JOURNAL OF ALGEBRA 14, 341–349 (1970)

Dominant Modules

TOYONORI KATO

College of General Education, Tôhoku University, Sendai, Japan

Submitted by P. M. Cohn

Received February 28, 1969

At the spring meeting of the Mathematical Society of Japan, in 1968, professors K. Morita and H. Tachikawa gave a lecture entitled "On semi-primary QF-3 rings," in which they have given a new equivalence between module categories (see Kato [3], Theorem 2). Successively, Tachikawa [12] has generalized and refined this type of equivalences and has communicated orally to the author an outline of his results [12], such that, the above equivalences are free from the "QF-3"-ness of rings and the self-injective dimension [$E(R_R)$-dominant dimension in our terms] of rings plays a vital role in the double centralizer property not only for faithful injective modules but also for faithful projective modules. These results are the origin of the present work.

Let R be a ring, $_RP$ a faithful, finitely generated projective left R module, $Q = \text{End}(_RP)$ the endomorphism ring of $_RP$. In case P_Q contains a copy of each simple right Q module, $_RP$ is called a dominant module. Denote by $\mathcal{M}_Q$ {respectively, $\mathcal{L}[E(R_R)]$} the category of right Q modules (respectively, of right R modules having $E(R_R)$-dominant dimension $\geqslant 2$, where $E(R_R)$ is the injective hull of R_R), and let $H = \text{Hom}(P_Q\, ,\ )_R$ be a functor $\mathcal{M}_Q \to \mathcal{M}_R$. Then our main Theorem 1 states that $_RP$ is a dominant module if, and only if, H is an equivalence $\mathcal{M}_Q \to \mathcal{L}[E(R_R)]$. As an interesting byproduct of Proof of Theorem 1, we have the following: if $E(R_R)/R \hookrightarrow \prod E(R_R)$, then every faithful, finitely generated projective left R module has the double centralizer property. Moreover, in case R has a left dominant module, the converse holds. Section 3 is devoted to examples of dominant modules. We show in Example 3 that, R has a left dominant module and domi. dim $R_R \geqslant 2$ if, and only if, R is the endomorphism ring of a generator-cogenerator in the category of right modules. The final Example 4 states that each semiperfect ring R, for which every nonzero right ideal has a nonzero socle, always has a left dominant module of the form Re, $e = e^2 \in R$.

Throughout this paper, rings will have a unit element and modules will be unital. We adopt the notational device of writing homomorphisms of modules on the side opposite the scalars.

1. Preliminaries

Let R be a ring, and $\mathscr{M}_R$ the category of right R modules. A_R (or $A_R \in \mathscr{M}_R$) will denote the fact that A is a right R module. Now let X_R and U_R be modules. In case $X_R \hookrightarrow \prod U_R$, where $\prod U_R$ is a direct product of copies of U_R, X_R is called U torsionless. If A_R has an injective resolution

$$0 \to A_R \to X_1 \to X_2 \to \cdots \to X_n$$

with the X_i all U torsionless, then we shall say that A_R has U-dominant dimension $\geqslant n$ (denoted by U-domi. dim $A_R \geqslant n$). In case $U_R = R_R$, "U-torsionless" is "torsionless", and U-domi. dim $A_R =$ domi. dim A_R. We shall denote by $\mathscr{L}(U_R)$ the category of right R modules having U dominant dimension $\geqslant 2$.

Let ${}_RP$ be faithful and finitely generated projective, and $Q = \operatorname{End}({}_RP)$ the endomorphism ring of ${}_RP$. Suppose that P_Q contains a copy of each simple right Q module. Let us call such a faithful, finitely generated projective left R module ${}_RP$ a dominant module. It is obvious that each left S ring (for definition, see Kato [2], p. 236, or Onodera [9], p. 404) has a left dominant module. On the other hand, a commutative ring R has a dominant module if, and only if, R is an S ring (see Morita [6]).

Let $B_R \subset A_R$ be modules. We denote by $A_R \,{}'\!\supset B_R$ the fact that A_R is an essential extension of B_R and by $E(B_R)$ the injective hull of B_R. $A_R = E(B_R)$ means that A_R is injective and $A_R \,{}'\!\supset B_R$. We must distinguish $A_R = E(B_R)$ from $A_R \approx E(B_R)$ ($A_R \approx E(B_R)$ does not necessarily imply $A'_R \supset B_R$). We shall have need of the following criterion for essential extensions.

Lemma 1. *Let $B_R \subset A_R$ be modules. Then the following statements are equivalent*:

(1) $A_R \,{}'\!\supset B_R$.

(2) *If $B_R \to A_R \to X_R$ is a monomorphism, then $A_R \to X_R$ is a monomorphism, where $B_R \to A_R$ is the inclusion map and $A_R \to X_R$ is arbitrary.*

2. Dominant Modules

Throughout this section, let R be a ring, ${}_RP$ a faithful, finitely generated projective left R module, $Q = \operatorname{End}({}_RP)$ the endomorphism ring of ${}_RP$, and $R' = \operatorname{End}(P_Q)$ the double centralizer of ${}_RP$. We shall regard R as a subring of R' by virtue of the faithfulness of ${}_RP$. We now define two covariant morphism functors H and H^* which play an important role in this work.

$$H(B_Q) = \operatorname{Hom}(P_Q, B_Q)_R \quad \text{for} \quad B_Q \in \mathscr{M}_Q,$$
$$H^*(A_R) = \operatorname{Hom}[({}_RP)^*{}_R, A_R]_Q \quad \text{for} \quad A_R \in \mathscr{M}_R,$$

where $({}_RP)^* = {}_Q\mathrm{Hom}({}_RP, {}_RR)_R$. It is worth noting that $H(B_Q)$ also acquires the structure of a right R' module by virtue of the bimodule structure of ${}_{R'}P_Q$. Motivated mainly by the new equivalence introduced by Morita and Tachikawa, we are now in a position to establish the following theorem.

THEOREM 1. *Let R, ${}_RP$, Q, R', and H, H^* be as above. Then we have*

(1) *H^*H is natural equivalent to the identity functor on $\mathscr{M}_Q$.*

(2) *$H[E(B_Q)]_R = E[H(B_Q)_R]$ for each $B_Q \in \mathscr{M}_Q$.*

(3) *R'_R is an essential extension of R_R, consequently, $E(R'_R) = E(R_R)$ and $E(R'_R)_{R'} = E(R'_{R'})$.*

(4) *$HH^*(A_R) \approx A_R$ for each $A_R \in \mathscr{L}[E(R_R)]$.*

(5) *$H^*[E(A_R)] = E[H^*(A_R)]$ for each $E(R_R)$*-torsionless *$A_R \in \mathscr{M}_R$.*

(6) *If $R_R \in \mathscr{L}[E(R_R)]$, then $R' = R$.*

(7) *H is an equivalence $\mathscr{L}[E(P_Q)] \to \mathscr{L}[E(R_R)]$.*

(8) *${}_RP$ is a dominant module if, and only if, H is an equivalence $\mathscr{M}_Q \to \mathscr{L}[E(R_R)]$. If such is the case, then $R'_{R'} \in \mathscr{L}[E(R'_{R'})]$.*

Remark. Since P_Q is a generator in $\mathscr{M}_Q$, by C. L. Walker and E. A. Walker ([13], Corollary 3.3), we have

(2′) $H[E(B_Q)]_{R'} = E[H(B_Q)_{R'}]$ for each $B_Q \in \mathscr{M}_Q$, which is an immediate consequence of (2) above.

Proof. First of all, we introduce a natural transformation η_A for each $A_R \in \mathscr{M}_R$,

$$\eta_A : A_R \to HH^*(A_R) = \mathrm{Hom}\{P_Q, \mathrm{Hom}[({}_RP)^*{}_R, A_R]_Q\}_R$$

by $[\eta_A(a)p]f = a(pf)$ for $a \in A_R$, $p \in {}_RP_Q$, and $f \in {}_Q({}_RP)^*{}_R$.

(1) For each $B_Q \in \mathscr{M}_Q$ we have isomorphisms $H^*H(B_Q) = \mathrm{Hom}[({}_RP)^*{}_R, \mathrm{Hom}(P_Q, B_Q)_R]_Q \approx \mathrm{Hom}[({}_RP)^* \otimes_R P_Q, B_Q]_Q \approx \mathrm{Hom}[\mathrm{Hom}({}_RP, {}_RP)_Q, B_Q]_Q$ (since ${}_RP$ is finitely generated projective) $\approx \mathrm{Hom}(Q_Q, B_Q)_Q \approx B_Q$. A routine varification shows that each of the above isomorphisms is natural.

(2) Since ${}_RP$ is projective, $H[E(B_Q)]_R$ is injective for $B_Q \in \mathscr{M}_Q$ (see Cartan and Eilenberg [1], Proposition 1.4, p. 107). Moreover, $H[E(B_Q)]_R \;{}'\!\supset H(B_Q)_R$. To see this, let $H(B_Q)_R \to H[E(B_Q)]_R \to X_R$ be a monomorphism. Then $H^*H(B_Q) \to H^*H[E(B_Q)] \to H^*(X_R)$ is also a monomorphism. But the following commutative diagram

$$\begin{array}{ccc} H^*H(B_Q) & \longrightarrow & H^*H[E(B_Q)] \\ \wr & & \wr \\ B_Q & \longrightarrow & E(B_Q) \end{array}$$

implies $H^*H[E(B_Q)]\ '\!\supset H^*H(B_Q)$. Consequently, the above map

$$H^*H[E(B_Q)] \to H^*(X_R)$$

must be a monomorphism by Lemma 1 and it leads to a commutative diagram

$$\begin{array}{ccc} HH^*H[E(B_Q)] & \hookrightarrow & HH^*(X_R) \\ {\scriptstyle \eta}\wr & & \uparrow{\scriptstyle \eta_X} \\ H[E(B_Q)] & \longrightarrow & X_R , \end{array}$$

concluding that $H[E(B_Q)] \to X_R$ is a monomorphism. Thus in view of Lemma 1, we have $H[E(B_Q)]_R\ '\!\supset H(B_Q)_R$. Therefore,

$$H[E(B_Q)]_R = E[H(B_Q)_R].$$

(3) Note that, for $A_R \in \mathcal{M}_R$ $A_R \approx H(B_Q)$ for some $B_Q \in \mathcal{M}_Q$ if, and only if, η_A is an isomorphism. This is verified making use of the natural equivalence $B_Q \approx H^*H(B_Q)$ given in (1). Therefore, since $R'_R = H(P_Q)$, $\eta_{R'} : R'_R \to HH^*(R'_R)$ is an isomorphism. Now the inclusion map $R_R \to R'_R$ induces an isomorphism $H^*(R_R) \to H^*(R'_R)$, for the composition map

$$P_Q \approx H^*(R_R) \to H^*(R'_R) = H^*H(P_Q) \approx P_Q$$

is identical. We are now ready to prove $R'_R\ '\!\supset R_R$. Keeping Lemma 1 in mind, if $R_R \to R'_R \to X_R$ is a monomorphism, so is $H^*(R_R) \approx H^*(R'_R) \to H^*(X_R)$. We have thus a commutative diagram

$$\begin{array}{ccc} HH^*(R'_R) & \longrightarrow & HH^*(X_R) \\ \wr & & \uparrow{\scriptstyle \eta_X} \\ R'_R & \longrightarrow & X_R \end{array}$$

with the upper horizontal map a monomorphism. It follows that $R'_R \to X_R$ is a monomorphism. It is now clear that $E(R'_R) = E(R_R)$ since $E(R'_R)\ '\!\supset R'_R\ '\!\supset R_R$. It remains to show that $E(R'_R)_{R'} = E(R'_{R'})$. In fact $E(R'_R) \approx E[HH^*(R'_R)] = H\{E[H^*(R'_R)]\}_{R'}$ is R' injective (${}_RP$ is finitely generated projective $\Rightarrow P_Q$ is a generator in $\mathcal{M}_Q \Rightarrow {}_{R'}P$ is finitely generated projective by Morita [5], Lemma 3.3) and $E(R'_R)_{R'}\ '\!\supset R'_{R'}$ $(E(R'_R)\ '\!\supset R'_R)$. Thus, $E(R'_R)_{R'} = E(R'_{R'})$.

(4) Since $H[E(P_Q)]_R = E[H(P_Q)_R] = E(R'_R) = E(R_R)$ by (2) and (3), $\eta : E(R_R) \to HH^*[E(R_R)]$ is an isomorphism, and hence, we have an isomorphism

$$\prod E(R_R) \approx \prod HH^*[E(R_R)] \approx HH^*\left[\prod E(R_R)\right].$$

It should be remarked that this isomorphism is nothing else but the η. In case C_R is $E(R_R)$ torsionless, η_C is a monomorphism by the commutativity of the following diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & C_R & \longrightarrow & \prod E(R_R) \\ & & {\scriptstyle \eta_C}\downarrow & & \wr\wr \\ 0 & \longrightarrow & HH^*(C_R) & \longrightarrow & HH^*\left[\prod E(R_R)\right]. \end{array}$$

In case A_R is injective and $E(R_R)$ torsionless, η_A is an isomorphism, for we have an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_R & \longrightarrow & \prod E(R_R) & \longrightarrow & C_R & \longrightarrow & 0 \\ & & {\scriptstyle \eta_A}\downarrow & & \wr\wr & & \downarrow{\scriptstyle \eta_C} & & \\ 0 & \longrightarrow & HH^*(A_R) & \longrightarrow & HH^*\left[\prod E(R_R)\right] & \longrightarrow & HH^*(C_R) & \longrightarrow & 0 \end{array}$$

with η_C a monomorphism. Finally, let $E(R_R)$-domi. dim $A_R \geqslant 2$. Then we have an exact sequence $0 \to A_R \to E(A_R) \to C_R \to 0$ with both $E(A_R)$ and C_R of $E(R_R)$ torsionless. This yields an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_R & \longrightarrow & E(A_R) & \longrightarrow & C_R & \longrightarrow & 0 \\ & & {\scriptstyle \eta_A}\downarrow & & \wr\wr & & \downarrow{\scriptstyle \eta_C} & & \\ 0 & \longrightarrow & HH^*(A_R) & \longrightarrow & HH^*[E(A_R)] & \longrightarrow & HH^*(C_R) & & \end{array}$$

with η_C a monomorphism. Hence, $\eta_A : A_R \approx HH^*(A_R)$ for each $A_R \in \mathscr{L}[E(R_R)]$.

(5) Let A_R be $E(R_R)$ torsionless. Then $A_R \hookrightarrow \prod E(R_R)$ and hence, $A_R \subset E(A_R) \hookrightarrow \prod E(R_R)$ by virtue of the injectivity of $\prod E(R_R)$. Thus $\eta : E(A_R) \approx HH^*[E(A_R)]$ by the above (4). Now, $H^*[E(A_R)]$ is injective, for $H(E\{H^*[E(A_R)]_Q\}) = E\{HH^*[E(A_R)]_R\} \approx E[E(A_R)] = E(A_R)$ and hence,

$$E\{H^*[E(A_R)]_Q\} \approx H^*H(E\{H^*[E(A_R)]_Q\}) \approx H^*[E(A_R)].$$

To show $H^*[E(A_R)] \,'\!\supset H^*(A_R)$, let $H^*(A_R) \to H^*[E(A_R)] \to X_Q$ be a monomorphism. We must show that $H^*[E(A_R)] \to X_Q$ is also a monomorphism. Keeping the results in (4) above in mind, we have the following commutative diagram

$$\begin{array}{ccccc} HH^*(A_R) & \longrightarrow & HH^*[E(A_R)] & \longrightarrow & H(X_Q) \\ {\scriptstyle \eta_A}\Big\uparrow\!\!\cup & & \wr\wr & & \\ A_R & \longrightarrow & E(A_R) & & \end{array}$$

with $HH^*(A_R) \to HH^*[E(A_R)] \to H(X_Q)$ a monomorphism. But, since $E(A_R) ' \supset A_R$, we conclude that $HH^*[E(A_R)] \to H(X_Q)$ is a monomorphism. Thus we have an exact commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow & H^*HH^*[E(A_R)] & \longrightarrow & H^*H(X_Q) \\ & \wr\wr & & \wr\wr \\ & H^*[E(A_R)] & \longrightarrow & X_Q, \end{array}$$

which shows that $H^*[E(A_R)] \to X_Q$ is a monomorphism. In view of Lemma 1, we have thus proved $H^*[E(A_R)] = E[H^*(A_R)]$.

(6) Suppose $E(R_R)$-domi. dim $R_R \geqslant 2$. Then $\eta_R : R_R \to HH^*(R_R)$ is an isomorphism by (4). Consequently we have an isomorphism

$$R_R \overset{\eta}{\approx} HH^*(R_R) \approx H(P_Q) = R'_R$$

which is the inclusion map $R_R \subset R'_R$. Hence, $R' = R$.

(7) Since we have proved (1) and (4), it suffices to show that

$$H(B_Q) \in \mathscr{L}[E(R_R)]$$

for each $B_Q \in \mathscr{L}[E(P_Q)]$ and $H^*(A_R) \in \mathscr{L}[E(P_Q)]$ for each $A_R \in \mathscr{L}[E(R_R)]$. In case X_R is $E(R_R)$ torsionless, $H^*(X_R)$ is $E(P_Q)$ torsionless. In fact, since $X_R \hookrightarrow \prod E(R_R)$, we have $H^*(X_R) \hookrightarrow H^*[\prod E(R_R)] \approx \prod H^*[E(R_R)] = \prod H^*[E(R'_R)] = \prod E[H^*(R'_R)] = \prod E[H^*H(P_Q)] \approx \prod E(P_Q)$ by (1), (3) and (5). Now let $A_R \in \mathscr{L}[E(R_R)]$ and let $0 \to A_R \to X_1 \to X_2$ be an exact sequence with X_i injective and $E(R_R)$ torsionless. This yields an exact sequence

$$0 \to H^*(A_R) \to H^*(X_1) \to H^*(X_2),$$

where $H^*(X_i)$ is injective by (5) and $E(P_Q)$ torsionless by the above. Thus, $H^*(A_R) \in \mathscr{L}[E(P_Q)]$. In a similar manner, we have $H(B_Q) \in \mathscr{L}[E(R_R)]$ for $B_Q \in \mathscr{L}[E(P_Q)]$, noting that $H[E(P_Q)] = E[H(P_Q)] = E(R'_R) = E(R_R)$ by (2) and (3).

(8) Suppose that ${}_RP$ is a dominant module. Then P_Q contains a copy of each simple right Q module and hence, $E(P_Q)$ is a cogenerator in $\mathscr{M}_Q$ (see Osofsky [10], Lemma 1). Therefore, $\mathscr{L}[E(P_Q)] = \mathscr{M}_Q$ and H is an equivalence $\mathscr{M}_Q \to \mathscr{L}[E(R_R)]$ by (7). Conversely, assume that H gives an equivalence $\mathscr{M}_Q \to \mathscr{L}[E(R_R)]$. Then, since $E(R_R)$ is a cogenerator in $\mathscr{L}[E(R_R)]$, $E(P_Q)$ is a cogenerator in $\mathscr{M}_Q$, or equivalently, P_Q contains a copy of each simple right Q module. Thus, ${}_RP$ is a dominant module. Finally, let $0 \to P_Q \to X_1 \to X_2$

be an injective resolution of P_Q with the X_i all $E(P_Q)$ torsionless. This leads to an injective resolution of $H(P_Q)_{R'} = R'_{R'}$,

$$0 \to R'_{R'} \to H(X_1)_{R'} \to H(X_2)_{R'},$$

where the $H(X_i)_{R'}$ is $E(R'_{R'})$ torsionless. Thus we have shown

$$R'_{R'} \in \mathscr{L}[E(R'_{R'})].$$

Remark. In view of Tachikawa [12], it is just the core of Theorem 1 that $H[E(P_Q)] = E(R_R)$.

COROLLARY. *If $E(R_R)/R \hookrightarrow \prod E(R_R)$, then every faithful, finitely generated projective left R module has the double centralizer property. In case R has a left dominant module, the converse holds good.*

It may be interesting to compare the above corollary with a result in Kato ([4], Corollary to Theorem 2) which states that each finitely-faithful, injective right R module has the double centralizer property if, and only if, $E(R_R)/R \hookrightarrow \prod E(R_R)$.

3. EXAMPLES

We are now ready to consider examples of dominant modules.

EXAMPLE 1 (Wedderburn). Let R be a simple Artinian ring, ${}_RP$ a minimal left ideal of R. Then ${}_RP$ is a dominant module, $Q = \mathrm{End}({}_RP)$ is a division ring, and $R' = \mathrm{End}(P_Q) = R$. In this case the functor $H : \mathscr{M}_Q \to \mathscr{L}[E(R_R)] = \mathscr{M}_R$ is the Morita equivalence.

EXAMPLE 2. Let R be a commutative ring. Then the following statements are equivalent:

(1) R has a dominant module.

(2) R is an S ring.

The following is closely related to Kato ([3], Theorem 2) and Mueller ([7], Lemma 9) and ([8], Theorem 2).

EXAMPLE 3. The following conditions on a ring R are equivalent:

(1) R has a left dominant module and domi. dim $R_R \geqslant 2$.

(2) R is the endomorphism ring of a generator-cogenerator in the category of right modules.

Proof. (1) implies (2). Let $_RP$ be a dominant module, and domi. dim $R_R \geqslant 2$. Let $Q = \text{End}\,(_RP)$, $R' = \text{End}(P_Q)$. Then P_Q is a generator in $\mathcal{M}_Q$, $R' = R$ by Theorem 1, (6). Since $E(R_R) \hookrightarrow \prod R_R$, Theorem 1, (5) yields

$$E(P_Q) \approx E[H^*(R_R)] = H^*[E(R_R)] \hookrightarrow H^*\left(\prod R_R\right) \approx \prod H^*(R_R) \approx \prod P_Q.$$

This shows that P_Q is a cogenerator in $\mathcal{M}_Q$ since P_Q contains a copy of each simple right Q module (see Sugano [11], Lemma 1). Thus $R = R' = \text{End}(P_Q)$ for a generator-cogenerator P_Q in $\mathcal{M}_Q$.

(2) implies (1). Assume $R = \text{End}(U_Q)$ for a generator-cogenerator U_Q in $\mathcal{M}_Q$. Then domi. dim $R_R \geqslant 2$ by Kato ([3], Theorem 2). Next, since U_Q is a generator in $\mathcal{M}_Q$, $_RU$ is faithful, finitely generated projective and $\text{End}(_RU) = Q$ by the Morita Theorem. Thus, $_RU$ is a dominant module, since U_Q is a cogenerator in $\mathcal{M}_Q$.

The final example tells us that each left perfect ring has a left dominant module.

EXAMPLE 4. Let R be a semiperfect ring for which each nonzero right ideal has nonzero socle. Pick out orthogonal idempotents $e_1, e_2, \ldots, e_n$ in R such that $e_1R/e_1J, e_2R/e_2J, \ldots, e_nR/e_nJ$ is one of each isomorphism type of simple right ideals of R, where J is the Jacobson radical of R. Let $e = e_1 + e_2 + \cdots + e_n$, then Re is a dominant module and the equivalence H is $\mathcal{M}_{eRe} \to \mathcal{L}[E(R_R)]$.

Proof. We first show that $_RRe$ is faithful. To see this, let $0 \neq a \in R$. Since aR contains a simple right ideal isomorphic to e_iR/e_iJ for some i, $aRe_i \neq 0$. Therefore, $aRe \neq 0$, and thus $_RRe$ is faithful. Next we show that Re_Q contains a copy of each simple right Q module, where $Q = \text{End}(_RRe) = eRe$. Note that e_iRe/e_iJe is a typical simple right Q module and that $l(J)e_i \neq 0$ since $l(J)e_i \approx \text{Hom}(e_iR/e_iJ, R_R) \neq 0$, where $l(\ )$ is the left annihilator in R. Thus,

$$\text{Hom}[(e_iRe/e_iJe)_Q, Re_Q] \neq 0 \qquad \text{for each} \quad i = 1, 2, \ldots, n.$$

It follows that Re is a dominant module.

ACKNOWLEDGMENT

The author is indebted to Professor H. Tachikawa for valuable suggestions and comments on the subject of this paper.

References

1. H. Cartan and S. Eilenberg, "Homological Algebra," Princeton Univ. Press, Princeton, N.J., 1956.
2. T. Kato, Torsionless modules, *Tôhoku Math. J.* **20** (1968), 233–242.
3. T. Kato, Rings of dominant dimension ≥ 1, *Proc. Japan. Acad.* **44** (1968), 579–584.
4. T. Kato, Rings of U-dominant dimension ≥ 1, *Tôhoku Math. J.* **21** (1969), 321–327.
5. K. Morita, Duality for modules and its applications to the theory of rings with minimum condition, *Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A* **6**, No. 150 (1958), 83–142.
6. K. Morita, On S-rings in the sense of F. Kasch, *Nagoya Math. J.* **27** (1966), 687–695.
7. B. J. Mueller, Dominant dimension of semiprimary rings, *Crelles J.*, Bd. **232** (1968), 173–179.
8. B. J. Mueller, The classification of algebras by dominant dimension, *Canad. J. Math.* **20** (1968), 398–409.
9. T. Onodera, Über Kogeneratoren, *Arch. Math.* **19** (1968), 402–410.
10. B. L. Osofsky, A generalization of quasi-Frobenius rings, *J. Algebra* **4** (1966), 373–387.
11. K. Sugano, A note on Azumaya's theorem, *Osaka J. Math.* **4** (1967), 157–160.
12. H. Tachikawa, On splitting of module categories, *Math. Z.* **111** (1969), 149–150.
13. C. L. Walker and E. A. Walker, "Quotient Categories of Modules," *in* "Proceedings of the La Jolla Conference on Categorical Algebra," pp. 404-420, Springer-Verlag, Berlin, 1966.