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Nonlinear Lie derivations of triangular algebras[☆]

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ABSTRACT

In this paper we prove that every nonlinear Lie derivation of triangular algebras is the sum of an additive derivation and a map into its center sending commutators to zero.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$, and let $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Recall that a left $\mathcal{A}$-module $\mathcal{M}$ is faithful if $a \in \mathcal{A}$ and $a\mathcal{M} = 0$ implies that $a = 0$. The $\mathcal{R}$-algebra

$$\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

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under the usual matrix operations is called a triangular algebra. The most important examples of triangular algebras are upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras. Cheung [4,5] described commuting maps and Lie derivations of these algebras. Benkovič and Eremita [2] studied commuting traces of biadditive maps and Lie isomorphisms of triangular algebras. Benkovič [3] investigated biderivations of triangular algebras. Wong [19] treated Jordan isomorphisms of triangular algebras, while Zhang and Yu [20] studied Jordan derivations.

Let $\mathcal{A}$ be an algebra on a commutative ring $\mathcal{R}$. A map $\delta : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if it is additive and satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. If there exists an element $a \in \mathcal{A}$ such that $\delta(x) = [x, a]$ for all $x \in \mathcal{A}$, where $[x, a] = xa - ax$ is the Lie product or the commutator of the elements $x, a \in \mathcal{A}$, then δ is said to be an inner derivation. Let $\varphi : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity assumption). We say that φ is a nonlinear Lie derivation if $\varphi([x, y]) = [\varphi(x), y] + [x, \varphi(y)]$ for all $x, y \in \mathcal{A}$.

The structure of additive or linear Lie derivations on rings or algebras has been studied by many authors. For example, see [1,11,13–18,21] and their references. Recently, Cheng and Zhang [6] described nonlinear Lie derivations of upper triangular matrix algebras. In this paper we will investigate nonlinear Lie derivations of triangular algebras.

2. Main result

Let $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and let $Z(\mathcal{U})$ be its centre. It follows from [4, Proposition 3] that

$$Z(\mathcal{U}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : am = mb \text{ for all } m \in \mathcal{M} \right\}. \quad (1)$$

Let us define two natural projections $\pi_{\mathcal{A}} : \mathcal{U} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{U} \to \mathcal{B}$ by

$$\pi_{\mathcal{A}} : \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mapsto a \quad \text{and} \quad \pi_{\mathcal{B}} : \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mapsto b.$$

Then $\pi_{\mathcal{A}}(Z(\mathcal{U})) \subseteq Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{U})) \subseteq Z(\mathcal{B})$, and there exists a unique algebra isomorphism $\tau : \pi_{\mathcal{A}}(Z(\mathcal{U})) \to \pi_{\mathcal{B}}(Z(\mathcal{U}))$ such that $am = m\tau(a)$ for all $m \in \mathcal{M}$.

Let $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ be identities of the algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, and let 1 be the identity of the triangular algebra $\mathcal{U}$. Throughout this paper we shall use following notation:

$$e_1 = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = 1 - e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}$$

and

$$\mathcal{U}_{ij} = e_i \mathcal{U} e_j \quad \text{for } 1 \leqslant i \leqslant j \leqslant 2.$$

It is clear that the triangular algebra $\mathcal{U}$ may be represented as

$$\mathcal{U} = e_1\mathcal{U}e_1 + e_1\mathcal{U}e_2 + e_2\mathcal{U}e_2 = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{22}. \quad (2)$$

Here $\mathcal{U}_{11}$ and $\mathcal{U}_{22}$ are subalgebras of $\mathcal{U}$ isomorphic to $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\mathcal{U}_{12} \subseteq \mathcal{U}$ is a $(\mathcal{U}_{11}, \mathcal{U}_{22})$-bimodule isomorphic to the bimodule $\mathcal{M}$. We also see that $\pi_{\mathcal{A}}(Z(\mathcal{U}))$ and $\pi_{\mathcal{B}}(Z(\mathcal{U}))$ are isomorphic to $e_1Z(\mathcal{U})e_1$ and $e_2Z(\mathcal{U})e_2$, respectively. Then there is an algebra isomorphism $\sigma : e_1Z(\mathcal{U})e_1 \to e_2Z(\mathcal{U})e_2$ such that $am = m\sigma(a)$ for all $m \in \mathcal{U}_{12}$.

In this section, we will prove the following theorem.

Theorem 2.1. *Let $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and let $\varphi : \mathcal{U} \to \mathcal{U}$ be a nonlinear Lie derivation. If $\pi_{\mathcal{A}}(Z(\mathcal{U})) = Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{U})) = Z(\mathcal{B})$, then φ is the sum of an additive derivation and a map into its center $Z(\mathcal{U})$ sending each commutator to zero.*

Next we assume that $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a triangular algebra with $\pi_{\mathcal{A}}(Z(\mathcal{U})) = Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{U})) = Z(\mathcal{B})$, and that $\varphi : \mathcal{U} \to \mathcal{U}$ is a nonlinear Lie derivation. From Eq. (1), we have the following lemma.

Lemma 2.1. *Let $x \in \mathcal{U}$. Then $x \in \mathcal{U}_{12} + Z(\mathcal{U})$ if and only if $[x, m] = 0$ for all $m \in \mathcal{U}_{12}$.*

Lemma 2.2. *$\varphi(0) = 0$ and there exists $n_0 \in \mathcal{U}$ such that $\varphi(e_1) - [e_1, n_0] \in Z(\mathcal{U})$.*

Proof. It is clear that $\varphi(0) = \varphi([0,0]) = [\varphi(0), 0] + [0, \varphi(0)] = 0$. For every $m \in \mathcal{U}_{12}$, we have

$$\begin{aligned}\varphi(m) &= \varphi([e_1, m]) = [\varphi(e_1), m] + [e_1, \varphi(m)] \\ &= \varphi(e_1)m - m\varphi(e_1) + e_1\varphi(m) - \varphi(m)e_1.\end{aligned}$$

It follows that $e_1\varphi(m)e_1 = e_2\varphi(m)e_2 = 0$. Then by Eq. (2),

$$\varphi(m) = e_1\varphi(m)e_1 + e_1\varphi(m)e_2 + e_2\varphi(m)e_2 = e_1\varphi(m)e_2 = [e_1, \varphi(m)]. \tag{3}$$

Hence $[\varphi(e_1), m] = 0$ for all $m \in \mathcal{U}_{12}$. It follows from Lemma 2.1 that

$$\varphi(e_1) = n_0 + z_0 \in \mathcal{U}_{12} + Z(\mathcal{U}) \tag{4}$$

for some $n_0 \in \mathcal{U}_{12}$ and $z_0 \in Z(\mathcal{U})$. From the fact $e_1z_0e_2 = 0$ and Eq. (4), we see that $n_0 = e_1\varphi(e_1)e_2 = [e_1, n_0]$. Thus $\varphi(e_1) - [e_1, n_0] = z_0 \in Z(\mathcal{U})$. The proof is completed. □

Remark 2.1. Let n_0 be as in Lemma 2.2, we define a map $\phi : \mathcal{U} \to \mathcal{U}$ by $\phi(x) = \varphi(x) - [x, n_0]$. Clearly, ϕ is also a nonlinear Lie derivation of $\mathcal{U}$. It follows from Lemma 2.2 that $\phi(e_1) \in Z(\mathcal{U})$. Therefore, without loss of generality, we can assume that $\varphi(e_1) \in Z(\mathcal{U})$.

Lemma 2.3. (a) *$e_1\varphi(x)e_2 = 0$ for all $x \in \mathcal{U}_{11} \cup \mathcal{U}_{22}$.*
(b) *$e_2\varphi(a)e_2 \in e_2Z(\mathcal{U})e_2$ for all $a \in \mathcal{U}_{11}$ and $e_1\varphi(b)e_1 \in e_1Z(\mathcal{U})e_1$ for all $b \in \mathcal{U}_{22}$.*

Proof. (a) Let $x \in \mathcal{U}_{11} \cup \mathcal{U}_{22}$. It follows from the facts $[x, e_1] = 0$ and $\varphi(e_1) \in Z(\mathcal{U})$ that

$$0 = \varphi(0) = \varphi([x, e_1]) = [\varphi(x), e_1] + [x, \varphi(e_1)] = \varphi(x)e_1 - e_1\varphi(x).$$

This implies that $e_1\varphi(x)e_2 = 0$ for all $x \in \mathcal{U}_{11} \cup \mathcal{U}_{22}$.
(b) Let $a \in \mathcal{U}_{11}$ and $b \in \mathcal{U}_{22}$. It follows from (a) and Eq. (2) that

$$\varphi(a) = e_1\varphi(a)e_1 + e_2\varphi(a)e_2 \tag{5}$$

and

$$\varphi(b) = e_1\varphi(b)e_1 + e_2\varphi(b)e_2. \tag{6}$$

On the other hand, we have

$$\begin{aligned}0 &= \varphi(0) = \varphi([a, b]) = [\varphi(a), b] + [a, \varphi(b)] \\ &= \varphi(a)b - b\varphi(a) + a\varphi(b) - \varphi(b)a.\end{aligned}$$

This together with Eqs. (5) and (6) gives us that

$$e_2\varphi(a)e_2b - be_2\varphi(a)e_2 + ae_1\varphi(b)e_1 - e_1\varphi(b)e_1a = 0.$$

It follows that

$$e_2\varphi(a)e_2b - be_2\varphi(a)e_2 = 0 \quad \text{for all } b \in \mathcal{U}_{22}$$

and

$$ae_1\varphi(b)e_1 - e_1\varphi(b)e_1a = 0 \quad \text{for all } a \in \mathcal{U}_{11}.$$

By $\pi_{\mathcal{B}}(Z(\mathcal{U})) = Z(\mathcal{B})$ and $\pi_{\mathcal{A}}(Z(\mathcal{U})) = Z(\mathcal{A})$, then $e_2\varphi(a)e_2 \in Z(\mathcal{U}_{22}) = e_2Z(\mathcal{U})e_2$ and $e_1\varphi(b)e_1 \in Z(\mathcal{U}_{11}) = e_1Z(\mathcal{U})e_1$. The proof is completed. □

Remark 2.2. For each $a \in \mathcal{U}_{11}$ and $b \in \mathcal{U}_{22}$, we define $h_1(a) = e_2\varphi(a)e_2$ and $h_2(b) = e_1\varphi(b)e_1$. It follows from Lemma 2.3(b) that $h_1 : \mathcal{U}_{11} \to e_2Z(\mathcal{U})e_2$ is a map with $h_1([x_1, y_1]) = 0$ for all $x_1, y_1 \in \mathcal{U}_{11}$ and $h_2 : \mathcal{U}_{22} \to e_1Z(\mathcal{U})e_1$ is a map with $h_2([x_2, y_2]) = 0$ for all $x_2, y_2 \in \mathcal{U}_{22}$. Let $\sigma : e_1Z(\mathcal{U})e_1 \to$

$e_2Z(\mathcal{U})e_2$ be the algebra isomorphism such that $am = m\sigma(a)$ for all $a \in e_1Z(\mathcal{U})e_1$ and $m \in \mathcal{U}_{12}$. For each $x \in \mathcal{U}$, we define

$$h(x) = h_2(e_2xe_2) + \sigma^{-1}(h_1(e_1xe_1)) + \sigma(h_2(e_2xe_2)) + h_1(e_1xe_1).$$

Then for every $m \in \mathcal{U}_{12}$,

$$(h_2(e_2xe_2) + \sigma^{-1}(h_1(e_1xe_1)))m = m(\sigma(h_2(e_2xe_2)) + h_1(e_1xe_1)).$$

It follows from Eq. (1) that $h(x) \in Z(\mathcal{U})$ for all $x \in \mathcal{U}$. Hence h is a map from $\mathcal{U}$ into its centre $Z(\mathcal{U})$. It is easy to verify that $h([x, y]) = 0$ for all $x, y \in \mathcal{U}$. Then the map $\psi : \mathcal{U} \to \mathcal{U}$ defined by $\psi(x) = \varphi(x) - h(x)$ is also a nonlinear Lie derivation and $\psi(e_1) \in Z(\mathcal{U})$.

Lemma 2.4. *Let ψ be as in Remark* 2.2. *Then $\psi(\mathcal{U}_{ij}) \subseteq \mathcal{U}_{ij}$ for $1 \leqslant i \leqslant j \leqslant 2$.*

Proof. From Eq. (3), we see that $\psi(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}$. Let $a \in \mathcal{U}_{11}$. It follows from Eq. (5) and the definition of ψ that

$$\begin{aligned}\psi(a) &= e_1\psi(a)e_1 + e_2\psi(a)e_2 = e_1\psi(a)e_1 + e_2\varphi(a)e_2 - e_2h(a)e_2\\ &= e_1\psi(a)e_1 + h_1(a) - h_1(a) = e_1\psi(a)e_1 \in \mathcal{U}_{11}.\end{aligned}$$

Hence $\psi(\mathcal{U}_{11}) \subseteq \mathcal{U}_{11}$. Similarly, we can show that $\psi(\mathcal{U}_{22}) \subseteq \mathcal{U}_{22}$. The proof is completed. $\square$

Lemma 2.5. *Let ψ be as in Remark* 2.2. *Then*

(a) $\psi(am) = \psi(a)m + a\psi(m)$ *for all* $a \in \mathcal{U}_{11}$ *and* $m \in \mathcal{U}_{12}$;
(b) $\psi(nb) = \psi(n)b + n\psi(b)$ *for all* $n \in \mathcal{U}_{12}$ *and* $b \in \mathcal{U}_{22}$.

Proof. (a) Let $a \in \mathcal{U}_{11}$ and $m \in \mathcal{U}_{12}$. Then $am = [a, m]$, and so we have from Lemma 2.4 that

$$\psi(am) = [\psi(a), m] + [a, \psi(m)] = \psi(a)m + a\psi(m).$$

Similarly, we can show that (b) holds. The proof is completed. $\square$

Lemma 2.6. *Let ψ be as in Remark* 2.2. *Then*

(a) $\psi(a + m) - \psi(a) - \psi(m) \in Z(\mathcal{U})$ *for all* $a \in \mathcal{U}_{11}$ *and* $m \in \mathcal{U}_{12}$;
(b) $\psi(n + b) - \psi(n) - \psi(b) \in Z(\mathcal{U})$ *for all* $n \in \mathcal{U}_{12}$ *and* $b \in \mathcal{U}_{22}$.

Proof. (a) Let $a \in \mathcal{U}_{11}$ and $m, n \in \mathcal{U}_{12}$. It follows from $[a, n] = [a + m, n]$ and Lemma 2.4 that

$$[\psi(a), n] + [a, \psi(n)] = [\psi(a + m), n] + [a + m, \psi(n)] = [\psi(a + m), n] + [a, \psi(n)].$$

Then $[\psi(a + m) - \psi(a), n] = 0$ for all $n \in \mathcal{U}_{12}$. By Lemma 2.1, $\psi(a + m) - \psi(a) \in \mathcal{U}_{12} + Z(\mathcal{U})$. This implies that

$$\psi(a + m) - \psi(a) - e_1(\psi(a + m) - \psi(a))e_2 \in Z(\mathcal{U}). \tag{7}$$

Since $\psi(e_1) \in Z(\mathcal{U})$ and $[e_1, x] = e_1xe_2$ for all $x \in \mathcal{U}$, we have

$$\psi(e_1xe_2) = [\psi(e_1), x] + [e_1, \psi(x)] = [e_1, \psi(x)] = e_1\psi(x)e_2. \tag{8}$$

By Eq. (8), then

$$e_1(\psi(a + m) - \psi(a))e_2 = \psi(e_1(a + m)e_2) - \psi(e_1ae_2) = \psi(m).$$

This and Eq. (7) give us that

$$\psi(a+m)-\psi(a)-\psi(m)\in Z(\mathcal{U})$$

for all $a\in\mathcal{U}_{11}$ and $m\in\mathcal{U}_{12}$. Similarly, we can show that (b) holds. The proof is completed. $\square$

Lemma 2.7. *Let ψ be as in Remark* 2.2. *Then*

(a) $\psi(m+n)=\psi(m)+\psi(n)$ *for all* $m,n\in\mathcal{U}_{12}$;
(b) $\psi(a+m+b)-\psi(a)-\psi(m)-\psi(b)\in Z(\mathcal{U})$ *for all* $a\in\mathcal{U}_{11}$, $m\in\mathcal{U}_{12}$ *and* $b\in\mathcal{U}_{22}$.

Proof. (a) Since $\psi(e_1)\in Z(\mathcal{U})$ and $[e_1,x]=[x,e_2]$ for all $x\in\mathcal{U}$, we have

$$\begin{aligned}[\psi(x),e_2]&=[e_1,\psi(x)]=[e_1,\psi(x)]+[\psi(e_1),x]=\psi([e_1,x])\\&=\psi([x,e_2])=[\psi(x),e_2]+[x,\psi(e_2)].\end{aligned}$$

Then $[x,\psi(e_2)]=0$ for all $x\in\mathcal{U}$, and so $\psi(e_2)\in Z(\mathcal{U})$.

Let $m,n\in\mathcal{U}_{12}$. It follows from $m+n=[e_1+m,n+e_2]$ and Lemmas 2.6 and 2.4 that

$$\begin{aligned}\psi(m+n)&=[\psi(e_1+m),n+e_2]+[e_1+m,\psi(n+e_2)]\\&=[\psi(e_1)+\psi(m),n+e_2]+[e_1+m,\psi(n)+\psi(e_2)]\\&=[\psi(m),n+e_2]+[e_1+m,\psi(n)]=\psi(m)+\psi(n).\end{aligned}$$

(b) Let $a\in\mathcal{U}_{11}$, $m\in\mathcal{U}_{12}$ and $b\in\mathcal{U}_{22}$. Then $[a+m+b,n]=[a,n]+[b,n]$ for all $n\in\mathcal{U}_{12}$. By (a) and Lemma 2.4, we get

$$\begin{aligned}&[\psi(a+m+b),n]+[a+m+b,\psi(n)]\\&\quad=\psi([a,n]+[b,n])=\psi([a,n])+\psi([b,n])\\&\quad=[\psi(a),n]+[a,\psi(n)]+[\psi(b),n]+[b,\psi(n)]\\&\quad=[\psi(a)+\psi(b),n]+[a+m+b,\psi(n)],\end{aligned}$$

and so $[\psi(a+m+b)-\psi(a)-\psi(b),n]=0$ for all $n\in\mathcal{U}_{12}$. It follows from Lemma 2.1 that

$$\psi(a+m+b)-\psi(a)-\psi(b)-e_1(\psi(a+m+b)-\psi(a)-\psi(b))e_2\in Z(\mathcal{U}).\tag{9}$$

On the other hand, we have from Eq. (8) that

$$e_1(\psi(a+m+b)-\psi(a)-\psi(b))e_2=\psi(m).$$

This and Eq. (9) show that $\psi(a+m+b)-\psi(a)-\psi(m)-\psi(b)\in Z(\mathcal{U})$. The proof is completed. $\square$

Remark 2.3. From Lemma 2.9(b), we define a map $g:\mathcal{U}\to Z(\mathcal{U})$ by

$$g(x)=\psi(x)-\psi(e_1xe_1)-\psi(e_1xe_2)-\psi(e_2xe_2).\tag{10}$$

Then $g(x)e_1=e_1g(x)e_1=e_1\psi(x)e_1-\psi(e_1xe_1)$, and for every $x,y\in\mathcal{U}$

$$\begin{aligned}g([x,y])e_1&=e_1\psi([x,y])e_1-\psi([e_1xe_1,e_1ye_1])\\&=[e_1\psi(x)e_1,e_1ye_1]+[e_1xe_1,e_1\psi(y)e_1]\\&\quad-[\psi(e_1xe_1),e_1ye_1]-[e_1xe_1,\psi(e_1ye_1)]\\&=[e_1\psi(x)e_1-\psi(e_1xe_1),e_1ye_1]+[e_1xe_1,e_1\psi(y)e_1-\psi(e_1ye_1)]\\&=[g(x)e_1,e_1ye_1]+[e_1xe_1,g(y)e_1]=0.\end{aligned}$$

Similarly, we can show that $g([x,y])e_2=0$. Thus,

$$g([x,y])=g([x,y])e_1+g([x,y])e_2=0$$

for all $x, y \in \mathcal{U}$. Now we define a map $\delta : \mathcal{U} \to \mathcal{U}$ by

$$\delta(x) = \psi(x) - g(x). \tag{11}$$

It is clear that $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ and $\delta(e_1) \in Z(\mathcal{U})$.

Lemma 2.8. *Let δ be as in Remark* 2.3., *and let $i \in \{1, 2\}$. Then δ is an additive derivation on $\mathcal{U}_{ii}$.*

Proof. Let $a, c \in \mathcal{U}_{11}$ and $m \in \mathcal{U}_{12}$. By Lemma 2.5(a), then

$$\delta(acm) = \delta(ac)m + ac\delta(m) \tag{12}$$

and

$$\delta((a + c)m) = \delta(a + c)m + (a + c)\delta(m). \tag{13}$$

From Lemmas 2.5(a) and 2.7(a), we have

$$\delta(acm) = \delta(a)cm + a\delta(cm) = \delta(a)cm + a\delta(c)m + ac\delta(m) \tag{14}$$

and

$$\delta((a + c)m) = \delta(am) + \delta(cm) = \delta(a)m + a\delta(m) + \delta(c)m + c\delta(m). \tag{15}$$

It follows from Eqs. (12), (14) and (13), (15) that

$$(\delta(ac) - \delta(a)c - a\delta(c))m = 0$$

and

$$(\delta(a + c) - \delta(a) - \delta(c))m = 0$$

for all $m \in \mathcal{U}_{12}$. Note that $\mathcal{U}_{12}$ is a faithful left $\mathcal{U}_{11}$-module, we get

$$\delta(ac) = \delta(a)c + a\delta(c) \quad \text{and} \quad \delta(a + c) = \delta(a) + \delta(c)$$

for all $a, c \in \mathcal{U}_{11}$. Hence δ is an additive derivation on $\mathcal{U}_{11}$. Similarly, we can show that δ is also an additive derivation on $\mathcal{U}_{22}$. The proof is completed. □

Now we are in a position to prove our main theorem.

Proof of Theorem 2.1. It follows from the definitions of ψ and δ that

$$\varphi(x) = \psi(x) + h(x) = \delta(x) + g(x) + h(x) = \delta(x) + f(x)$$

for all $x \in \mathcal{U}$, where $f = g + h$ is a map from $\mathcal{U}$ into its center sending each commutator to zero. From Eqs. (10) and (11), we see that $\delta(e_i x e_j) = e_i\delta(x)e_j$ for all $x \in \mathcal{U}$ $(1 \leqslant i \leqslant j \leqslant 2)$. Then for every $a \in \mathcal{U}_{11}$, $m \in \mathcal{U}_{12}$ and $b \in \mathcal{U}_{22}$,

$$\delta(a + m + b) = \delta(a) + \delta(m) + \delta(b). \tag{16}$$

Let $x, y \in \mathcal{U}$, then $x = a + m + b$ and $y = c + n + d$ where $a, c \in \mathcal{U}_{11}$, $m, n \in \mathcal{U}_{12}$ and $b, d \in \mathcal{U}_{22}$. By Eq. (16) and Lemmas 2.7(a) and 2.8,

$$\begin{aligned}\delta(x + y) &= \delta(a + c) + \delta(m + n) + \delta(b + d)\\ &= \delta(a) + \delta(c) + \delta(m) + \delta(n) + \delta(b) + \delta(d)\\ &= \delta(x) + \delta(y).\end{aligned}$$

This and Lemmas 2.4, 2.5 and 2.8 give us that

$$\begin{aligned}\delta(xy) &= \delta(ac) + \delta(an) + \delta(md) + \delta(bd)\\ &= \delta(a)c + a\delta(c) + \delta(a)n + a\delta(n) + \delta(m)d + m\delta(d) + \delta(b)d + b\delta(d)\\ &= (\delta(a) + \delta(m) + \delta(b))y + x(\delta(c) + \delta(n) + \delta(d))\\ &= \delta(x)y + x\delta(y).\end{aligned}$$

We conclude that $\varphi = \delta + f$ is the sum of an additive derivation δ and a center-valued map f sending commutators to zero. The proof is completed. $\square$

Next we give an application of Theorem 2.1 to certain special classes of triangular algebras, such as block upper triangular matrix algebras and nest algebras.

Let $\mathcal{R}$ be a commutative ring with identity and let $M_{n\times k}(\mathcal{R})$ be the set of all $n \times k$ matrices over $\mathcal{R}$. For $n \geqslant 2$ and $m \leqslant n$, The block upper triangular matrix algebra $T_n^{\bar{k}}(\mathcal{R})$ is a subalgebra of $M_n(\mathcal{R})$ of the form

$$\begin{pmatrix} M_{k_1}(\mathcal{R}) & M_{k_1\times k_2}(\mathcal{R}) & \cdots & M_{k_1\times k_m}(\mathcal{R}) \\ 0 & M_{k_2}(\mathcal{R}) & \cdots & M_{k_2\times k_m}(\mathcal{R}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k_m}(\mathcal{R}) \end{pmatrix}$$

where $\bar{k} = (k_1, k_2, \ldots, k_m)$ is an ordered m-vector of positive integers such that $k_1 + k_2 + \cdots + k_m = n$.

Let X be a Banach space over the real or complex field $\mathbb{F}$. Recall that a nest on X is a chain $\mathcal{N}$ of closed subspaces of X containing $\{0\}$ and X which is closed under arbitrary intersection and closed span. The nest algebra associated to $\mathcal{N}$, denoted by $\tau(\mathcal{N})$, is the weakly closed operator algebra consisting of all bounded linear operators that leave $\mathcal{N}$ invariant, i.e.,

$$\tau(\mathcal{N}) = \{T \in \mathcal{B}(X) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

A nest $\mathcal{N}$ is called trivial if $\mathcal{N} = \{0, X\}$. If X is a Hilbert space, then every nontrivial nest algebra is a triangular algebra. However, it is not always the case for a nest $\mathcal{N}$ on a general Banach space X as $N \in \mathcal{N}$ may be not complemented. We also refer the reader to [8] for the theory of nest algebras.

It is clear that every nontrivial nest algebra on a finite dimensional space is isomorphic to a block upper triangular matrix algebra. From Theorem 2.1 and the result of [7,12], we have the following corollary.

Corollary 2.1. *Let $T_n^{\bar{k}}(\mathcal{R})$ be a block upper triangular matrix algebra and $\varphi : T_n^{\bar{k}}(\mathcal{R}) \to T_n^{\bar{k}}(\mathcal{R})$ be a nonlinear Lie derivation. Then there exist $T \in T_n^{\bar{k}}(\mathcal{R})$, an additive derivation $\alpha : \mathcal{R} \to \mathcal{R}$ and a map $f : T_n^{\bar{k}}(\mathcal{R}) \to \mathcal{R}$ sending commutators to zero such that $\varphi(A) = AT - TA + A_\alpha + f(A)I_n$ for all $A = (a_{ij}) \in T_n^{\bar{k}}(\mathcal{R})$, where $A_\alpha = (\alpha(a_{ij}))$ and I_n is the identity of $T_n^{\bar{k}}(\mathcal{R})$.*

For the infinite dimensional case, we have the following corollary.

Corollary 2.2. *Let X be an infinite dimensional Banach space over the real or complex field $\mathbb{F}$, and let $\mathcal{N}$ be a nest on X which contains a nontrivial element complemented in X. Assume that $\varphi : \tau(\mathcal{N}) \to \tau(\mathcal{N})$ is a nonlinear Lie derivation. Then there exist $T \in \tau(\mathcal{N})$ and a functional $f : \tau(\mathcal{N}) \to \mathbb{F}$ with $f([A, B]) = 0$ for every $A, B \in \tau(\mathcal{N})$ such that $\varphi(A) = AT - TA + f(A)I$ for all $A \in \tau(\mathcal{N})$.*

Proof. Let $N \in \mathcal{N}$ be the complemented element. Then $X = N \dot{+} M$ for some closed subspace M. Let $\mathcal{N}_1 = \{N' \cap N : N' \in \mathcal{N}\}$ and $\mathcal{N}_2 = \{N' \cap M : N' \in \mathcal{N}\}$. It follows that

$$\tau(\mathcal{N}) = \begin{pmatrix} \tau(\mathcal{N}_1) & \mathcal{B}(M, N) \\ 0 & \tau(\mathcal{N}_2) \end{pmatrix}$$

is a triangular algebra satisfying the conditions of Theorem 2.1, and so there exist an additive derivation δ of $\tau(\mathcal{N})$ and a functional $f : \tau(\mathcal{N}) \to \mathbb{F}$ sending commutators to zero such that $\varphi(A) = \delta(A) + f(A)I$ for all $A \in \tau(\mathcal{N})$. By the results of [9,10], δ is a linear derivation. Then there exists $T \in \tau(\mathcal{N})$ such that $\delta(A) = AT - TA$ for all $A \in \tau(\mathcal{N})$. Hence $\varphi(A) = AT - TA + f(A)I$ for all $A \in \tau(\mathcal{N})$. The proof is completed. $\square$

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