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Gorenstein semigroup algebras of weighted trees and ordered points on the projective line

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ABSTRACT

We determine exactly which semigroup algebras of weighted trees are Gorenstein. These algebras arise as toric degenerations of projective coordinate rings of moduli of weighted points on the projective line. As a corollary, we find exactly when these families of algebras are Gorenstein as well.

Contents

1. Introduction

We study a class of semigroup algebras which has made appearances in the moduli of point arrangements on the projective line, blowups of point arrangements on projective space, moduli of principal $SL_2(\mathbb{C})$ bundles on rational curves, and integrable systems of linkages in $\mathbb{R}^3$. In particular

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we will decide exactly when these algebras have the Gorenstein property. As is the case with many results in combinatorial commutative algebra, the answer to this classification problem is polyhedral in nature.

Members of the family of semigroup algebras we study are constructed from two pieces of information, a trivalent tree $\mathcal{T}$ with n labeled leaves, and a choice $\mathbf{r}$ of n non-negative integers. For a fixed tree $\mathcal{T}$, we let $V(\mathcal{T})$ denote the set of vertices, $E(\mathcal{T})$ denote the set of edges, $L(\mathcal{T})$ denote the set of edges incident on a leaf, and $I(\mathcal{T})$ denote the set of internal edges $E(\mathcal{T}) \setminus L(\mathcal{T})$. For each trinode $\tau \in V(\mathcal{T})$ we number the three edges incident on τ in some way with $\{1, 2, 3\}$, and we denote the i-th such edge by (τ, i). Our constructions do not depend on the particulars of this assignment, it is done purely for book-keeping purposes.

Definition 1.1. The semigroup $S_{\mathcal{T}}(\mathbf{r})$ is graded, and the k-th graded component is the set of weightings $\omega : E(\mathcal{T}) \to \mathbb{Z}_+$ defined by the following conditions.

(1) For all $\tau \in V(\mathcal{T})$ the numbers $\omega(\tau, i)$ satisfy $|\omega(\tau, 1) - \omega(\tau, 2)| \leqslant \omega(\tau, 3) \leqslant |\omega(\tau, 1) + \omega(\tau, 2)|$.
(2) $\Sigma_{i=1}^{3} \omega(\tau, i)$ is even.
(3) For all $v_m \in L(\mathcal{T})$, $\omega(v_m) = k\mathbf{r}_m$.

The expressions in item 1 above are called the triangle inequalities, item 2 is referred to as the parity condition. From now on, when three numbers A, B and C satisfy both the parity condition and the triangle inequalities, we write $\Delta_2(A, B, C)$. Notice that for a single trinode τ, these conditions are symmetric in the (τ, i). The graded components of the semigroups $S_{\mathcal{T}}(\mathbf{r})$ are subsets of the semigroup $S_{\mathcal{T}}$ of non-negative integer weightings of $\mathcal{T}$ which satisfy conditions (1) and (2). This semigroup can be described as the lattice points in the convex cone $P_{\mathcal{T}} \subset \mathbb{R}^{E(\mathcal{T})}$ defined by positive real vectors satisfying (1), with respect to the lattice $L_2(\mathcal{T})$ of integer vectors satisfying (2). When $\mathcal{T}$ only has three leaves we denote this cone by P_3. The members of the first graded piece of $S_{\mathcal{T}}(\mathbf{r})$ are the lattice points in a cross-section $P_{\mathcal{T}}(\mathbf{r})$ of $P_{\mathcal{T}}$, defined by specializing the weights in $L(\mathcal{T})$ to the entries in the vector $\mathbf{r}$. In what follows, the semigroup algebra of a semigroup S_P of lattice points in a cone P will be denoted $\mathbb{C}[S_P]$. For convenience and in sympathy with the symplectic uses of semigroup algebras of weighted trees we will work over $\mathbb{C}$, but our results hold over any algebraically closed field of characteristic 0.

Presentations of these semigroups and their associated semigroup algebras were constructed by the author in [M]. In [HMSV], Howard, Millson, Snowden, and Vakil independently constructed presentations of $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ in order to find presentations of a projective coordinate ring of the moduli space of $\mathbf{r}$-weighted points on $\mathbb{P}^1$, denoted $M_{\mathbf{r}}$. The associated embedding they studied comes from the homeomorphism of projective varieties, $M_{\mathbf{r}} \cong Gr_2(\mathbb{C}^n)//_{\mathbf{r}}T$, where the right-hand side is the $\mathbf{r}$-weight variety of the Grassmannian variety $Gr_2(\mathbb{C}^n)$ of 2-dimensional subspaces of $\mathbb{C}^n$. They constructed a toric degeneration of this algebra, $\mathbb{C}[M_{\mathbf{r}}]$, to $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ for each tree $\mathcal{T}$ by means of the Speyer and Sturmfels [SpSt] toric deformations of $Gr_2(\mathbb{C}^n)$. These degenerate the projective coordinate ring of $Gr_2(\mathbb{C}^n)$ given by the Plücker embedding to $\mathbb{C}[S_{\mathcal{T}}]$, see [HMSV] and [SpSt] for details.

Sturmfels and Xu [StXu] have shown that the Cox ring $R_{n-1,n-2}$ of the blow up of $\mathbb{P}^{n-3}$ at $n-1$ points in general position is isomorphic (as a multigraded algebra) to the Plücker algebra of $Gr_2(\mathbb{C}^n)$. See [StXu], Theorem 3.5. The multigrading, given by the Picard group of this blow-up, corresponds to the multigrading on $\mathbb{C}[S_{\mathcal{T}}]$ given by specializing to the weights on elements of $L(\mathcal{T})$. This then implies that the subring of $R_{n-1,n-2}$ obtained by taking invariants with respect to a character of the corresponding "Picard torus" is isomorphic to one of the algebras $\mathbb{C}[M_{\mathbf{r}}]$.

The algebras $\mathbb{C}[M_{\mathbf{r}}]$ can also be realized as projective coordinate rings of the moduli $\mathcal{M}_{\mathbb{P}^1,\vec{p}}(SL_2(\mathbb{C}))$ of quasi-parabolic $SL_2(\mathbb{C})$ principal bundles on an n-marked $\mathbb{P}^1$. For this construction see [M], and [StXu]. The global sections of line bundles on this moduli problem are so-called conformal blocks from conformal field theory. The algebra $\mathbb{C}[M_{\mathbf{r}}]$ can be considered as a subset of these objects with a multiplication structure. In this way, the Hilbert function of $\mathbb{C}[M_{\mathbf{r}}]$ encodes the rule for enumerating conformal blocks.

The spaces $M_{\mathbf{r}}$ and the toric varieties $M_{\mathbf{r}}^{\mathcal{T}}$ defined by taking *Proj* of the semigroup algebra $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ are also the subject of the paper [HMM]. Here the symplectic geometry of polygonal linkages in $\mathbb{R}^3$ is studied using the polytopes $P_{\mathcal{T}}(\mathbf{r})$.

1.1. The Gorenstein property for semigroup algebras

Let $\mathfrak{k}$ be a field, then a $\mathbb{Z}$-graded $\mathfrak{k}$-algebra R is said to be *Gorenstein* if the Matlis Dual

$$H_m^{dim(R)}(R)^* = \underline{Hom}_{\mathfrak{k}}\big(H_m^{dim(R)}(R), \mathfrak{k}\big), \tag{1}$$

is isomorphic to grade-shifted copy $R(-a)$ of R. Here m is the maximal ideal generated by elements in R of positive degree, and $\underline{Hom}_{\mathfrak{k}}(-,-)$ is the functor of graded $\mathfrak{k}$-morphisms. The number a is called the a-invariant of the graded Gorenstein algebra R. We refer the reader to the book by Bruns and Herzog, [BH] for this and all other relevant definitions.

Gorenstein algebras are in a sense the nicest class of commutative algebras beyond complete intersections. They are of finite injective dimension, and their free resolutions have a Poincare-duality property. Combinatorial features of Gorenstein algebras are also very nicely behaved. In particular, their Hilbert functions have a duality property expressed by the following theorem, which is essentially due to Stanley, see [BH, Corollary 4.4.6].

Theorem 1.2 *(Stanley). Let $R = \bigoplus_{n \geqslant 0} R_n$ be a graded domain over $\mathbb{C}$, and let the following be its Hilbert function.*

$$H_R(t) = \frac{\sum_{i=0}^{s} h_i t^i}{\prod_{j=1}^{d}(1 - t^{a_j})}. \tag{2}$$

Then R is Gorenstein if and only if $H_R(t) = (-1)^d t^{a(R)} H_R(t^{-1})$.

A corollary of this theorem is that the Gorenstein property is stable over degenerations of domains. Combining this with the results of Speyer and Sturmfels, [SpSt] and Howard, Millson, Snowden, and Vakil, [HMSV, Section 3], we have the following useful result.

Theorem 1.3. *The graded algebra $\mathbb{C}[M_{\mathbf{r}}]$ is Gorenstein if and only if $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ is Gorenstein, for any tree $\mathcal{T}$.*

Proof. Since $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ is a flat degeneration of $\mathbb{C}[M_{\mathbf{r}}]$, these algebras have the same Hilbert function. Both algebras are domains, so we may apply Theorem 1.2. $\square$

We also get the following corollary.

Corollary 1.4. *$\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ is Gorenstein if and only if $\mathbb{C}[S_{\mathcal{T}'}(\mathbf{r})]$ is Gorenstein for any trivalent trees $\mathcal{T}, \mathcal{T}'$ with the same number of leaves.*

Since we are dealing with algebras generated by the lattice points of convex cones we are able to use the following proposition, which is a consequence of Corollary 6.3.8 in [BH].

Proposition 1.5. *Let S_P be the semigroup given by the lattice points in a convex cone P. Then the algebra $\mathbb{C}[S_P]$ is Gorenstein if and only if there is a lattice point $\omega \in int(P)$ with $int(P) = \omega + S_P$. Furthermore, in the presence of a grading, we have $a(\mathbb{C}[S_P]) = deg(\omega)$.*

This proposition follows from the fact that the ideal $(int(P))\mathbb{C}[S_P]$ can identified with the *canonical module* of the algebra $\mathbb{C}[S_P]$ (resp. the $*$-canonical module in the presence of a grading), see [BH, Corollary 6.3.8]. We wish to prove this property for $S_{\mathcal{T}}(\mathbf{r})$, seen as the lattice points in the cone over $P_{\mathcal{T}}(\mathbf{r}) \times \{1\}$ in $\mathbb{R}^{I(\mathcal{T})} \times \mathbb{R}$ with respect to the product lattice $L_2 \times \mathbb{Z}$.

1.2. Statement of results

In order to classify which $\mathcal{T}, \mathbf{r}$ give a Gorenstein semigroup algebra, we break the problem into two parts. First, we analyze when some Minkowski sum of $P_{\mathcal{T}}(\mathbf{r})$ contains a unique interior lattice point, a necessary but not sufficient condition for the Gorenstein property.

Theorem 1.6. *$P_{\mathcal{T}}(\mathbf{r})$ has a unique interior point if and only if $\mathbf{r} = \vec{2} + \vec{R}$ where $\vec{R}$ is of one of the following types.*

(1) *$R_i = \Sigma_{j \neq i} R_j$ for some i.*
(2) *$\Delta_2(R_i, R_j, R_k)$ holds for some i, j, k and $R_\ell = 0$ for all $\ell \neq i, j, k$.*

This will be proved in Section 3. Next, we find when every other interior point of the cone has the unique interior point of the appropriate Minkowski sum of $P_{\mathcal{T}}(\mathbf{r})$ as a summand. Theorem 1.6 allows us to significantly narrow our search for $P_{\mathcal{T}}(\mathbf{r})$ which satisfy this condition. In order to carry this out we bring in an alternative description of the weightings $\omega \in P_{\mathcal{T}}(\mathbf{r})$, which can be found in [HMM] and [HMSV]. We call this the *piping model* of points in $P_{\mathcal{T}}(\mathbf{r})$.

The piping model allows us to associate a planar graph $T_{\mathcal{T}}(\omega)$ on n vertices to a lattice point $\omega \in P_{\mathcal{T}}(\mathbf{r})$. The valences of these vertices are given by the vector $\mathbf{r}$. The piping model gives a way to compare the lattice points in the polytopes $P_{\mathcal{T}}(\mathbf{r})$ as the tree $\mathcal{T}$ varies. We will describe an operation whereby a graph $T_{\mathcal{T}}(\omega)$ can be placed inside a different tree $\mathcal{T}'$ to give a lattice point in $P_{\mathcal{T}'}(\mathbf{r})$. This operation can be shown to be piecewise-linear on the polytopes $P_{\mathcal{T}}(\mathbf{r})$, and gives a combinatorial method for comparing our semigroup algebras.

We let $N_{ij}(\omega)$ be the multiplicity of edges between i and j in the graph $T_{\mathcal{T}}(\omega)$. We are now ready to state our main theorem. From now on we denote the unique minimal degree internal weighting for the cone on $P_{\mathcal{T}}(\mathbf{r})$ by $\omega_{\mathbf{r}}(\mathcal{T})$, if it exists. Also, we let $2_{\mathcal{T}}$ be the weighting which assigns 2 to each edge of $\mathcal{T}$, an example is illustrated in Fig. 2. In general, $T_{\mathcal{T}}(2_{\mathcal{T}})$ is always a complete planar cycle on the set $L(\mathcal{T})$.

Theorem 1.7. *The algebra $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ is Gorenstein if and only if the following conditions hold*:

(1) *$a\mathbf{r}$ is as in Theorem 1.6 for some a.*
(2) *In this degree, $N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}}) \geqslant n - 4$ when it is nonzero.*

This will be proved in Section 4. A consequence of this theorem is that the same conditions determine exactly when the algebra $\mathbb{C}[M_{\mathbf{r}}]$ is Gorenstein. Note that for a fixed a-invariant, the points $\mathbf{r}$ which satisfy these conditions are solutions to inequalities and linear equations. It would be nice to have a conceptual justification for why the classification takes this form. We finish with a theorem which restricts the a-invariant of $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$.

Theorem 1.8.

$$a\big(\mathbb{C}\big[S_{\mathcal{T}}(\mathbf{r})\big]\big) \,\big|\, 2(n-2).$$

This is proved in Section 5.

2. The piping model

In this section we describe the piping model of lattice points in $P_{\mathcal{T}}(\mathbf{r})$. We will show how to construct the graph $T_{\mathcal{T}}(\omega)$ for $\omega \in P_{\mathcal{T}}(\mathbf{r})$. We start by considering a weighting ω on a single trinode, τ. Since $\Delta_2(\omega(\tau,1), \omega(\tau,2), \omega(\tau,3))$ holds for weightings in P_3, we may apply the 1–1 transformation of cones

$$T : P_3 \to \mathbb{R}^3_+ \tag{3}$$

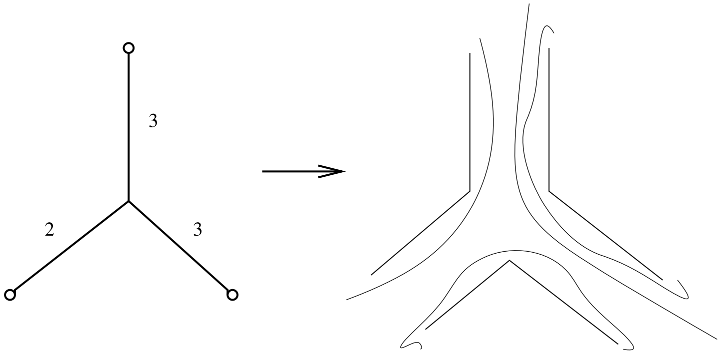

Fig. 1. Piping model for a trinode.

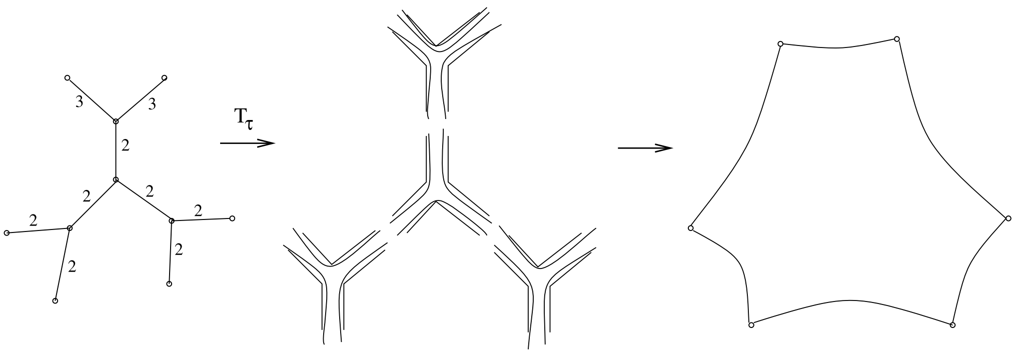

Fig. 2. Piping model for a general $\mathcal{T}$.

given by $T(\omega)(x_{ij}) = \frac{1}{2}(\omega(\tau, i) + \omega(\tau, j) - \omega(\tau, k))$. This is an isomorphism of semigroups when $\mathbb{R}^3$ is given the standard lattice. We represent the image of ω as a graph on the leaves of this tree, where the number of edges ("pipes") going from i to j is $T(\omega)(x_{ij})$. See Fig. 1 below, and also Figs. 2 and 5.

The map T has an inverse S given by $S(\eta)(\tau, i) = \eta(x_{ij}) + \eta(x_{ik})$. Using the map T we may go from weightings on the trinode τ to planar graphs on the set $\{(\tau, i)\}$. The transformation T is useful as it clarifies divisibility issues in the semigroup of lattice points in P_3: ω divides ω' if and only if $T(\omega)(x_{ij}) \leqslant T(\omega')(x_{ij})$ for all i, j. For a general tree $\mathcal{T}$, something similar happens for graphs on the set $L(\mathcal{T})$. Given a graph G on the set $L(\mathcal{T})$, we may construct a simultaneous weighting of the edges of each trinode $\tau \in V(\mathcal{T})$ as follows. For each edge $e \in G$ we consider the unique path γ in $\mathcal{T}$ joining the end points of e, each trinode edge (τ, i) traversed by this path gets weight $+1$. This defines a weighting of the edges of $\mathcal{T}$, and we call the resulting map $S_{\mathcal{T}}$.

There is a section to this map, $T_{\mathcal{T}}$, which is defined by the following algorithm. Given a weighting ω of $\mathcal{T}$, consider the simultaneous weighting of trinodes given by restricting ω to each $\tau \in V(\mathcal{T})$. Apply T to each of these weightings, and join up the ends of the resulting pipes in the unique way such that the resulting graph is planar. We leave it to the reader to show that this is well-defined. An example is illustrated in Fig. 2.

The map $T_{\mathcal{T}}$ cannot be an inverse as there are many (non-planar) graphs that give the same weighting under $S_{\mathcal{T}}$. We will make use of the following map.

$$S_{\mathcal{T}'} \circ T_{\mathcal{T}} : P_{\mathcal{T}}(\mathbf{r}) \to P_{\mathcal{T}'}(\mathbf{r}). \tag{4}$$

This operation can be shown to be piecewise linear, but that is not important for our purposes. With reference to the term-order degenerations constructed in [SpSt] and used in [HMSV], this trans-

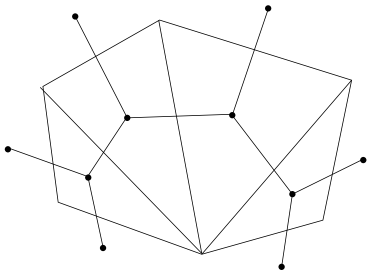

Fig. 3. The edges of $\mathcal{T}$ receive weight equal to the length of the dual diagonal in $\mathcal{P}$.

formation encodes the rule for changing which monomial in a relation on certain elements of a spanning set of $C[M_{\mathbf{r}}]$ has highest weight. The reader will also note that a weighting ω divides another weighting ω' if and only if $\omega|_\tau$ divides $\omega'|_\tau$ for all $\tau \in V(\mathcal{T})$, so the map T is still very useful for questions of divisibility in the case of general $\mathcal{T}$.

Note that the piping model only makes sense for a tree $\mathcal{T}'$ which has a specified embedding into $\mathbb{R}^2$, because of our need to deal with planar graphs. For this reason we restrict our attention to trees $\mathcal{T}$ with such an embedding assumed. This has no effect on our results because the isomorphism class of $P_{\mathcal{T}}$ as a multigraded algebra is determined by the topological type of $\mathcal{T}$, and therefore any $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ is isomorphic to some $\mathbb{C}[S_{\mathcal{T}'}(\mathbf{r}')]$ with $\mathcal{T}'$ planar and $\mathbf{r}'$, a permutation of the entries of $\mathbf{r}$.

3. Proof of Theorem 1.6

In this section we will classify the polytopes $P_{\mathcal{T}}(\mathbf{r})$ that have a unique interior point. Let $P_{\mathcal{T}}$ denote the cone of weightings ω on the tree $\mathcal{T}$ such that

$$\Delta_2\big(\omega(\tau,1), \omega(\tau,2), \omega(\tau,3)\big) \tag{5}$$

holds for each internal vertex $\tau \in V(\mathcal{T})$. A weighting ω is on a face of this cone if and only if one of these triangle inequalities is an equality for some vertex τ. Let D_n be the cone of side lengths for n-sided polygons (so $D_3 = P_3$), there is a map of cones $\pi : P_{\mathcal{T}} \to D_n$ given by forgetting the weights on internal edges of $\mathcal{T}$. We have the following identification, see Prop. 4.4 of [HMSV].

$$P_{\mathcal{T}}(\mathbf{r}) = \pi^{-1}(\mathbf{r}). \tag{6}$$

We study the dimension of these fibers when $\mathbf{r}_i \neq 0$ for all i. For $n = 3$ there is nothing to say, so suppose $n > 3$. Recall that a tree $\mathcal{T}$ defines a triangulation of any n-gon. If for every side length $\mathbf{r}_i < \Sigma_{j \neq i} \mathbf{r}_j$ then we may form a planar n-gon $\mathcal{P}$ with side lengths $\mathbf{r}$ in $\mathbb{R}^2$. We may find such an n-gon where all triangles in $\mathcal{P}$ formed by the diagonals coming from $\mathcal{T}$ and the sides have non-zero area. Let ω be the (not necessarily integer) weighting of $\mathcal{T}$ given by the diagonal lengths and side lengths of $\mathcal{P}$. (See Fig. 3.)

Observe that for any diagonal d of $\mathcal{P}$ specified by $\mathcal{T}$ which borders two triangles with non-zero area, we may stretch and contract the length of d within some neighborhood ϵ without changing the lengths of any other sides and specified diagonals non-zero, see Fig. 4.

This implies that there is a small neighborhood of dimension $|I(\mathcal{T})|$ in $P_{\mathcal{T}}(\mathbf{r})$ which contains ω. On the other hand, if some entry $\mathbf{r}_j$ of $\mathbf{r}$ has $\mathbf{r}_j = \sum_{i \neq j} \mathbf{r}_i$ then $P_{\mathcal{T}}(\mathbf{r})$ is a single point. Thus we conclude that the dimension of $P_{\mathcal{T}}(\mathbf{r})$ can be either $|I(\mathcal{T})|$ or 0.

Proposition 3.1. *Let $\mathbf{r}_i \neq 0$ for all i. A weighting ω is in the interior of $P_{\mathcal{T}}(\mathbf{r}) = \pi^{-1}(\mathbf{r})$ only if it is in the interior of $P_{\mathcal{T}}$.*

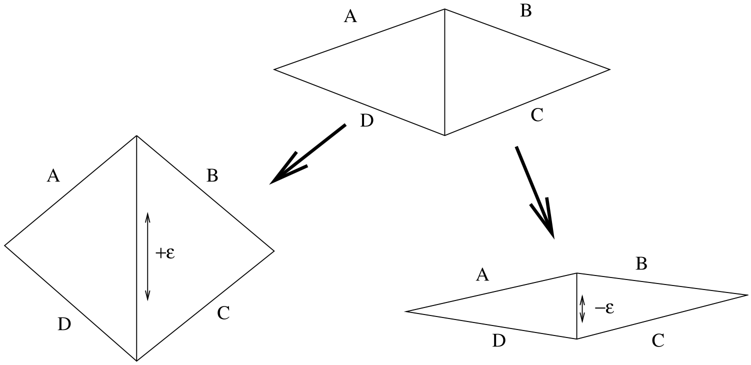

Fig. 4. Creating a neighborhood of ω. The case for a general polygon can be reduced to the case of a quadrilateral by excising all but the 4 edges incident on the diagonal in question.

Proof. First note that if $\mathbf{r}$ has some entry equal to the sum of the other entries, then $P_{\mathcal{T}}(\mathbf{r})$ is a fiber over a point in a facet of D_n, therefore the unique point of $P_{\mathcal{T}}(\mathbf{r})$ is on a facet of $P_{\mathcal{T}}$, so suppose this is not the case. If ω is an interior point of $P_{\mathcal{T}}(\mathbf{r})$ then there is a $|I(\mathcal{T})|$-dimensional neighborhood of ω in $P_{\mathcal{T}}(\mathbf{r})$, this implies that the weighting $\omega(e)$ of each edge $e \in E(\mathcal{T})$ can be expanded and contracted while keeping the weight in $P_{\mathcal{T}}(\mathbf{r})$ without changing the other weights on the members of $L(\mathcal{T})$. If some triangle inequality is an equality, then some triangle τ in the polygon defined by ω is degenerate. Let e be the edge of $\mathcal{T}$ dual to the longest edge in τ, by the degeneracy of τ, the length of this edge cannot be increased without also increasing the lengths of one of the other two edges in τ. This implies that all triangle inequalities must be strict on ω, so this weight is an internal point in $P_{\mathcal{T}}$. $\square$

The content of this proposition can also be found in [KM]. The following gives an algebraic characterization of the interior lattice points of $P_{\mathcal{T}}$ and $P_{\mathcal{T}}(\mathbf{r})$.

Proposition 3.2. *A non-negative integer weighting $\omega \in P_{\mathcal{T}}$ is in the interior if and only if $\omega = \eta + 2_{\mathcal{T}}$ for some $\eta \in P_{\mathcal{T}}$.*

Proof. If ω is in the interior of $P_{\mathcal{T}}$ then all inequalities defined by the condition Δ_2 are strict. After converting to the piping model, we must have $T_{\mathcal{T}}(\omega)(x_{ij}(\tau)) \geqslant 1$, for each trinode $\tau \in V(\mathcal{T})$. This implies that ω has $2_{\mathcal{T}}$ as a factor. Running this argument in reverse gives the converse. $\square$

Corollary 3.3. *If $\omega \in P_{\mathcal{T}}(\mathbf{r})$ is an interior point, then $\omega = 2_{\mathcal{T}} + \eta$ for some $\eta \in P_{\mathcal{T}}(\mathbf{r} - \vec{2})$*

Corollary 3.4. *The toric algebra $\mathbb{C}[P_{\mathcal{T}}]$ is Gorenstein.*

This second corollary also follows from the same theory that gave us Theorem 1.3, and the fact that the algebra of the Plücker embedding of $Gr_2(\mathbb{C}^n)$ is Gorenstein. As a consequence of Proposition 3.2 and Corollary 3.3 we get the following proposition.

Proposition 3.5. *$P_{\mathcal{T}}(\mathbf{r})$ has a unique interior point if and only if $\mathbf{r} = \vec{2} + \vec{R}$ such that $P_{\mathcal{T}}(\vec{R})$ is a single point.*

The next proposition classifies all $\vec{R}$ which have this property.

Proposition 3.6. *$P_{\mathcal{T}}(\vec{R})$ is exactly one point if and only if $\vec{R}$ satisfies one of the following*:

(1) *$R_i = \Sigma_{i \neq j} R_j$ for some i.*
(2) *$\Delta_2(R_i, R_j, R_k)$ holds for some i, j, k and $R_\ell = 0$ for all $\ell \neq i, j, k$.*

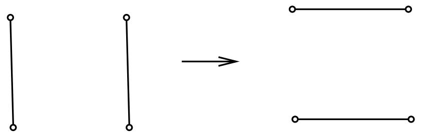

Fig. 5. Creating another weighting.

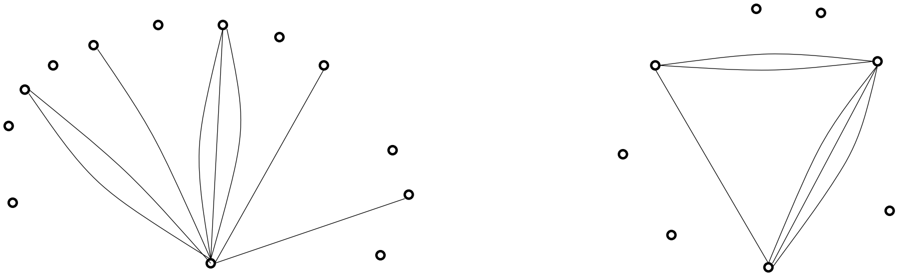

Fig. 6. Associated graphs for $\vec{R}$ from the proof of Proposition 3.6.

Proof. For sufficiency, note that there is exactly one polygon fitting the description given by both cases above. For necessity, we consider the piping model $T_{\mathcal{T}}(\omega)$ of the tree weighting. We suppose $P_{\mathcal{T}}(\mathbf{r})$ is a single point and classify the pipe arrangements allowed for weightings of $\mathcal{T}$. Suppose we were allowed an arrangement of pipes where two edges do not share any common vertex. Then we may swap pipes while maintaining the edge weights, as in Fig. 5. This implies that any pair of pipes must share a common vertex. There are exactly two ways for this to happen, the reader can verify that $S_{\mathcal{T}} \circ T_{\mathcal{T}}(\omega) = \omega$ must satisfy the edge weight conditions in the statement of the theorem. □

Corollary 3.7. *$P_{\mathcal{T}}(\mathbf{r})$ has a unique interior lattice point if and only if $P_{\mathcal{T}'}(\mathbf{r})$ has a unique interior lattice point, for all $\mathcal{T}'$.*

When we convert the above weighting conditions to their graphical representation on the set $L(\mathcal{T})$, we get the possibilities represented below in Fig. 6. One possibility is a graph where every pipe shares a common incident vertex, the second possibility has exactly three vertices with incident pipes. Propositions 3.5 and 3.6 then prove Theorem 1.6.

4. Proof of Theorem 1.7

Theorem 1.6 gives a necessary condition for $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ to be Gorenstein. Now we see what must be added in order to ensure that all interior lattice points carry the unique interior lattice point $\omega_{\mathbf{r}}(\mathcal{T})$ as a summand. We will make use of the piping model for most of this section. For the cases presented in the statement of Theorem 1.6, the first case has $N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}}) = R_j$ and $N_{kj}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}}) = 0$ for all k, $j \neq i$, and the second case has $N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}}) = \frac{1}{2}(R_i + R_j - R_k)$ and $N_{m\ell}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}}) = 0$ for ℓ or $m \neq i, j, k$.

Proposition 4.1. *Let $\mathbf{r} = \vec{2} + \vec{R}$, where $\vec{R}$ satisfies the conditions of Proposition* 3.6. *$\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ is Gorenstein if and only if there is no interior weighting ω in degree $k \geqslant a$ such that $N_{ij}(\omega - 2_{\mathcal{T}}) < N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}})$ for all i, j.*

Proof. After converting ω to the piping model and removing the complete cycle on $L(\mathcal{T})$ corresponding to $2_{\mathcal{T}}$ we get the graph of $\omega - 2_{\mathcal{T}}$. It is clear that if $N_{ij}(\omega - 2_{\mathcal{T}}) \geqslant N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}})$ for all i, j then $\omega_{\mathbf{r}}(\mathcal{T})$ is a summand of ω.

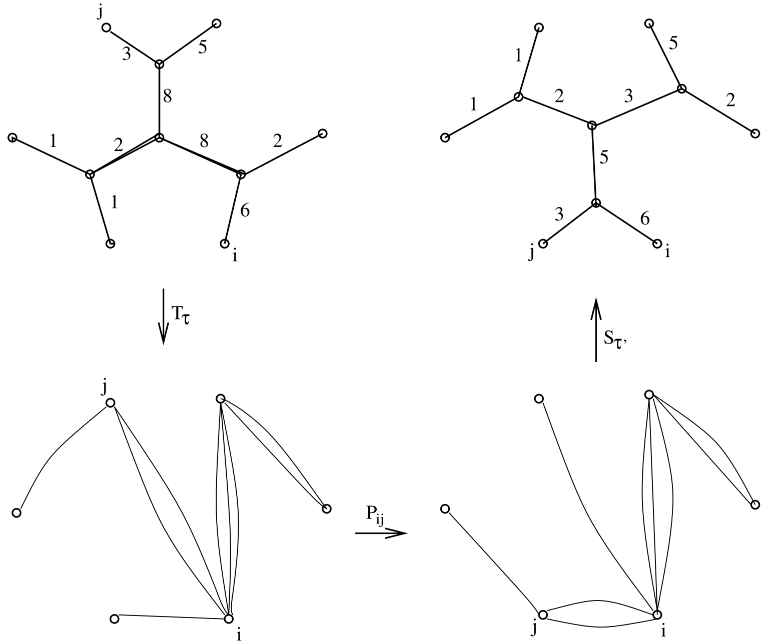

Fig. 7. Proof of Theorem 1.7.

For the converse, suppose $N_{ij}(\omega - 2_{\mathcal{T}}) < N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}})$ for some $i, j \in L(\mathcal{T})$. We find a weighting ω' on a new tree $\mathcal{T}'$ which has a pair of leaves i' and j' connected to a common trinode τ, with the number of pipes between i' and j' in the trinode equal to $N_{ij}(\omega)$. To do this, simply exchange members of $L(\mathcal{T})$ with a permutation σ so that $\sigma(i) = i'$ and $\sigma(j) = j'$ are next to each other, and choose a $\mathcal{T}'$ such that these leaves are now incident on a common internal vertex τ. Carrying the graph corresponding to ω along with the permutation σ produces a new graph which may have crossings, but this does not matter, as no crossings can be introduced between i' and j'.

We consider the weighting $S_{\mathcal{T}'} \circ \sigma \circ T_{\mathcal{T}}(\omega) = \omega'$. By Corollary 3.7, there exists a unique internal lattice point $\omega_{\sigma(\mathbf{r})}(\mathcal{T}')$ in the polytope $P_{\mathcal{T}'}(\sigma(\mathbf{r}))$ with $N_{i'j'}(\omega_{\sigma(\mathbf{r})}(\mathcal{T}')) = N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}))$. By construction $N_{i'j'}(\omega' - 2_{\mathcal{T}'}) < N_{i'j'}(\omega_{\sigma(\mathbf{r})}(\mathcal{T}') - 2_{\mathcal{T}'})$, this implies $\omega_{\sigma(\mathbf{r})}(\mathcal{T}')|_\tau$ cannot divide $\omega'|_\tau$. It follows that $\omega_{\sigma(\mathbf{r})}(\mathcal{T}')$ cannot divide ω', and that $\mathbb{C}[S_{\mathcal{T}'}(\sigma(\mathbf{r}))]$ is not Gorenstein.

The permutation group $\mathcal{S}_n$ acts on the algebra of global sections of $Gr_2(\mathbb{C}^n)$ given by the Plücker embedding by permuting the entries of the multigrading, so we get $\mathbb{C}[M_{\mathbf{r}}] \cong \mathbb{C}[M_{\sigma(\mathbf{r})}]$. Now by Theorem 1.3 and Corollary 1.4, $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ cannot be Gorenstein either. □

Now we are ready to prove Theorem 1.7, this is accomplished with the next proposition. (It is shown in Fig. 7.)

Proposition 4.2. *For any $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ such that some multiple of $\mathbf{r}$ satisfies the criteria of Theorem 1.6, there is a weighting ω which has $N_{ij}(\omega - 2_{\mathcal{T}}) < N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}})$ if and only if $N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}})$ is less than $n - 4$ when it is nonzero.*

Proof. We must show that a weighting ω can be created with $N_{ij}(\omega - 2_{\mathcal{T}}) < N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}})$ if and only if $N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}})$ is less than $n - 4$. First we note that it is necessary to have

$$\sum_{\ell \neq i,j} \left[\frac{k}{a}(R_\ell + 2) - 2\right] - \left[\frac{k}{a}(R_i + 2) - 2\right] - \left[\frac{k}{a}(R_j + 2) - 2\right] + 2N_{ij}\big(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}}\big) > 0, \quad (7)$$

where a is the degree of $\omega_{\mathbf{r}}(\mathcal{T})$ and k is the degree of ω. To see this, note that the $\omega - 2_{\mathcal{T}}$ weight on the ℓ-th leaf of $\mathcal{T}$ must be

$$kr_\ell = \frac{k}{a}ar_\ell - 2 = \left[\frac{k}{a}(R_\ell + 2) - 2\right]. \tag{8}$$

It then follows that the sum of the $\omega - 2_{\mathcal{T}}$ weights on i and j,

$$\left[\frac{k}{a}(R_i + 2) - 2\right] + \left[\frac{k}{a}(R_j + 2) - 2\right] \tag{9}$$

must be less than or equal to the doubled count of edges between i and j, $2N_{ij}(\omega - 2_{\mathcal{T}})$, plus the sum of valences of the other vertices, $\sum_{\ell \neq i,j}[\frac{k}{a}(R_\ell + 2) - 2]$. Since we assumed $N_{ij}(\omega_{\bar{r}}(\mathcal{T}) - 2_{\mathcal{T}}) > N_{ij}(\omega - 2_{\mathcal{T}})$, we obtain the inequality (7).

It remains to show how this inequality reduces to $N_{ij}(\omega_{\mathbf{r}}(\mathcal{T}) - 2_{\mathcal{T}}) < n - 4$. In the case where $R_1 = \sum_{j \neq 1} R_j$, we have $R_j = N_{1j}(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}})$, so the inequality reduces to

$$2\left[\frac{k}{a} - 1\right](n-4) > 2\left[\frac{k}{a} - 1\right]\big(N_{1j}\big(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}}\big)\big) \tag{10}$$

for any of the non-zero nonzero $N_{1j}(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}})$. This clearly implies$N_{1j}(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}}) < n - 4$.

Conversely, if $N_{1j}(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}}) < n - 4$ then we can recover this inequality for $k = 2a$, and construct a graph G with the desired properties as follows. We assume without loss of generality that $j = 2$, and between the leaves 1 and 2 we put R_2 edges, note that this is less than the required $R_2 + 1$ for $\omega_{\mathcal{T}}(\mathbf{r})$ to divide $\omega(G)$. Between 1 and $j \neq 2$ we put $2R_j$ edges, and we add a complete planar cycle. The resulting graph requires 2 more edges at each vertex $\neq 1, 2$ and $R_2 + 2$ more edges at vertices 1 and 2 to have the correct multi-degree. We have assumed that $R_2 < n - 4$, so it follows that $2R_2 + 4 < 2n - 4 = 2(n-2)$. This ensures that there are enough spots left to assign edges to G in order to obtain the correct multidegree.

In the second case, where we have that $\Delta_2(R_1, R_2, R_3)$ holds with all other $R_\ell = 0$, we may assume without loss of generality that $a = 1$. Assuming $i = 2$, $j = 3$, the above inequality becomes

$$2k\frac{(R_1 - R_2 - R_3)}{2} + 2(k-1)(n-4) + 2N_{23}\big(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}}\big) > 0. \tag{11}$$

We use the identity $\frac{(R_1 - R_2 - R_3)}{2} = -N_{23}(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}})$ to obtain

$$2[k-1](n-4) > 2[k-1]N_{23}\big(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}}\big). \tag{12}$$

Since k must be greater than 1, this implies the inequality.

Conversely, if $N_{23}(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}}) < n - 4$ then we may recover the inequality above for $k = 2$, and construct a graph H with the desired properties as follows. Between 2 and 3 we place $N_{23}(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}})$ edges, note that this is less than the number needed for $\omega(H)$ to carry $\omega_{\mathcal{T}}(\mathbf{r})$ as a divisor. We complete this to a cycle by adding a single edge between each consecutive pair $(3,4), \ldots, (n-1,n), (n,1), (1,2)$. Now we add $2N_{12}(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}}) + 1$, and $2N_{12}(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}}) + 1$ edges between $1, 2$ and $1, 3$ respectively. To finish, we must place edges in such a way that 2 and 3 each receive $N_{23}(\omega_{\mathcal{T}}(\mathbf{r}) - 2_{\mathcal{T}}) + 4 \leqslant n - 1$ more edges. There are $2(n-2)$ spots left to fill from the remaining vertices, so this is always possible. $\square$

5. The *a*-invariant

Since the polytopes $P_{\mathcal{T}}(\mathbf{r})$ are the fibers of π, a morphism of convex cones induced by ambient linear map, we get $P_{\mathcal{T}}(k\mathbf{r}) = kP_{\mathcal{T}}(\mathbf{r})$. This allows us to prove Theorem 1.8. This theorem is implied by the following proposition.

Proposition 5.1. *If $P_{\mathcal{T}}(k\mathbf{r})$ has a unique internal lattice point then k must divide* $2(n-2)$.

Proof. If $n \leqslant 3$ then the semigroup algebra is isomorphic to $\mathbb{C}[x]$. Furthermore, if any $k\mathbf{r}_i = 2$ then $k = 1$ or 2. This takes care of all cases except when $R_i = \Sigma_{i \neq j} R_j$ and all $R_j > 0$. In this case, k must divide the expression

$$\Sigma_{i \neq j}(R_j + 2) - (R_i + 2) = R_i + 2(n-1) - R_i - 2 = 2(n-2). \qquad \square \tag{13}$$

Example 5.2. (Gorenstein property first shown by B. Howard and M. Herring, [HH].) Consider the case $\mathbf{r} = (1, \ldots, 1) = \vec{1}$. This case satisfies all the conditions of theorem 1.7, with the unique interior point occurring in the polytope $P_{\mathcal{T}}(\vec{2})$, the lattice points of which give the degree 2 part of the algebra. Therefore $\mathbb{C}[S_{\mathcal{T}}(\vec{1})]$ and $\mathbb{C}[M_{\vec{1}}]$ are Gorenstein, with a-invariant equal to 2. The latter algebra is of particular importance in [HMSV].

Example 5.3. In order to see the range of possible a-invariants, we'll look at a small example. Consider the weights $(1, 1, 2, 4, 6)$, the third graded component of $\mathbb{C}[M_{(1,1,2,4,6)}]$ has weights $(3, 3, 6, 12, 18) = (2, 2, 2, 2, 2) + (1, 1, 4, 10, 16)$. Since $16 = 1 + 1 + 4 + 10$, and each number is greater than or equal to $5 - 4 = 1$, this algebra is Gorenstein with the generator of the canonical module in degree 3.

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