \mathscr{B}$ such that $eqB \subset U$. Let $\tilde{\mathscr{B}}$ be the family of all those members B of $\mathscr{B}$ thus obtained. It is easy to show that $\tilde{\mathscr{B}}$ is a countable Q-neighborhood of e.

The converse of Proposition 3.1 is generally not true. We shall construct, in the following, a C_{II}-space, which is of course Q-C_{I}, but is not C_{I}.

DEFINITION 3.2 [6, 5]. Let $(X, \mathscr{U})$ be an ordinary topological space. Let $F(\mathscr{U}) = \{f \in [0, 1]^x \mid f \text{ is lower semi-continuous}\}$; $(X, F(\mathscr{U}))$ is called the induced fuzzy topological space of $(X, \mathscr{U})$:

LEMMA 3.1. *Let $(X, \mathscr{U})$ be a completely regular (crisp) topological space. For every $h \in F(\mathscr{U})$, there exists a family $\mathscr{F} \subset [0, 1]^X$ each member of which is continuous with respect to $\mathscr{U}$, such that $h = \sup\{f \mid f \in \mathscr{F}\}$. In other words, the family $\mathscr{B} = \{f \mid f:(X, \mathscr{U}) \to [0, 1]$ is continuous$\}$ forms a base for the induced fuzzy topology.*

For the proof of this lemma, refer to the proof of Proposition 5 in [11 Chap. 9, Sect. 1].

In the following, let $(X, \mathscr{U})$ be the subspace $[0, 1]$ of the real axis. Let T denote the totality of all rationals in $(-\infty, 1]$. For each positive integer n, subdivide X into 2^n equal parts. The $2^n + 1$ points of subdivision are successively denoted by $x_n{}^k$ $(k = 0, 1, \ldots, 2^n)$. For a fixed n, let $\tilde{\mathscr{B}}_n = \{f \mid f(x_n{}^k) = f_k \in T,$ f is linear in $\Delta_k = [x_n{}^k, x_n^{k+1}]$, for each $k \in \{0, 1, \ldots, 2n\}\}$. Each f is obviously continuous and $\tilde{\mathscr{B}}_n$ is countable. For each $f \in \tilde{\mathscr{B}}_n$, let $f^+(x) = \max\{f(x), 0\}$, then $f^+ \colon X \to [0, 1]$ is obviously continuous. Let $\mathscr{B}_n = \{f^+ \mid f \in \tilde{\mathscr{B}}_n\}$, $\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}_n$, For this countable family $\mathscr{B}$ of continuous functions, we have

LEMMA 3.2. *Let $g: X \to [0, 1]$ be continuous and $\epsilon > 0$; then there exists $f^+ \in \mathscr{B}$ such that*

$$g(x) - \epsilon < f^+(x) \leqslant g(x), \qquad x \in X$$

and hence g can be expressed as the least upper bound of a countable member of functions in $\mathscr{B}$.

Proof. For $\epsilon > 0$, since g is uniformly continuous on X, there exists a positive integer n, such that on each of 2^n equally subdivided intervals, the difference between the greatest value of g and the least value of g (that is the oscillation) is less than $\epsilon/8$. For each x_n^k, take $f_k \in T$ such that $g(x_n^k) - \epsilon/8 > f_k > g(x_n^k) - \epsilon/4$. Let f denote the function which takes the value f_k at x_n^k and is linear in each of the equally subdivided parts. Then f^+ is the required function. In fact, consider an arbitrary interval $\Delta_k = [x_n^k, x_n^{k+1}]$ and we may assume $f_k \geqslant f_{k+1}$ without loss of generality, then f_k is the greatest value of f on Δ_k. Noting that the oscillation of g on Δ_k is less than $\epsilon/8$, for each $x \in \Delta_k$, we have

$$g(x) > g(x_n^k) - \epsilon/8 > f_k \geqslant f(x).$$

Hence for $x \in X$, $g(x) > f(x)$. On the other hand, the oscillation of f on Δ_k is $f_k - f_{k+1} < (g(x_n^k) - \epsilon/8) - (g(x_n^{k+1}) - \epsilon/4) = g(x_n^k) - g(x_n^{k+1}) + \epsilon/8 < \epsilon/4$, and hence, for $x \in \Delta_k$, we have

$$\begin{aligned} |f(x) - g(x)| &\leqslant |f(x) - f_k| + |f_k - g(x_n^k)| + |g(x_n^k) - g(x)| \\ &\leqslant \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon. \end{aligned}$$

It follows that for $x \in X$, $g(x) - \epsilon < f(x) < g(x)$. Since $g(x) \geqslant 0$, it is clear that $g(x) - \epsilon < f^+(x) \leqslant g(x)$.

In order to prove the later part of the lemma, it suffices to take $\epsilon = 1/n$ and denote the corresponding f^+ by $f_n^+ \in \mathscr{B}$. Then g is easily seen to be $\sup\{f_n^+ \mid f_n^+ \in \mathscr{B}\}$.

THEOREM 3.1. *Let $(X, \mathscr{U})$ be the subspace $[0, 1]$ of the real axis, and let $\mathscr{T} = F(\mathscr{U})$ be the induced fuzzy topology for $\mathscr{U}$. Then $(X, F(\mathscr{U}))$ is C_{II}, but is not C_{I}.*

Proof. From Lemmas 3.1 and 3.2 the countable family $\mathscr{B} \subset \mathscr{T}$, given in Lemma 3.2, obviously forms a countable base; that is, $(X, \mathscr{T})$ is C_{II}. We now take any point $x \in X$ and may assume x to be $0 \in [0, 1]$ without loss of generality. Consider the crisp singleton $e = x_1$ i.e., the value of e at x is 1. If a countable member of open sets $B_n \in \mathscr{T}$ $(n = 1, 2,\ldots)$ forms a neighborhood base of e, since $e \in B_n$, we have $B_n(x) = 1$. From the lower semi-continuity, for $\epsilon = 1/n$, there exists a $\mathscr{U}$-open neighborhood G_n of x such that for $y \in G_n$, we always have $B_n(y) > 1 - \epsilon = 1 - 1/n$. Therefore for positive integer n, we can inductively take $y_n \in X$ such that $0 < y_n \in G_n$ and $y_n < y_{n-1}/2$ (with the convention $y_0 = 2$). Now let us construct a fuzzy set B in X as follows:

$$B(y) = \begin{cases} 1 - 1/n, & \text{for } y = y_n, \\ \text{a linear function}, & \text{for } y \in [y_n, y_{n-1}], \\ = 0, & \text{for } y \geqslant 1, \\ = 1, & \text{for } y = x\ (= 0). \end{cases}$$

Evidently B is continuous and takes values in [0, 1] and hence $B \in \mathscr{T}$. Since $B(x) = 1$, B is an open neighborhood of e. But since $B_n(y_n) > 1 - 1/n = B(y_n)$, any B_n is not contained in B. This contradicts the fact that $\{B_n\}$ is a neighborhood base of e and hence $(X, \mathscr{T})$ is not a C_{I} space.

4. Closure and Kuratowski's Theorem on 14 Sets

DEFINITION 4.1. Let A be a fuzzy set in $(X, \mathscr{T})$ and the union of all the $\mathscr{T}$-open sets contained in A is called the interior of A, denoted by A^0 *or* by $\text{Int}_{\mathscr{T}} A$. Evidently A^0 is the largest open set contained in A and $(A^0)^0 = A^0$.

DEFINITION 4.1′. The intersection of all the $\mathscr{T}$-closed sets containing A is called the closure of A, denoted by $\bar{A}$, or by $\text{cl}_{\mathscr{T}} A$. Obviously $\bar{A}$ is the smallest $\mathscr{T}$-closed set containing A and $\overline{(\bar{A})} = \bar{A}$.

THEOREM 4.1. *A fuzzy point $e \in A^0$ iff e has a neighborhood contained in A. The proof, being straightforward, is omitted.*

THEOREM 4.1′. *A fuzzy point $e = x_\lambda \in \bar{A}$ iff each Q-neighborhood of e is quasi-coincident with A.*

Proof. $x_\lambda \in \bar{A}$ iff, for every closed set $F \supset A$, $x_\lambda \in F$, or $F(x) \geqslant \lambda$. By taking complement, this fact can be stated as follows: $x_\lambda \in \bar{A}$ iff, for every open set $B \subset A'$, $B(x) \leqslant 1 - \lambda$. In other words, for every open set B satisfying $B(x) > 1 - \lambda$, B is not contained in A'. From Proposition 2.1, B is not contained in A' iff B is quasi-coincident with $(A')' = A$. We have thus proved that $x_\lambda \in \bar{A}$ iff every open Q-neighborhood B of x_λ is quasi-coincident with A, which is evidently equivalent to what we want to prove.

DEFINITION 4.2. A fuzzy point e is called an adherence point of a fuzzy set A iff, every Q-neighborhood of e is quasi-coincident with A.

COROLLARY. *$\bar{A}$ is the union of all the adherence points of A.*

THEOREM 4.2. $A^0 = \overline{((A'))}'$, $\bar{A} = ((A')^0)'$, $(\bar{A})' = (A')^0$, $\overline{(A')} = (A^0)'$

Proof. Let $\mathscr{A} = \{A_\alpha \mid A_\alpha \in \mathscr{T}$ and $A_\alpha \subset A\}$; then $A^0 = \bigcup \mathscr{A}$. Evidently, $\mathscr{A}' = \{A' \mid A_\alpha \in \mathscr{A}\}$ is the family of all the closed sets containing A' and hence $\bar{A}' = \bigcap (\mathscr{A}')$. From De Morgan's law, we have $\overline{((A'))} = (\bigcap (\mathscr{A}'))' = \bigcup \{(A')'\} = \bigcup \mathscr{A} = A^0$. The first formula is thus obtained. The other three can be similarly derived or from the first formula.

THEOREM 4.3. (The 14-set theorem). *If A is a fuzzy set in $(X, \mathscr{T})$, then at most 13 fuzzy sets can be constructed from A by successive applications, in any order, of interior, closure and complementation. Moreover, there is a crisp set A in a crisp topological space from which 14 different sets can be constructed by these three operations.*

Proof. The last part of the theorem is well known in general topology. The first part can also be proved in a manner similar to that in general topology, since in the original proof, $(A')' = A$, $\overline{(\bar{A})} = \bar{A}$, $(A^0)^0 = A^0$ and only the formulae in Theorem 4.2 are used (cf. [10, note on p. 45, Problem 1.E].

DEFINITION 4.3. A mapping f: $[0, 1]^X \to [0, 1]^X$ is called a fuzzy closure operator on X iff f satisfies the following Kuratowski closure axioms: (1) $f(\varnothing) = \varnothing$, (2) $A \subset f(A)$, (3) $f(f(A)) = f(A)$, (4) $f(A \cup B) = f(A) \cup f(B)$.

In a fuzzy topological space, it is easily seen that $\overline{A \cup B} = \bar{A} \cup \bar{B}$ (cf. [10, pp. 42–43]) and hence the mapping g: $[0, 1]^X \to [0, 1]^X$ defined by $g(A) = \bar{A}$ is a fuzzy closure operator on X. Conversely, any fuzzy closure operator on X can determine some fuzzy topology for X. For this, we have

THEOREM 4.4. *Let f be a fuzzy closure operator on X, let $\mathscr{F} = \{A \mid A \in [0, 1]^X$ such that $f(A) = A\}$ and let $\mathscr{T} = \{A' \mid A \in \mathscr{F}\}$, then $\mathscr{T}$ is a fuzzy topology for X and for every $B \in [0, 1]^X$, $\mathrm{cl}_{\mathscr{T}} B = f(B)$. The topology $\mathscr{T}$ thus determined as. above will be called the fuzzy topology associated with a fuzzy closure operator*

The proof may be carried out by repeating verbatim the proof of Theorem 1.8 in [10, p. 43] with the corresponding modifications of symbols. But the simple fact "when $A \subset B$, $f(A) \subset f(B)$," used in the proof, has to be proved as follows: from $A \subset B$, we have $B = A \cup B$ and hence $f(B) = f(A \cup B) = f(A) \cup f(B) \supset f(A)$.

DEFINITION 4.4. A fuzzy point e is called a boundary point of a fuzzy set A iff $e \in \bar{A} \cap \bar{A}'$. The union of all the boundary points of A is called a boundary of A, denoted by $b(A)$.

It is clear that $b(A) = \bar{A} \cap \bar{A}'$.

PROPOSITION 4.1. *$\bar{A} \supset A \cup b(A)$, where the inclusion symbol cannot be replaced by an equality.*

The first part of the proposition is obvious from the definition of $b(A)$. The last part will be shown by an example. It should be noticed that in general topology, we have $\bar{A} = A \cup b(A)$, which is a departure from fuzzy topology.

EXAMPLE. Let $x \in X$, $\mathscr{T} = \{X, \Phi, x_{1/2}\}$, $A = x_{2/3}$, and $e = x_{3/4}$; then the Q-neighborhood of e in $(X, \mathscr{T})$ are X and $x_{1/2}$, which are all quasi-coincident with A. Hence, by Theorem 4.1′, $e \in \bar{A}$. On the other hand $e \notin A$ and the Q-neighborhood of $\{x_{1/2}\}$ is not quasi-coincident with A', i.e., $e \notin b(A)$ and hence $e \notin A \cup b(A)$.

5. ACCUMULATION POITS: GENERALIZATION OF C. T. YANG'S THEOREM

DEFINITION 5.1. A fuzzy point e is called an accumulation point of a fuzzy set A iff e is an adherence point of A and every Q-neighborhood of e and A are quasi-coincident at some point different from supp(e), whenever $e \in A$. The union of all the accumulation points of A is called the derived set of A, denoted by A^d. It is evident that $A^d \subset \bar{A}$.

THEOREM 5.1. *$\bar{A} = A \cup A^d$, where A^d is the derived set of A.*

Proof. Let $\Omega = \{e \mid e$ is an adherence point of $A\}$. Then, from Theorem 4.1′, $\bar{A} = \bigcup \Omega$. On the other hand, $e \in \Omega$ is either "$e \in A$" or "$e \notin A$;" for the latter case, by Definition 5.1, $e \in A^d$, hence $\bar{A} = \bigcup \Omega \subset A \cup A^d$. The inverse inclusion relation is obvious.

COROLLARY. *A fuzzy set A is closed iff A contains all the accumulation points of A.*

Noting that A is closed iff $A = \bar{A}$, we obtain the corollary by Theorem 5.1.

LEMMA 5.1. *In $(X, \mathscr{T})$, let $A = x_\lambda$; then* (1) *for $y \neq x$, $\bar{A}(y) = A^d(y)$.* (2) *If $\bar{A}(x) > \lambda$, $\bar{A}(x) = A^d(x)$.* (3) *$\bar{A}(x) = \lambda$ iff $A^d(x) = 0$.*

Proof. The conclusions of (1), (2) and the sufficiency of (3) follow from Theorem 5.1. Now let $A(x) = \lambda$. We claim that any fuzzy point x_μ is not an accumulation point of A and hence $A^d(x) = 0$. In fact, when $\mu > \lambda$, $x_\mu \notin \bar{A}$ and hence $x_\mu \notin A^d$; when $\mu \leqslant \lambda$, $x_\mu \in A$. But then any Q-neighborhood of x_μ and A can not be quasi-coincident at a point different from x. Therefore x_μ is not an accumulation point of A.

PROPOSITION 5.1. *In $(X, \mathscr{T})$, let $A = \{x_\lambda\}$; then* (1) *when $A^d(x) > 0$, $A^d = \bar{A}$ is closed.* (2) *when $A^d(x) = 0$, A^d is closed iff there exists an open set B^* such that $B^*(x) = 1$ and for $y \neq x$, $B^*(y) = (\bar{A})'(y) = (A^d)'(y)$.* (3) *$A^d(x) = 0$ iff there exists an open set B such that $B(x) = 1 - \lambda$.*

Proof. (1) From (3) of Lemma 5.1 and the fact that $\bar{A}(x) \geqslant A(x) = \lambda$, we have $\bar{A}(x) > \lambda$. It follows from (2) and (1) of Lemma 5.1 that $\bar{A} = A^d$. (2) A^d is closed iff $(A^d)'$ is open. When $A^d(x) = 0$, in view of (1) and (3) of Lemma 5.1, it follows that $(A^d)'$ is the required B^*. The proof is thus completed. (3) From (3) of Lemma 5.1, $A^d(x) = 0$ iff $\bar{A}(x) = \lambda = A(x)$. This implies that there exists a closed F such that $F(x) = \lambda$, or equivalently, there exists an open B such that $B(x) = 1 - \lambda$.

The following is a generalization of Yang's result [10, p. 56], the proof of which in fuzzy topology is more complicated than usual.

THEOREM 5.2. *The derived set of each fuzzy set is closed iff the derived set of each point is closed.*

Proof. The necessity is obvious. We shall now show its sufficiency. Let H be an arbitrary fuzzy set. In the light of the corollary of Theorem 5.1, in order to show that $H^d \equiv D$ is closed, it suffices to show that for an arbitrary accumulation point x_λ of D, $x_\lambda \in D$. Since $x_\lambda \in \bar{D} = \overline{(H^d)} \subset \overline{(\bar{H})} = \bar{H}$, by Theorem 4.1′, x_λ is an adherence point of H. If $x_\lambda \notin H$ then x_λ is an accumulation point of H, i.e., $x_\lambda \in D$. We may assume that $x_\lambda \in H$, i.e., $\lambda \leqslant H(x) = \rho$ without loss of generality. Consider the fuzzy point $x_\rho = A$ and the two possibilities concerning A^d: (*I*) Let $A^d(x) = \rho_1 > 0$. From Lemma 5.1, $\rho_1 > A(x) = H(x)$, and hence $x_{\rho_1} \notin H$. But since $x_{\rho_1} \in A^d \subset \bar{A} \subset \bar{H}$, x_{ρ_1} is an accumulation point of H, $x_{\rho_1} \in D$. Moreover, $\lambda \leqslant H(x) < \rho_1$, i.e., $x_\lambda \in D$. (*II*) Let $A^d(x) = 0$. Let B be an arbitrary open Q-neighborhood of x_λ. We shall show that B and H are quasi-coincident at some point different from x and hence we know that $x_\lambda \in D$. In view of (2) of Proposition 5.1, there exists an open set B^* such that $B^*(x) = 1$ and for $y \neq x$, $B^*(y) = (\bar{A})'(y)$. Let $C = B \cap B^*$; then $C(x) = B(x) > 1 - \lambda$, and C is also an open Q-neighborhood of x_λ. Because x_λ is an accumulation point of D, C is quasi-coincident with D at some point z, i.e., $D(z) + C(z) > 1$. Owing to the fact that D is the union of all the accumulation points of H, there is an accumulation point z_μ such that $\mu + C(z) > 1$. Therefore C is also an open Q-neighborhood of z_μ. The proof will be carried out, according to the three possible cases concerning z_μ, as follows: (1) When $z = x$ and $\mu \leqslant \rho$, then $z_\mu \in H$. But since z_μ is an accumulation point of H, the Q-neighborhood of x_μ (and hence B) and H are quasi-coincident at some point different from $z = x$. (2) When $z = x$ and $\mu > \rho$, then $z_\mu \notin H$. From (3) of Proposition 5.1, there is an open set $\tilde{B}$ such that $\tilde{B}(x) = 1 - \rho > 1 - \mu$. Therefore $G = C \cup \tilde{B}$ is also an open Q-neighborhood of z_μ and G and H are quasi-coincident at some point w. Since $G(x) \leqslant \tilde{B}(x) = 1 - \rho = 1 - H(x)$, $w \neq x$, i.e., G (and hence C and B) and H are quasi-coincident at some point different from x, (3) Wnen $z \neq x$, by (2) of Proposition 5.1, $B^*(z) = (\bar{A})'(z)$. But $(\bar{A})' = (A')^0$ (Theorem 4.2). Since $(A')^0(z) = B^*(z) \geqslant C(z) > 1 - \mu$, by Definition 4.1, there exists an open set $\tilde{B} \subset A'$ such that $(A')^0(z) \geqslant \tilde{B}(z) > 1 - \mu$. Therefore $B \cap \tilde{B}$ is also an open Q-neighborhood, which is quasi-coincident with H at some point w. Since $\tilde{B} \subset A'$, $\tilde{B}(x) \leqslant 1 - A(x) = 1 - H(x)$, $w \neq x$, we have thus proved that $B \cap \tilde{B}$ (and hence B) is quasi-coincident with H at some point different from x.

6. Separation Axioms

Definition 6.1. $(X, \mathscr{T})$ is called a fuzzy quasi-T_0-space iff for every $x \in X$, and $\lambda \neq \mu$, λ, $\mu \in [0, 1]$, either $x_\lambda \notin \bar{x}_\mu$ or $x_\mu \notin \bar{x}_\lambda$.

Since when $\mu < \lambda$, $x_\mu \in \bar{x}_\lambda$, we have: $(Z, 3)$ is a quasi-T_0 space iff, for every $x \in S$, and $0 < \mu < \lambda \leqslant 1$, $x_\lambda \in \bar{x}_\mu$.

Definition 6.2. $(X, \mathscr{T})$ is called a fuzzy T_0 space iff, for any two fuzzy points e and d such that $e \neq d$, either $e \notin \bar{d}$ or $d \notin \bar{e}$.

Definition 6.3. $(X, \mathscr{T})$ is called a fuzzy T_1 space iff every fuzzy point is a closed set.

The following implications are obvious: $T_1 \Rightarrow T_0 \Rightarrow$ quasi-T_0. Every ordinary (crisp) topological space vacuously satisfies condition of being quasi-T_0 and hence the quasi-T_0-separation is a particularity in fuzzy topology.

Let $(X, \mathscr{T})$ be a quasi-T_0 space, let $x \in X$ and $\Delta = (\rho_1, \rho_2)(0 \leqslant \rho_1 < \rho_2 < 1)$; then there exists $B \in \mathscr{T}$ such that $B(x) \in \Delta$. In fact, let $\lambda = 1 - \rho_1$, $\mu = 1 - \rho_2$; then $\lambda > \mu > 0$. Since $(X, \mathscr{T})$ is a quasi-T_0 space, $x_\lambda \notin \bar{x}_\mu$ and hence there exists some open Q-neighborhood b $(B(x) > 1 - \lambda = \rho_1)$ which is not quasi-coincident with x_μ, i.e., $B(x) \leqslant 1 - \mu = \rho_2$. Hence $B(x) \in \Delta$.

The following property concerning quasi-T_0 spaces can be sharpened as follows.

Theorem 6.1. *$(X, \mathscr{T})$ is a quasi-T_0 space off for every $x \in X$ and $\rho \in [0, 1]$, there exists a $B \in \mathscr{T}$ such that $B(x) = \rho$.*

Proof. Necessity. When $\rho = 0$, it suffices to take $B = \Phi$; when $0 < \rho \leqslant 1$, take a strictly monotonic increasing sequence of positive real numbers converging to ρ. Let $\Delta_n = (\rho_n, \rho_{n+1}]$ $(n = 1, 2, \ldots)$; from the property just proved above there exist $B_n \in \mathscr{T}$ such that $B_n^{(x)} \in \Delta_n$ for each n. Therefore $B = \bigcup_{n=1}^{\infty} B_n$ is open and $B(x) = \rho$.

Sufficiency. For two fuzzy points x_λ and x_μ with $\mu < \lambda$, there exists from hypothesis an open set B such that $B(x) = 1 - \mu > 1 - \lambda$. It is evident that B is an open Q-neighborhood of x_λ but is not quasi-coincident with $\{x_\mu\}$. Hence it follows from Theorem 4.1′ that $x_\lambda \notin \bar{x}_\mu$.

Theorem 6.2. *$(X, \mathscr{T})$ is a T_0 space iff $(X, \mathscr{T})$ is quasi-T_0 and for any two distinct points x, y in X and for any ρ, $\nu \in [0, 1]$; then there exists $B \in \mathscr{T}$ such that $B(x) = \rho$ and $B(y) > \nu$, or $B(x) > \rho$ and $B(y) = \nu$.*

Proof. Necessity. When $(X, \mathscr{T})$ is T_0, it is also quasi-T_0. For $x \neq y$ and $\rho, \nu \in [0, 1)$, putting $\lambda = 1 - \rho$ and $\mu = 1 - \nu$, we obtain two distinct fuzzy points x_λ and y_μ. If $x_\lambda \notin \bar{y}_\mu$, there exists an open Q-neighborhood B_1 ($B_1(x) > 1 - \lambda = \rho$) which is not quasi-coincident with $\{y_\mu\}$, i.e., $B_1(y) \leqslant 1 - \mu = \nu$. In view of Theorem 6.1, there is $B_2 \in \mathscr{T}$ such that $B_2(y) = \nu$. Then the fuzzy open set $B = B_1 \cup B_2$ is the required one. If $y_\lambda \notin \{\bar{x}_\lambda\}$, the argument can be carried out in a similar way.

Sufficiency. Since $(X, \mathscr{T})$ is quasi-T_0, it suffices to consider the separation of two fuzzy points x_λ and y_μ with $x \neq y$. Putting $\rho = 1 - \lambda$, $\nu = 1 - \mu$, from the hypothesis, we may assume that there exists $B \in \mathscr{T}$ such that $B(x) = \rho$ and $B(y) > \nu$. Then B is a Q-neighborhood of y_μ which is not quasi-coincident with $\{x_\lambda\}$. Hence $y_\mu \notin \bar{x}_\lambda$.

THEOREM 6.3. *$(X, \mathscr{T})$ is a T_1 space iff, for each $x \in X$ and each $\lambda \in [0, 1]$, there exists $B \in \mathscr{T}$ such that $B(x) = 1 - \lambda$ and $B(y) = 1$ for $y \neq x$.*

Proof. Necessity. When $\lambda = 0$, it suffices to take $B = X$. When $\lambda > 0$, x_λ being a fuzzy point is a closed set by hypothesis, then $B = 1 - x_\lambda$ is the required open set.

Sufficiency. Let x_λ be an arbitrary fuzzy point. Then, by hypothesis, there exists a $B \in \mathscr{T}$ such that $B(x) = 1 - \lambda$ and $B(y) = 1$ for $y \neq x$. It follows that $x_\lambda = B'$ is closed.

DEFINITION 6.4. $(X, \mathscr{T})$ is fuzzy T_2 (Hausdorff) space iff, for any two fuzzy points e and d satisfying supp $e \neq$ supp d, there exist Q-neighborhoods B and C of e and d, respectively, such that $B \cap C = \Phi$.

PROPOSITION 6.1. *Let $(X, \mathscr{T})$ be a T_2 space, then any accumulation of a fuzzy point y_μ in $(X, \mathscr{T})$ is of the form y_λ $(\lambda > \mu)$.*

Proof. When $\lambda \leqslant \mu$, $y_\lambda \in y_\mu$, but since any Q-neighborhood of y_λ can be quasi-coincident with y_μ at most at y, y_λ is not an accumulation point of y_μ. When $x \neq y$, from the property of being fuzzy T_2, there exist Q-neighborhoods B and B_1 of x_λ and y_μ, respectively, such that $B \cap B_1 = \Phi$. But since $B_1(y) > 1 - \mu \geqslant 0$, $B(y) = 0$, i.e., B is not quasi-coincident with y_μ at y and hence x_λ is not an accumulation point of y_μ. We have thus proved that the only possible form of an accumulation point of y_μ is of the type y_λ with $\lambda > \mu$.

Since the separation axiom, T_2 is concerned only with those fuzzy points with different supports, it is possible, as the following example shows, that a T_2 space need not be quasi-T_0, to say nothing of being T_1.

EXAMPLE. Let $X = \{y, z\}$, when $y \neq z$. Let $\mathscr{T}$ be the fuzzy topology on X which has $\mathscr{B} = \{y_\lambda \mid \lambda \in (2/3, 1]\} \cup \{z_\lambda \mid \lambda \in (0, 1]\} \cup \{\Phi\}$ as a base. Obviously,

$(X, \mathscr{T})$ is a fuzzy T_2 space. But since there is no $\mathscr{T}$-open set which takes the value $\frac{1}{2}$ at y, in view of Theorem 6.1$\cdot$ $(X, \mathscr{T})$ is not quasi-T_0.

THEOREM 6.4. *If $(X, \mathscr{T})$ is both T_2 and quasi-T_0, then it is also T_1.*

Proof. Let y_μ be an arbitrary fuzzy point. From Proposition 6.1, an accumulation point, if any, of y_μ is of the form y_λ $(\lambda > \mu)$. In the light of the property of $(X, \mathscr{T})$ being T_0 and Theorem 6.1, there exists a $B \in \mathscr{T}$ such that $B(y) = 1 - \mu > 1 - \lambda$, i.e., B is a Q-neighborhood of y_λ and is not quasi-coincident with y_μ. Hence y_λ $(\lambda > \mu)$ cannot be an accumulation point of y_μ and therefore y_μ has no accumulation point. Owing to the corollary of Theorem 5.1, y_μ is closed. This means that $(x, \mathscr{T})$ is T_1.

Since the derived set of every fuzzy point in a T_1 space is obviously Φ, we obtain from Theorem 5.2 the following.

THEOREM 6.5. *The derived set of every fuzzy set on a T_1 space is closed.*

7. Ω-ACCUMULATION POINTS: LINDELÖF PROPERTY

DEFINITION 7.1. A fuzzy set A in X is said to be uncountable iff supp A is an uncountable subset of X.

PROPOSITION 7.1. *Every uncountable fuzzy set on a C_{II} space has an uncountable number of accumulation points the supports of which are mutually different.*

This proposition is a corollary of Theorem 7.1, below.

DEFINITION 7.2. A fuzzy point e is called an Ω-accumulation point of A iff the set consisting of all the points at each of which every Q-neighborhood of e and A are quasi-coincident is uncountable.

DEFINITION 7.3. A fuzzy set D in $(X, \mathscr{T})$ is said to have the Lindelöf property iff every open cover of D has a countable subcover. $(X, \mathscr{T})$ is called a hereditarily Lindelöf space iff every fuzzy set in X has the Lindelöf property. $(X, \mathscr{T})$ is called a Lindelöf space iff X has the Lindelöf property.

PROPOSITION 7.2. *Every C_{II} space is a hereditarily Lindelöf space.*

The proof is similar to that of Theorem 1.15 of [10], p. 49].

THEOREM 7.1. *Let $(X, \mathscr{T})$ be a hereditarily Lindelöf space. Let A be an uncountable fuzzy set on X. Let $B = \bigcup \{e \mid e$ is an Ω-accumulation point of A such that $e \in A\}$. Then $X_1 =$ supp $A -$ supp B is a countable subset of X.*

Proof. If X_1 is non-empty, taking any point $y \in X_1$, and putting $\lambda = A(y) > 0$, we know the fuzzy point $y_\lambda \in A$. Since $y \notin \text{supp } B$, y_λ is not an Ω-accumulation point of A. Then there exists some open Q-neighborhood B_y such that the set of all points at each of which B_y is quasi-coincident with A is at most countable, i.e., the set $\{z \mid z \in X \text{ such that } B_y(z) > 1 - A(z)\}$ is at most countable. Now put $\mathscr{B} = \{B_y \mid y \in X_1\}$ and define D as follows:

$$D(y) = B_y(y), \qquad \text{for} \quad y \in X_1,$$
$$= 0, \qquad \text{for} \quad y \notin X_1,$$

It is clear that $\mathscr{B}$ is an open cover of D. Since D has the Lindelöf property, there are a countable number of $B_{y_i} \in \mathscr{B}$, $i = 1, 2, \ldots$, such that $W = \bigcup_{i=1}^{\infty} B_{y_i} \supset D$. The set $\{z \in X \mid W(z) > 1 - A(z)\}$ is evidently at most countable. On the other hand, for every $y \in X_1$, $W(y) \geqslant D(y)$, and B_y is a Q-neighborhood of y_λ, i.e., $B_y(y) > 1 - \lambda = 1 - A(y)$. Therefore $W(y) > 1 - A(y)$ and X_1 is at most countable.

8. Subspaces

Definition 8.1. Let $(X, \mathscr{T})$ be a fuzzy topological space, and $Y \subset X$; then we call the family $\mathscr{U}$, defined by $\mathscr{U} = \{A \mid Y \mid A \in \mathscr{T}\}$, which is obviously a fuzzy topology for Y, the relative fuzzy topology, or the relativization of $\mathscr{T}$ to Y. Such a fuzzy topological space $(Y, \mathscr{U})$ is called a subspace of $(X, \mathscr{T})$. More formally, a fuzzy topological space $(Y, \mathscr{U})$ is called a subspace of another fuzzy topological space $(X, \mathscr{T})$ iff $Y \subset X$ and $\mathscr{U}$ is the relativization of $\mathscr{T}$ to Y. A $\mathscr{U}$-open (resp. $\mathscr{U}$-closed) set is also called a relative open (resp. $\mathscr{U}$-closed) set on Y.

Since the terms "space" and "subspace" often occur simultaneously in an argument, in order to simplify the exposition, we shall adopt the following conventions under conditions such that no confusion can arise: (1) For a subspace $(Y, \mathscr{U})$ we often omit the relative topology $\mathscr{U}$ and simply say the subspace Y. (2) A fuzzy set A on Y is automatically considered as a fuzzy set on X in the sense that A takes the value 0 on $X \backslash Y$. And conversely, any fuzzy set on X, taking value 0 on $X \backslash Y$, can also be considered a fuzzy set on Y. (3) For a fuzzy set A on the subspace $(Y, \mathscr{U})$, the closures of A with respect to $\mathscr{U}$ and $\mathscr{T}$ are respectively denoted by $\text{cl}_Y A$ and $\text{cl}_X(A)$.

Proposition 8.1. *Let $(Y, \mathscr{U})$ be a subspace of $(X, \mathscr{T})$, and $A \in [0, 1]^Y$; then* (1) *A is $\mathscr{U}$-closed iff there is a $\mathscr{T}$-closed set B such that $A = B \mid Y$.* (2) *A fuzzy point y_λ in Y is an accumulation point of A with respect to $\mathscr{U}$ iff y_λ is an accumulation point of A with respect to $\mathscr{T}$.* (3) $\text{cl}_Y A = Y \cap \text{cl}_X A$.

Proof. Property (1) follows directly from the definition of the relative topology and the operation of complementation.

(2) Since the Q-neighborhood system of y_λ with respect to $\mathscr{U}$ is obtained from that of y_λ with respect to $\mathscr{T}$ by restriction to Y, noting that A is a fuzzy set on Y, we can directly obtain the result from the definition of accumulation point.

Property (3) follows directly from Theorem 5.1, (2), and the definition of derived set.

9. Separation

DEFINITION 9.1. Two fuzzy sets A_1 and A_2 in $(X, \mathscr{T})$ are said to be separated iff there exist $U_i \in \mathscr{T}$ $(i = 1, 2)$ such that $U_i \supset A_i$ $(i = 1, 2)$ and $U_1 \cap A_2 = \Phi = U_2 \cap A_1$.

DEFINITION 9.1′. Two fuzzy sets A_1 and A_2 in a fuzzy topological space $(X, \mathscr{T})$ are said to be Q-separated iff there exist $\mathscr{T}$-closed sets H_i $(i = 1, 2)$ such that $H_i \supset A_i$ $(i = 1, 2)$ and $H_1 \cap A_2 = \varnothing = H_2 \cap A_1$. It is obvious that A_1 and A_2 are Q-separated iff $\bar{A}_1 \cap A_2 = \Phi = \bar{A}_2 \cap A_1$.

PROPOSITION 9.1. *Separation and Q-separation do not imply each other.*

This is obvious by observing the following example.

EXAMPLE I. Let $X = X_1 \cup X_2$, where X_1 and X_2 are (crisp) non-empty sets such that $X_1 \cap X_2$ is empty. Let $\lambda_i \in [0, 1]$ and define

$$
\begin{aligned}
C_{\lambda_1,\lambda_2}(x) &= \lambda_1, &&\text{for } x \in X_1,\\
&= \lambda_2, &&\text{for } x \in X_2.
\end{aligned}
$$

Obviously, C_{λ_1,λ_2} is a fuzzy set on X.

(1) Let $\mathscr{B} = \{X, \Phi, C_{2/3,0}, C_{0,2/3}\}$, let $\mathscr{T}_1$ be the topology for X, which takes $\mathscr{B}$ as a base, and let $A_1 = C_{1/2,0}$ and $A_2 = C_{0,1/2}$. Then A_1 and A_2 are separated with respect to $\mathscr{T}_1$. But $\bar{A}_1 = C_{1,1/3}$ and $\bar{A}_1 \cap A_2 \neq \Phi$, A_1 and A_2 are not Q-separated.

(2) Let $\mathscr{B}_1 = \{X, \Phi C_{1/2,1}, C_{1,1/2}\}$ and let $\mathscr{T}_2$ be the topology for X, which takes $\mathscr{B}_1$ as a base, then A_1 and A_2 are $\mathscr{T}_2$-closed, do not intersect and hence are, of course, Q-separated. But they are obviously not separated.

Separation and Q-separation coincide in (crisp) general topology. Moreover, we have

PROPOSITION 9.2. *Let D_1 and D_2 be two crisp sets of a fuzzy topological space $(X, \mathscr{T})$. Then D_1 and D_2 are separated iff they are Q-separated.*

Proof. Let D_1 and D_2 be Q-separated, then, by definition, there exist closed sets $H_i \supset D_i$ and $H_i \cap D_j = \varnothing$ $(i \neq j,\, i, j = 1, 2)$. Therefore H_i has to take the value 1 on the subset D_i of X, and the value 0 on the subset D_j of X. And hence $H_i' = G_i$ takes the value 0 on D_i, and the value 1 on D_j. Then we have $G_i \supset D_j$ and $G_i \cap D_j = \varnothing$. This shows that D_1 and D_2 are separated. In order to show that separation of D_1 and D_2 implies their Q-separation, it suffices to interchange the positions of the two terms "open sets" and "closed sets" occurring in the above argument.

We recall that $A_0 = \operatorname{supp} A$.

THEOREM 9.1. *Two fuzzy sets A and B are Q-separated iff* $A_0 \cap B_0 = \varnothing$, $\mathrm{cl}_{A_0 \cup B_0} A = \mathrm{cl}_{A_0} A$, $\mathrm{cl}_{A_0 \cup B_0} B = \mathrm{cl}_{B_0} B$.

Proof. Necessity. Let A and B be Q-separated, then $B \cap \mathrm{cl}_X A = \varnothing$. Then $A_0 \cap B_0 = \varnothing$ and $B_0 \cap \mathrm{cl}_X A = \varnothing$. From (3) of Proposition 8.1, we have

$$\mathrm{cl}_{A_0 \cup B_0} A = (A_0 \cup B_0) \cap \mathrm{cl}_X A = A_0 \cap \mathrm{cl}_X A = \mathrm{cl}_{A_0} A.$$

Similarly we can prove that $\mathrm{cl}_{A_0 \cup B_0} B = \mathrm{cl}_{B_0} B$.

Sufficiency. From the hypotheses and Proposition 8.1, it follows that $B \cap \mathrm{cl}_{A_0 \cup B_0} A = B \cap \mathrm{cl}_{A_0} A \subset B_0 \cap A_0 = \varnothing$. Therefore

$$B \cap \mathrm{cl}_X A = B \cap (B_0 \cup A_0) \cap \mathrm{cl}_X A = B \cap \mathrm{cl}_{A_0 \cup B_0} A = \varnothing.$$

Similarly, $A \cap \mathrm{cl}_X B = \varnothing$. That is, A and B are Q-separated.

PROPOSITION 9.3. *Two fuzzy sets A and B are Q-separated iff $\mathrm{cl}_{A_0 \cup B_0} A$ and $\mathrm{cl}_{A_0 \cup B_0} B$ are Q-separated.*

Proof. The sufficiency holds obviously. We shall now show its necessity. Let $C = \mathrm{cl}_{A_0 \cup B_0} A$, $D = \mathrm{cl}_{A_0 \cup B_0} B$. From Theorem 9.1, $C = \mathrm{cl}_{A_0} A$, $D = \mathrm{cl}_{B_0} B$, and hence $C_0 = A_0$, $D_0 = B_0$. Then $C_0 \cap D_0 = \varnothing$. Moreover, $\mathrm{cl}_{C_0} C \subset \mathrm{cl}_{C_0 \cup D_0} C = \mathrm{cl}_{C_0 \cup D_0}(\mathrm{cl}_{A_0 \cup B_0} A) = \mathrm{cl}_{A_0 \cup B_0} A = \mathrm{cl}_{A_0} A = \mathrm{cl}_{C_0} A \subset \mathrm{cl}_{C_0} C$, i.e., $\mathrm{cl}_{C_0} C = \mathrm{cl}_{C_0 \cup B} C$. Similarly, we have $\mathrm{cl}_{D_0} D = \mathrm{cl}_{C_0 \cup D_0} D$. It follows from Theorem 9.1 that C and D are Q-separated.

DEFINITION 9.2. Let A and B be two fuzzy sets in X. The operation $A \sim B$ is defined as follows:

$$\begin{aligned} (A \sim B)(x) &= A(x), && \text{for } x \in \{ y \in X \mid A(y) > B(y)\}, \\ &= 0, && \text{otherwise.} \end{aligned}$$

The operation "$\sim$" defined above reduces evidently to the difference "$-$" between two ordinary sets in ordinary set theory. In general topology, there is a theorem concerning separation of sets, which says as follows: If Y and Z are subsets of a topological space X and both Y and Z are closed or both are open, then $Y\backslash Z$ is separated from $Z\backslash Y$ (cf. [10 p. 52. Theorem 1.17]). This result is no longer true in fuzzy topology, as is shown by the following example. This is the only theorem in [10, Chap. I], which cannot be generalized in fuzzy topology.

EXAMPLE 2. Let $X = X_1 \cup X_2$, X_1 and X_2 being non-empty sets with empty intersection. The fuzzy set C_{λ_1,λ_2} in X is defined as in Example 1. Let $\mathscr{T} = \{C_{\lambda,\mu} \mid \lambda, \mu \in [0, \frac{1}{2}]\} \cup \{X\}$, which is obviously a fuzzy topology for X. Let $Y = C_{2/3,1}$, $Z = C_{1,2/3}$. Then both Y and Z are $\mathscr{T}$-closed, but $Y \sim Z = C_{0,1}$ and $Z \sim Y = C_{1,0}$ are not separated, since the only open set containing $Y \sim Z$ is X. They are also not Q-separated, because $\overline{(Y \sim Z)} = C_{1/2,1}$, which intersects $Z \sim Y$.

PROPOSITION 9.4. *Let A and B be two fuzzy sets in X such that $A \cup B = X$; then* (1) *$A \sim B$ and $B \sim A$ are crisp sets on X; and* (2) *$A \sim B$ and $B \sim A$ are separated iff they are Q-separated.*

Proof. (1) Suppose that $(A \sim B)(x) > 0$. By the definition of $A \sim B$, $A(x) > B(x)$. But $\max(A(x), B(x)) = 1$ and hence $A(x) = 1 = (A \sim B)(x)$. Therefore $A \sim B$ takes only the value 0 and 1 on X. By definition, $A \sim B$ is a crisp set in X. Similarly, $B \sim A$ can be proved to be crisp.

(2) Noting Proposition 9.2, we see at once the equivalence of separation and Q-separation between the crisp sets $A \sim B$ and $B \sim A$.

THEOREM 9.2. *Let A, B and D be fuzzy sets in a* fts. $(X, \mathscr{T})$ *such that $A \cup B = X$, and $A \sim B$ and $B \sim A$ are Q-separated (or separated); then*

$$\mathrm{cl}_X D = \mathrm{cl}_{A_0}(D \cap A) \cup \mathrm{cl}_{B_0}(D \cap B).$$

Proof. In view of Proposition 9.4, we need only consider the case in which $A \sim B$ and $B \sim A$ are Q-separated.

It is evident that $\mathrm{cl}_X D \supset \mathrm{cl}_{A_0}(D \cap A) \cup \mathrm{cl}_{B_0}(D \cap B)$. We shall now show that $\mathrm{cl}_X D \subset \mathrm{cl}_{A_0}(D \cap A) \cup \mathrm{cl}_{B_0}(D \cap B)$. From $A \cup B = X$, we have $A \cup (B \sim A) = X$. Then $D = (D \cap A) \cup (D \cap (B \sim A))$, $A \cap \mathrm{cl}_X D = (A \cap \mathrm{cl}_Y(D \cap A)) \cup (A \cap \mathrm{cl}_X(D \cap (B \sim A)))$. The first term of the right side is obviously contained in $A_0 \cap \mathrm{cl}_X(D \cap A) = \mathrm{cl}_{A_0}(D \cap A)$; We shall now show that the second term of this side is contained in $\mathrm{cl}_{B_0}(A \cap D)$. In fact, $\mathrm{cl}_X(B \sim A)$ and $A \sim B$ do not intersect. Then for every $x \in X$ such $A(x) > B(x)$, $\mathrm{cl}_X(B \sim A)$ takes the value 0. Therefore it may take the non-zero value only at those points x, where $B(x) \geqslant A(x)$. But when $B(x) \geqslant A(x)$, it is easily seen that $B(x) = 1$, because $A \cup B = Z$. This means that $\mathrm{cl}_X(B \sim A)$ may take non-zero values at points x, where $(Bx) = 1$, and hence $\mathrm{cl}_X(B \sim A) \subset B_0$.

Then we have

$$\mathrm{cl}_X(D \cap (B \sim A)) \subset \mathrm{cl}_X(D \cap B) \cap B_0 = \mathrm{cl}_{B_0}(D \cap B),$$

$$A \cap \mathrm{cl}_X D \subset \mathrm{cl}_{A_0}(D \cap A) \cup \mathrm{cl}_{B_0}(D \cap B).$$

Similarly we can prove

$$B \cap \mathrm{cl}_X D \subset \mathrm{cl}_{A_0}(D \cap A) \cup \mathrm{cl}_{B_0}(D \cap B),$$

consequently,

$$\mathrm{cl}_X D = (A \cap \mathrm{cl}_X D) \cup (B \cap \mathrm{cl}_X D) \subset \mathrm{cl}_{A_0}(D \cap A) \cup \mathrm{cl}_{B_0}(D \cap B).$$

THEOREM 9.3. *Let A and B be two fuzzy sets in $(Z, \mathscr{T})$ such that $A \cup B = Z$ and $A \sim B$ and $B \sim A$ are Q-separated or separated. Let D be a fuzzy set in $(Z, \mathscr{T})$ such that $D \cap A$ and $D \cap B$ are relative closed (open) in the subspace A_0 and B_0, respectively; then D is $\mathscr{T}$-closed (resp. $\mathscr{T}$-open).*

Proof. The closedness of D follows directly from Theorem 9.2. Since the lattice $[0, 1]^X$ is not a Boolean Algebra, especially because $A \cap A'$ need not be $\varnothing$, $A \cup A'$ need not be X, the openness of D can hardly be derived from the closedness of D' by taking complements. The following is a proof of the openness of D, which has some features of lattice theory. Let $X_1 = \{x \in X \mid B(x) = 0\}$, $X_2 = \{x \in X \mid 1 = A(x) > B(x) > 0\}$, $X_3 = \{x \in X \mid 1 = B(x) = A(x)\}$, $X_4 = \{x \in X \mid 1 = B(x) > A(x) > 0\}$ and $X_5 = \{x \in X \mid A(x) = 0\}$ since $A \cup B = X$, at least one of $A(x)$ and $B(x)$ is 1. Moreover, it is obvious that $X = \bigcup_{i=1}^{5}, X_i$, and $\{X_1, X_2, X_3, X_4, X_5\}$ is a pairwise disjoint collection (some X_i may be empty). Let f be a fuzzy set in X. Let the restriction of f to X_i be denoted by f_i. Then f may be also denoted by $(f_1, f_2, f_3, f_4, f_5)$. For a fuzzy set f which takes the value 0 or 1 on X_i, the corresponding component in the above representation of f will be directly written as 0 or 1. Using this representation and noting the definition of X_i, we see easily that $A = (1, 1, 1, a, 0)$, $B = (0, b, 1, 1,)$, where a and b are restrictions of A and B to X_4 and to X_2, respectively (a and b need not be constant). Let $D = (d_1, d_2, d_3, d_4, d_5)$; then $D \cap A = (d_1, d_2, d_3, a \cap d_4, 0)$, $D \cap B = (0, d_2 \cap b, d_3, d_4, d_5)$. By hypothesis, $D \cap A$ and $D \cap B$ are relative open sets on $A_0 = X_1 \cup X_2 \cup X_3 \cup X_4$ and $B_0 = X_2 \cup X_3 \cup X_4 \cup X_5$, respectively, and hence there exist $\mathscr{T}$-open sets V_1 and V_2 such that $V_1 \cap D_0 = D \cap A$, respectively. Let g denote the restriction of V_1 to X_5, and let h denote the restriction of V_2 to X; then we have $V_1 = (d_1, d_2, d_3, a \cap d_4, g)$, $V_2 = (h, b \cap d_2, d_3, d_4, d_5)$. On the other hand, from Proposition 9.4, we may assume that $A \sim B$ and $B \sim A$ are separated. Then there exist $\mathscr{T}$-open sets G_1, G_2 such that $G_1 \supset A \sim B$, $G_2 \supset B \sim A$ and $G_1 \cap (B \sim A) = \varnothing = G_2 \cap (A \sim B)$. But it is easily seen that $A \sim B = (1, 1, 0, 0, 0)$, $B \sim A = (0, 0, 0, 1, 1)$ and therefore $G_1 = (1, 1, s, 0, 0)$ and $G_2 = (0, 0, t, 1, 1)$. We have thus reduced our problem to a problem concerning only lattice operations: Given V_1, V_2, G_1, G_2 and D as above, express D in terms of a combination of unions and intersections V of those V_i's and G_i's. This can be done as follows. Let $V = V_1 \cup V_2 = (d_1 \cup h, d_2, d_3, d_4, d_5 \cup g), F_1 = G_1 \cup V_2 = (1, 1, s \cup d_3, d_4, d_5)$, $F_2 = G_2 \cup V_1 = (d_1, d_2, t \cup d_3, 1, 1) F_1 \cap F_2 = (d_1, d_2, d_3 \cup (s \cap t), d_4, d_5)$; finally, we have the required expression:

$$(F_1 \cap F_2) \cap V = (d_1, d_2, d_3, d_4, d_5) = D.$$

By replacing "open sets" by "closed sets" and separation by Q-separation, in the foregoing argument, a similar proof for the closedness of D is obtained.

10. Connectedness

Definition 10.1. A fuzzy set D in $(X, \mathscr{T})$ is called disconnected iff; there exist two non-empty sets A and B in the subspace D_0 (i.e., supp D) such that A and B are Q-separated and $D = A \cup B$. A fuzzy set is called connected iff it is not disconnected.

Lemma 10.1. *A fuzzy set D is disconnected iff there are relative closed sets in the subspace D_0 such that $A \cap D \neq \Phi$, $B \cap D \neq \Phi$, $A \cap B = \Phi$* and $A \cup B \supset D$.

Proof. The necessity can be obtained by means of $D_0 = A_0 \cup B_0$ and Proposition 9.3. We shall now show its sufficiency as follows. For the given closed sets A and B, let $\tilde{A} = A \cap D$, $\tilde{B} = B \cap D$; it is evident that $\tilde{A} \cup \tilde{B} = (A \cup B) \cap D = D$, $\tilde{A} \neq \Phi$, $\tilde{B} \neq \Phi$, since it is easily seen that A and B are Q-separated and $\tilde{A} \subset A$, $\tilde{B} \subset B$, $\tilde{A}$ and $\tilde{B}$ are also Q-separated in the subspace D_0.

Theorem 10.1. *Let D be a connected set in $(X, \mathscr{T})$; then $\mathrm{cl}_X D$ and $\mathrm{cl}_{D_0} D$ are also connected.*

Proof. Suppose $\mathrm{cl}_X D = E$ is disconnected. Then, from Lemma 10.1, there are relative closed set A and B in the subspace E_0 such that $A \cup B \supset E$, $A \cap E \neq \Phi$, $B \cap E \neq \Phi$, $A \cap B = \Phi$. Obviously, $A \cup B \supset D$. From the connectedness of D, we may assume $A \cap D = \Phi$ (for the case where $B \cap D = \Phi$, a similar argument holds). That is, $D \subset B$. It follows that $E = E \cap E_0 = (\mathrm{cl}_X D) \cap E_0 = \mathrm{cl}_{E_0} D \subset \mathrm{cl}_{D_0} B = B$, but since $A \cap B = \Phi$, $A \cap E = \Phi$, which is a contradiction.

The connectnesness of $\mathrm{cl}_{D_0} D$ can be proved analogously by indirect method. (At this time take $E = \mathrm{cl}_{D_0} D$ and note that $E_0 = (\mathrm{cl}_{D_0} D)_0 = D_0$.)

Theorem 10.2. *Let $\mathscr{A}$* be a family of connected fuzzy sets in a fts $(X, \mathscr{T})$.

If no two members of $\mathscr{A}$ are Q-separated in the subspace $(\bigcup \mathscr{A})_0$, then $\bigcup \mathscr{A}$ is connected.

Proof. For simplicity, let $D = \bigcup \mathscr{A}$. Suppose D is disconnected. From Lemma 10.1, there are relative closed sets A and B in D_0 such that $A \cup B \supset D$, $A \cap D \neq \Phi$, $B \cap D \neq \Phi$, $A \cap B = \Phi$. Let H be any member of $\mathscr{A}$. Since B is connected and $A \cup B \supset H$, we may assume that $A \cap H = \Phi$ (For the case in which $H \cap B = \Phi$, the argument can be carried out in a similar way.) Then $H \subset B$. We have thus proved that any member of $\mathscr{A}$ is contained either in A or in B. Moreover, since $A \cap D \neq \Phi$ and $B \cap D \neq \Phi$, we may suppose $H_i \in \mathscr{A}$ $(i = 1, 2)$ such that $H_1 \subset A$, $H_2 \subset B$. In view of $\text{cl}_{D_0} H_1 \subset \text{cl}_{D_0} A = A$, $\text{cl}_{D_0} H_2 \subset B$, it is obvious that H_1 and H_2 are Q-separated in D_0, which is a contradiction.

DEFINITION 10.2. Let D be a fuzzy set on $(X, \mathscr{T})$. The maximal connected fuzzy set contained in D is called a component of D.

THEOREM 10.3. *Let D be a fuzzy set on $(X, \mathscr{T})$, each connected fuzzy set contained in D is contained in some component of D, and any two distinct components of D are Q-separated in the subspace D_0.*

Proof. The last part of the theorem is a direct consequence of Theorem 10.2 and the maximality of a component. We shall now prove the first part of the theorem. When $D = \Phi$, the case is trivial. Suppose $D \neq \Phi$, and A is a connected set contained in D. First suppose $A \neq \Phi$. Let $\mathscr{B} = \{B \mid B$ is connected such that $A \subset B \subset D\}$. Any two members of $\mathscr{B}$ contain a non-empty set A and hence are not Q-separated. By Theorem 10.2, $\bigcup \mathscr{B} = E$ is connected. It is obvious that E is a component of D. For the case where $A = \Phi$, since $D \neq \Phi$, take any fuzzy point $e \in D$; from the definition of a connected set, it is evident that e is connected. According to this proof, e and hence Φ is contained in a component of D.

A component of a fuzzy set D need not be relative closed on D_0, which is another departure from general topology.

EXAMPLE. Suppose X is composed of a single point x. Let $\mathscr{T} = \{X, \Phi, x_{1/3}\}$. Let $D = x_{1/3}$; then D is of course a component of D but is not a closed set in $D_0 = X$.

PROPOSITION 10.1. *If D is a closed fuzzy set in $(X, \mathscr{T})$, then every component of D is a closed set on X.*

Proof. Let A be a component of D. In view of Theorem 10.1, $\text{cl}_X A$ is connected. From $A \subset \text{cl}_X A \subset \text{cl}_X D = D$ and the maximality of a component, we have $A = \text{cl}_X A$; i.e., A is closed in X.

At the end of this section we shall point out that the theorem on separation sets (cf. [10, p. 60, Problem 1.Q]) can be generalized to fuzzy topological spaces. The proof of this generalization, though a little longer, is straightforward and hence is left for the readers.

THEOREM 10.4 (Theorem on Q-separated sets). *Let A and B be connected fuzzy sets and $A \supset B$. If $A \sim B = C \cup D$, and C and D are Q-separated in $(X, \mathscr{T})$, then $B \cup C$ and $B \cup D$ are connected.*

11. NETS

From this section on, we turn to the investigation of Moore-Smith convergence in the fuzzy topological spaces. All the theorems in [10; Chap. II] have been generalized to fuzzy topology in the following pages.

DEFINITION 11.1. Let D be a non-void set. Let $\geqslant$ be a semi-order on D. The pair $(D, \geqslant)$ is called a directed set, directed by $\geqslant$, iff, for every pair m, $n \in D$, there exists a $p \in D$ such that $p \geqslant m$ and $p \geqslant n$.

DEFINITION 11.2. Let $(D, \geqslant)$ be a directed set. Let X be an ordinary set. Let $\mathscr{J}$ be the collection of all the fuzzy points in X. The function S: $D \to \mathscr{J}$ is called a fuzzy net in X. In other words, a fuzzy net is a pair $(S, \geqslant)$ such that S is a function: $D \to \mathscr{J}$ and $\geqslant$ directs the domain of S. For $n \in D$, $S(n)$ is often denoted by S_n and hence a net S is often denoted by $\{S_n, n \in D\}$.

DEFINITION 11.3. Let $S = \{S_n, n \in D\}$ be a fuzzy net in X. S is said to be quasi-coincident with A iff, for each $n \in D$, S_n is quasi-coincident with A. S is said to be eventually quasi-coincident with A iff there is an element m of D such that, if $n \in D$ and $n \geqslant m$, then S_n is quasi-coincident with A. S is said to be frequently quasi-coincident with A iff for each m in D there is an n in D such that $n \geqslant m$ and S_n is quasi-coincident with A. S is said to be in A, iff for each $n \in D$, $S_n \in A$.

DEFINITION 11.4. A net S in a fts $(X, \mathscr{T})$ is said to converge to a point e in X relative to $\mathscr{T}$ iff S is eventually quasi-coincident with each Q-neighborhood of e.

THEOREM 11.1. *In a fts $(X, \mathscr{T})$, a fuzzy point $e \in \bar{A}$ iff there is a fuzzy net S in A such that S converges to e.*

Proof. Necessity. Let $D = \{B \in [0, 1]^X \mid$ there exists $0 \in \mathscr{T}$ such that eq. $0 \subset B\}$. Then $\{D, \subset\}$ is a directed set (cf. Proposition 2.2, (2)). For each $B \in D$, since $e \in \bar{A}$, by Theorem 4.1′, there exists a point $z \in X$ such that $B(z) + A(z) > 1$. Hence $A(z) = \mu > 0$, and fuzzy point $z_\mu \in A$ and $z_\mu q B$. Now we have the function S: $D \to \{\text{fuzzy points } e \text{ in } A\}$ such that for every $B \in D$, $S(B) = z_\mu \in A$. Then the net $(S_B\, ;\, B \in D)$ is the required one.

Sufficiency. If there is a fuzzy net S in A converging to e, then, by Definition 11.4, for every Q-neighborhood B of e, S is eventually quasi-coincident with B. Hence there is some $S_n = z_\mu$ such that $z_\mu q A$ and $z_\mu \in B$., i.e., $B(z) + \mu > 1$ and $\mu \leqslant A(z)$. It follows that $B(z) + A(z) > 1$. Hence B and A are quasi-coincident. From Theorem 4.1', $e \in \bar{A}$.

THEOREM 11.2. *A fuzzy subset A in a fuzzy topological space $(X, \mathscr{T})$ is closed iff every fuzzy net S in A cannot converge to a fuzzy point not belonging to A.*

Noting that A is closed iff $\bar{A} = A$, we see that this is a direct consequence of Theorem 11.1.

In Definition 9.2, we introduced the operation $A \sim B$ as a generalization of difference between sets in the ordinary set theory. We shall now introduce the operation $A - B$ between two fuzzy sets A and B, which is another generalization.

DEFINITION 11.4. Let A and B be two fuzzy sets in X. Then $A - B$ is defined as follows.

$$\begin{aligned}(A - B)(x) &= 0, &\quad &\text{for } x \in \{y \in X \mid A(y) \geqslant B(y) > 0\}\\ &= A(x), &\quad &\text{otherwise.}\end{aligned}$$

Intuitively speaking, for $x \in X$, when the fuzzy point $x_{B(x)}$ exists (i.e., $B(x) > 0$) and belongs to A (i.e., $A(x) \geqslant B(x)$), then the value of A at x is reduced to 0 and remains unchanged otherwise.

Evidently, when $A(x) < \lambda$, $A - x_\lambda = A$; when $A(x) \geqslant \lambda$, $\operatorname{supp}(A - x_\lambda) = \operatorname{supp} A - \{x\}$.

THEOREM 11.3. *In a fuzzy topological space $(X, \mathscr{T})$, x_λ is an accumulation point of A iff there is a fuzzy net in $A - x_\lambda$ converging to x_λ.*

Proof. When $A(x) < \lambda$, $A - x_\lambda = A$ and $x_\lambda \notin A$. Hence x_λ is an accumulation point of A iff $x_\lambda \in \bar{A}$. In view of Theorem 11.1, this is also equivalent to the fact that there is a fuzzy net in $A = A - x_\lambda$ which converges to x_λ.

When $A(x) \geqslant \lambda$, $x_\lambda \in A$. At this time, x_λ is an accumulation point of A iff every Q-neighborhood B of x_λ is quasi-coincident with A at a point different from x, i.e., B and $H = A - x_\lambda$ are quasi-coincident. In other words, x_λ is an accumulation point of A iff $x_\lambda \in \bar{H}$. From Theorem 11.1, this is equivalent to the fact that there is a fuzzy net in $H = A - x_\lambda$ converging to x_λ.

12. Uniqueness of Convergence Theorem on Iterated Limits

As the following proposition shows, the set of fuzzy points to which a fuzzy net converges is, in general, infinite. But if we put certain restrictions on the supports, we can obtain some appropriate result concerning the uniqueness of convergence.

Proposition 12.1. *In a fuzzy topological space* $(X, \mathscr{T})$, *if a fuzzy net* S *converges to a fuzzy point* x_λ, *then, for every* $\mu \in (0, \lambda]$, S *converges also to* x_μ.

Proof. Since $\mu \leqslant \lambda$, then $1 - \mu \geqslant 1 - \lambda$, and hence any Q-neighborhood of x_μ is also a Q-neighborhood of x_λ. From this fact, it is evident that S converges to x_μ.

Definition 12.1. Suppose for each $\alpha \in I$, $(D_\alpha, >_\alpha)$ be a directed set. Recall that the Cartesian product $D = \times \{D_\alpha : \alpha \in I\}$ is the set of all functions d on I such that d_α $(= d(\alpha)) \in D_\alpha$ for each $\alpha \in I$. The product order $>$ on D is defined as follows: for $d, f \in D$, $d > f$ iff, for each $\alpha \in I$, $d_\alpha >_\alpha f_\alpha$. The product directed set of the collection of directed sets $\{(D_\alpha, >_\alpha) \mid \alpha \in I\}$ is defined as the pair $(D, >)$, where D is the Cartesian product of $\{D_\alpha \mid \alpha \in I\}$ and $>$ is the product order on D. (It is easily verified that $>$ directs D.)

Theorem 12.1. *In a fuzzy topological space* $(X, \mathscr{T})$, *every fuzzy net does not converges to two fuzzy points with different supports iff* $(X, \mathscr{T})$ *is a fuzzy* T_2 *space.*

Proof. If there is a fuzzy net $S = (S_n, n \in D)$ in X converging to two fuzzy points e_1, e_2 with $\operatorname{supp} e_1 \neq \operatorname{supp} e_2$. For any Q-neibhborhood B_i of e_i $(i = 1, 2)$, in view of Definition 11.4, there is some S_m which is quasi-coincident with both B_1 and B_2. Let $\operatorname{supp} S_m = z$. obviously $B_1(z) > 0$, $B_2(z) > 0$. Hence B_1 and B_2 intersect. This contradicts the fact that $(X, \mathscr{T})$ is T_2. The sufficiency is thus proved. We shall now show the necessity. If $(X, \mathscr{T})$ is not T_2, then there are fuzzy points e_1 and e_2 with different supports such that for any Q-neighborhoods, A_i of e_i $(i = 1, 2)$, $A_1 \cap A_2 \neq \Phi$, therefore, there exists $z \in X$ such that $A_1(z) > 0$, $A_2(z) > 0$. Let $\lambda = \min(A_1(z), A_2(z))$; then the fuzzy point $z_{1-\lambda/2}$ is quasi-coincident with both A_1 and A_2. Denote this fuzzy point by $S(A_1, A_2)$. Let $\mathscr{U}_{e_i}$ be the Q-neighborhood system of e_i; then both $\mathscr{U}_{e_1}$ and $\mathscr{U}_{e_2}$ are directed by $\subset$ and then we construct the product directed set $\{\mathscr{U}_{e_1} \times \mathscr{U}_{e_2}, \geqslant\}$ according to the above definition. To each number $A_1 \times A_2 \in \mathscr{U}_{e_1} \times \mathscr{U}_{e_2}$ corresponds the fuzzy point $S(A_1, A_2)$ described above. This correspondence gives a fuzzy net S in X. It is evident that S converges to both e_1 and e_2 whose supports are different. Contradiction.

The following theorem on iterated limits concerning the convergence of nets has its own interest in mathematical analysis. To give the generalized form of this theorem here is to make a preparation for investigating the relationship between topologies and convergence classes.

DEFINITION 12.2. Let D and E_m, $m \in D$ be directed sets. If, corresponding to each $m \in D$ and each $n \in E_m$, there exists a fuzzy point $S(m, n)$, we call the function S an iterated fuzzy net in X: $\{S(m, n) \mid m \in D, n \in E_m\}$, If in a fts $(X, \mathscr{T})$, for a given $m \in D$, the fuzzy net $\{S(m, n)\ n \in E_m\}$ converges to a fuzzy point S_m in X, then we have a fuzzy net $\{S_m, m \in D\}$. If $\{S_m, m \in D\}$ converges to a fuzzy point e in X, we say that the iterated limit e of the iterated fuzzy net exists, or we simply say that S converges to e.

THEOREM 12.2 (Iterated limit theorem). *If, in a fts $(X, \mathscr{T})$, an iterated fuzzy net $\{S(m, n), m \in D, n \in E_m\}$ converges to e, let F denote the product order $D \times (\times \{E_m \mid m \in D\})$. A function R with domain F is defined as follows: for each $(m, f) \in F$, $R(m, f) = (m, f(m))$. Then $S \circ R$ is a fuzzy net with domain F and converges to e.*

Proof. $S \circ R$ is obviously a fuzzy net with domain F. For any Q-neighborhood B of e, since S converges to e, there is an $m \in D$ such that for every $p \in D$ and $p \geqslant m$, S_p is quasi-coincident with B. Let x_p denote the support of S_p. Then $B(x_p) + S_p(x_p) > 1$ and hence B is also an open Q-neighborhood of S_p. Since by hypothesis, $\{S(p, n), n \in E_p\}$ converges to S_p (cf. Definition 12.2), there corresponds an $f(p) \in E_p$ such that for every $n \in E_p$ and $n \geqslant f(p)$, $S(p, n)$ is quasi-coincident with B. For each $p \geqslant m$, we have thus defined $f(p) \in E_p$. If $p \in D$, which does not follow m, let $f(p)$ be an arbitrary member of E_p, then we obtain $f \in \times \{E_p, p \in D\}$ and $(m, f) \in F$. When $(p \cdot q) \in F$ and $(p, q) \geqslant (m, f)$, we have $p \geqslant m$, $g(p) \geqslant f(p)$. In view of the foregoing construction, we know that $S(p, g(p))\ qB$, and hence $S \circ R\ (p, q) = S(p, g(p))\ qB$, i.e., $S \circ R$ is eventually quasi-coincident with B. Hence $S \circ R$ converges to e.

13. SUBNETS AND SUBSEQUENCES

DEFINITION 13.1. A fuzzy net $T = \{T_m, m \in E\}$ in X is called a fuzzy subset of a fuzzy net $S = \{S_n, n \in D\}$ in X iff there is a function N: $E \to D$ such that

(1) $T = S \circ N$, that is, for each $i \in E$, $T_i = S_{N(i)}$.

(2) For each $n \in D$, there exists some $m \in E$ such that, if $E \ni p \geqslant m$, $N(p) \geqslant n$.

THEOREM 13.1. *Let $S = \{S_n, n \in D\}$ be a fuzzy net in X and let $\mathscr{A}$ be a family of fuzzy sets in X such that the intersection of any two members of $\mathscr{A}$ contains a member of $\mathscr{A}$, and such that S is frequently quasi-coincident with each* *member of $\mathscr{A}$. Then there is a subnet T of S which is eventually quasi-coincident with each member of $\mathscr{A}$.*

Proof. The intersection of any two members of $\mathscr{A}$ contains a member of $\mathscr{A}$ and therefore $\mathscr{A}$ is directed by $\subset$. Let $D_1 = \{A, \subset\}$ and E, the set of all pairs (m, A) such that $m \in D$, $A \in \mathscr{A}$, and $S_m q A$. Since S is frequently quasicoincident with each member of $\mathscr{A}$, E is evidently nonvoid. It is clear that E is a subset of $D \times D_1$. The product order for $D \times D_1$ restricted on E_1 gives a semi-order $\geqslant$ for E. Then E is directed by $\geqslant$. In fact, for members (m, A) and (n, B) of E, there is $G \in \mathscr{A}$ such that $G \subset A \cap B$. Moreover, since S is frequently quasi-coincident with G, there is $p \in D$ such that $p \geqslant m$, $p \geqslant n$ and $S_p q G$ and hence $(p, G) \in E$ and (p, G) follows both (m, A) and (n, B). In the directed set $(E, \geqslant)$, define a function N: $E \to D$ such that $N(m, A) = m$. It is easily seen that condition (2) of Definition 13.1 is satisfied. Therefore $T = S \circ N$ is a fuzzy subnet of S. Finally, if A is a member of $\mathscr{A}$, since S is frequently quasi-coincident with A, there is some $m \in D$ such that $S_m q A$, and hence $(m, A) \in E$. For $(n, B) \in E$ and $(n, B) \geqslant (m, A) = S \circ N(n, B) = S_n$, $S_n q B$ and hence $S_n q A$. Therefore T is eventually quasi-coincident with A.

DEFINITION 13.2. In a fts $(X, \mathscr{T})$, a fuzzy point e is called a cluster point of a fuzzy net S iff for every Q-neighborhood B of e, S is frequently quasi-coincident with B.

THEOREM 13.2. *In a fts $(X, \mathscr{T})$ a fuzzy point e is a cluster point of a fuzzy net S iff S has a fuzzy subnet T converging to e.*

Proof. The sufficiency follows directly from Definitions 13.2 and 13.1. Suppose e is a cluster point of S. In view of the fact that the Q-neighborhood system of e satisfies the conditions concerning the family $\mathscr{A}$ in Theorem 13.1, the necessity follows directly.

THEOREM 13.3. *Let $S = \{S_n, n \in D\}$ be a fuzzy net on X. For each $n \in D$, let A_n be the union of all fuzzy points S_m $(m \geqslant n)$: $A_n = \bigcup_{m \leqslant n} S_m$, then in the fts $(X, \mathscr{T})$, a fuzzy point e is a cluster point of S iff $e \in \bar{A}_n$ for each $n \in D$.*

Proof. If e is cluster point of S, from Definition 13.2, for each n, A_n is quasi-coincident with each Q-neighborhood of e because S is frequently quasi-coincident with each Q-neighborhood of e. Therefore $e \in \bar{A}_n$ for each n. Conversely, let $e \in \bar{A}_n$ for each n. Take any Q-neighborhood B of e and any $m \in D$. Since $e \in \bar{A}_m$, by Theorem 4.1′, B and A_m are quasi-coincident at some point z, i.e., $B(z) + A_m(z) > 1$. By the definition of A_m, $A_m = \bigcup_{p \leqslant m} S_p$, we have $\sup_{p \leqslant m} S_p(z) > 1 - B(z)$. Hence there is $n \geqslant m$ such that $A_m(z) \geqslant S_n(z) > 1 - B(z)$, i.e., $B(z) + S_n(z) > 1$, that is to say, $S_n q B$. Hence S is frequently quasi-coincident with B. It follows that e is a cluster point of S.

When the domain of a fuzzy net $\{S_n, n \in D\}$ in X consists of the positive integers, with the natural order between positive integers as semi-order, we call this net a fuzzy sequence. Similar to the definition of subnets, we may define the concept of fuzzy subsequences.

THEOREM 13.4. *Let $(X, \mathscr{T})$ be a fuzzy C_{I} space or Q-C_{I} space. A fuzzy set in X, and e a fuzzy point in X. Then*

(1) *$e \in \bar{A}$ iff there is a fuzzy sequence converging to e.*

(2) *A is closed iff every fuzzy sequence cannot converge to a fuzzy point not belonging to A.*

(3) *e is an accumulation point of A iff there is a fuzzy sequence in $A - e$ converging to e.*

(4) *e is a cluster point of a fuzzy sequence iff S has a subsequence converging to e.*

Proof. In view of Proposition 3.1, it suffices to suppose $(X, \mathscr{T})$ to be a $Q - C_{\text{I}}$ space. Let e be an arbitrary fuzzy point and $\mathscr{B} = \{B_n\}$ $(n = 1, 2,\ldots)$ a countable open Q-neighborhoods of e. By taking appropriate intersections, we may assume $B_n \supset B_{n+1}$ without loss of generality. The Q-neighborhood base satisfying the above condition is said to be monotonic. We shall use the monotonic Q-neighborhood system instead of the Q-neighborhood system of a fuzzy point. Condition (1) may be proved by a method similar to that of Theorem 11.1; (2) can be obtained from (1) following the proof of Theorem 11.3; (3) can be similarly proved (use (1) just obtained to replace Theorem 11.1 in the proof of Theorem 11.3). The proof of (4) may be simpler than that of Theorem 13.2, which is given as follows: If $S = \{S_n, n \in D\}$ has a subsequence converging to a fuzzy point e, e is obviously a cluster point of S. Conversely, if S has e as a cluster point, let $\{B_i\}$ $(i = 1, 2,\ldots)$ be a monotonic open Q-neighborhood base of e. Evidently, we can inductively take S_{n_i}, for each B_i, such that S_{n_i} is quasi-coincident with B_i and $n_i > n_{i-1}$ (n_0 is understood as 0, and n_1 may be arbitrarily taken, only requiring that S_{n_1}, be quasi-coincident with B_1). Then $\{S_{n_i}\}$, $i = 1, 2,\ldots$ is the required fuzzy subsequence converging to e.

14. A ONE-TO-ONE CORRESPONDENCE BETWEEN CONVERGENCE CLASSES AND FUZZY TOPOLOGIES

DEFINITION 14.1. Let $\mathscr{G}$ be a class consisting of pairs (S, e), where S is a fuzzy net in X and e a fuzzy point in X. We say that $\mathscr{G}$ is a convergence class for X iff it satisfies the four conditions listed below. For convenience, we also say that S converges $(\mathscr{G})$ to e or that $\lim_n S_n = e(\mathscr{G})$ iff (S, e)

(1) If $S = (S_n, n \in D)$ is a fuzzy net such that $S_n = e$, a fuzzy point, for each n, then $(S, e) \in \mathscr{G}$.

(2) If S converges ($\mathscr{G}$) to e, then so does each fuzzy subnet of S.

(3) If S does not converge ($\mathscr{G}$) to e, then there is a fuzzy subnet T of S, no fuzzy subnet of which converges ($\mathscr{G}$) to e.

(4) Let D be a directed set, and for each $m \in D$, let E_m be a directed set. Let $S = \{S(m, n): m \in D, n \in E_m\}$ (cf. Definition 12.2) be the iterated fuzzy net in X. For each $m \in D$, let the fuzzy net $\{S(m, n), n \in E_m\}$ converge ($\mathscr{G}$) to some fuzzy point S_m and let the fuzzy net $\{S_m, m \in D\}$ thus obtained, converge ($\mathscr{G}$) to a to a fuzzy point e. Let F be product directed set $D \times (\times \{E_m, m \in D\})$ For each $(m, f) \in F$, let $R(m, f) = (m, f(m))$. Then $S \circ R$ is a fuzzy net with domain F and converges ($\mathscr{G}$) to e.

PROPOSITION 14.1. *In a fts $(X, \mathscr{T})$, if a fuzzy net $S = \{S_n, n \in D\}$ fails to converge to a fuzzy point e. Then there exists an open Q-neighborhood B and a fuzzy subnet $T = \{T_m, m \in E\}$ of S such that T_m is not quasi-coincident with B for each $m \in E$ and hence any fuzzy subnet of T does not converge to e.*

Proof. Since S does not converge to e, by Definition 11.4, there exists an open n-neighborhood B such that for any $n \in D$, there exists $m \in D$ such that $m \geqslant n$, and S_m is not quasi-coincident with B. Let $E = \{m \in D \mid S_m$ is not quasi-coincident with $B\}$, E is a cofinal subset of D. Let N denote the identity mapping $N: E \to D$. It is evident that $S \circ N = T$ is a subnet of S, which satisfies the requirements.

It has previously been proved that convergence in a fts $(X, \mathscr{T})$ satisfies (1), (2), (3) and (4) (especially cf. Theorem 12.2 and Proposition 14.1). Hence the class consisting of all pairs (S, e), where S is a fuzzy net converging to e, relative to $\mathscr{T}$, is actually a convergence class. This convergence class is determined by the fuzzy tkpology $\mathscr{T}$ and is hence denoted by $\chi(\mathscr{T})$. Conversely, as the following theorem shows, every convergence class $\mathscr{G}$ for X can also determine a fuzzy topology $\psi(\mathscr{T})$ for X.

THEOREM 14.1. *Let $\mathscr{G}$ be a convergence class for a set X, and for each fuzzy set A in X let A^c denote the union of all the fuzzy points e such that, for some fuzzy net in A, S converges ($\mathscr{G}$) to e. Then c is a closure operator for X. (cf. Definition 4.3) and $(S, e) \in \mathscr{G}$ iff S converges to e relative to the fuzzy topology $\psi(\mathscr{G})$ associated with the closure operator c.*

Proof. That c is a closure operator and that if S converges to e relative to $\psi(\mathscr{G})$, then $(S, e) \in \mathscr{G}$ can be proved by repeating the corresponding arguments in the proof of Theorem 9 in [10] with a slight modification in terminology. We shall now show that if $(S, e) \in \mathscr{G}$, then S converges to e relative to the fuzzy topology $\psi(\mathscr{G})$ associated with c. Suppose S fails to converge to e relative to

$\psi(\mathscr{G})$, from Proposition 14.1, there exists an open Q-neighborhood of e and subnet $T = \{T_m: m \in E\}$ of S such that each T_m is not quasi-coincident with B. From Proposition 2.1; $T_m \subset B'$ for each $m \in E$, i.e., $T = \{T_m, m \in E\}$ is in B'. From property (2) of convergence class, $(T, e) \in \mathscr{G}$, and hence, by definition, $e \in (B')^c$. Since B' is closed, $(B')^c = B'$ and $e \in B'$. By using Proposition 2.1, e is not quasi-coincident with $(B')' = B$, which contradicts the fact that B is a Q-neighborhood of e.

Theorem 14.1 not only establishes the fact that to every convergence class $\mathscr{G}$ for X, there corresponds a fuzzy topology $\psi(\mathscr{G})$ for X, but also proves that the convergence class $\chi(\psi(\mathscr{G}))$ determin.d by the fuzzy topology $\psi(\mathscr{G})$ is exactly $\mathscr{G}$, that is, $\chi\psi = 1$. Then the correspondence χ is a surjection ("onto" mapping). From Theorem 11.1, it is easily seen that for distinct topologies $\mathscr{T}_i$ $(i = 1, 2)$, the corresponding convergence classes $\chi(\mathscr{T}_i)$ $(i = 1, 2)$ are also distinct, i.e., χ is injective. Hence χ is a bijection between the family of fuzzy topologies for X and that of the convergence classes for X and $\chi^{-1} = \psi$. It is also evident that the correspondence χ is order inverting, i.e., if $\mathscr{T}_1 \supset \mathscr{T}_2$, $\chi(\mathscr{T}_1) \subset \chi(\mathscr{T}_2)$.

Acknowledgment

The authors wish to express their sincere thanks of Professor Kwan Chao-chih at the Mathematics Institute of the Academia Sinica for his enthusiastic support and encouragement in their launching of an investigation on fuzzy topology.

References

1. L. A. Zadeh, Fuzzy sets, *Inform. and Contr.* **8** (1965), 338–353.
2. C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.* **24** (1968), 182–189.
3. J. A. Goguen, The fuzzy Tychonoff theorem, *J. Math. Anal. Appl.* **43** (1973), 734–742.
4. B. Hutton, Normality in fuzzy topological spaces, *J. Math. Anal. Appl.* **50** (1975), 74–79.
5. M. D. Weiss, Fixed points. separation and induced topology for fuzzy sets, *J. Math. Anal. Appl.* **50** (1975), 142–150.
6. C. K. Wong, Covering properties of fuzzy topological spaces, *J. Math. Anal. Appl.* **43** (1973), 697–704.
7. C. K. Wong, Fuzzy topology: Product and quotient theorems, *J. Math. Anal. Appl.* **45** (1974), 312–521.
8. C. K. Wong, Fuzzy points and local properties of fuzzy topology, *J. Math. Anal. Appl.* **46** (1974), 316–328.
9. J. A. Goguen, *L*-fuzzy sets, *J. Math. Anal. Appl.* **18** (1967), 145–174.
10. J. L. Kelley, "General Topology," Princeton Univ. Press, Princeton, 1955.
11. N. Bourbaki, Topologie générale, *Actualiés Sci. Indust.*, 1045 (1948).