Kähler magnetic flows for a product of complex space forms

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Abstract

In this paper we study semi-conjugacy of Kähler magnetic flows for a product of complex space forms.

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1. Introduction

A closed 2-form $B$ on a complete Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ with Riemannian metric $(\cdot, \cdot)$ is called a magnetic field. We define a skew symmetric operator $\Omega_B : TM \to TM$ on the tangent bundle $TM$ of $M$ by $B(u, v) = \langle u, \Omega_B(v) \rangle$ for all tangent vectors $u, v \in T_xM$ at an arbitrary point $x \in M$. A smooth curve $\gamma$ parametrized by its arclength is called a trajectory for $B$ if it satisfies the equation $\nabla_{\dot{\gamma}} \dot{\gamma} = \Omega_B(\dot{\gamma})$, where $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along $\gamma$ with respect to the Riemannian connection $\nabla$ on $M$. As $M$ is complete, we find that every trajectory for $B$ is defined on the whole real line $\mathbb{R}$.

The equation for trajectories is a generalization of the Newton equation for a motion of a charged particle under a static magnetic field. When $B$ is trivial, that is, $B$ is the null 2-form $B = 0$, we see that trajectories for $B$ are nothing but geodesics.

An important class which is easy to treat is the class of uniform magnetic fields, which are fields with parallel skew symmetric operator (i.e., $\nabla \Omega_B = 0$). A typical example of uniform magnetic fields is a constant multiple of the Kähler form on a Kähler manifold.

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(M, , J) with complex structure J. We shall call such magnetic fields Kähler magnetic fields. Just like geodesics on M induce the geodesic flow on the unit tangent bundle UM of M we can define magnetic flow B_{\psi_t} on UM by B_{\psi_t}(v) = \gamma_\psi_t(t), where \gamma_\psi_t denotes the trajectory for B with \gamma_\psi_t(0) = v. Although we have only a few results on Kähler magnetic fields, the author hopes that they play an important role in the study of Kähler manifolds from a Riemannian geometric point of view (cf. [2,5]). In the preceding paper [1] we study Kähler magnetic fields on complex space forms, which are complex projective spaces, complex Euclidean spaces and complex hyperbolic spaces. In this paper we study some basic properties on Kähler magnetic flows for a product manifold of complex space forms. We refer the reader to [4] for magnetic flows on a torus and [6] for uniform magnetic fields on a Kähler C-space.

2. Trajectories on a product of complex space forms

We shall start by discussing some properties on trajectories for Kähler magnetic fields on a product of complex space forms. We call a smooth curve γ parametrized by its arclength closed if there is a non-zero constant T with γ(t + T) = γ(t) for all t. For a closed curve we call the minimum positive T with such a property its length. A smooth curve is said to be simple if it does not have self-intersection points. For a smooth curve γ on a Hadamard manifold M we shall call unbounded in both directions if both of the images γ([0, ∞)) and γ((−∞, 0]) are unbounded sets. For such a curve γ we define its limit points at infinity by

\[ γ(∞) = \lim_{t \to ∞} γ(t), \quad γ(−∞) = \lim_{t \to −∞} γ(t) ∈ ∂M \]

if they exist in the ideal boundary ∂M of M.

Let \( B_J \) be the Kähler form on a Kähler manifold (M, , J). For a constant \( \kappa \) we denote by \( B_\kappa \) the Kähler magnetic field \( \kappa B_J \). A trajectory γ for a Kähler magnetic field \( B_\kappa \) is a smooth curve parametrized by its arclength with \( \nabla \dot{γ} \dot{γ} = \kappa J \dot{γ} \), hence it is a circle (see [3] for the definition and some basic properties).

On a complex projective space \( \mathbb{C}P^m(\alpha) \) of constant holomorphic sectional curvature \( \alpha \), every trajectory for \( B_\kappa \) is simple and closed with length \( 2\pi/\sqrt{\kappa^2 + \alpha} \). On a complex Euclidean space \( \mathbb{C}^r \), every trajectory for \( B_\kappa \) is a circle of radius \( 1/|\kappa| \) in the sense of Euclidean geometry, hence it is simple and closed with length \( 2\pi/|\kappa| \). On a complex hyperbolic space \( \mathbb{C}H^n(−β) \) of constant holomorphic sectional curvature −β, every trajectory for \( B_\kappa \) is

1. simple and closed with length \( 2\pi/\sqrt{\kappa^2 − β} \) if \( |\kappa| > \sqrt{β} \),
2. simple and unbounded in both directions if \( |\kappa| ≤ \sqrt{β} \),
3. when \( |\kappa| = \sqrt{β} \), it is so-called a horocycle, that is, it has single point at infinity (i.e., \( γ(∞) = γ(−∞) \)), and if it crosses to a geodesic \( \sigma \) with \( \sigma(∞) = γ(∞) \) then they cross orthogonally at their crossing point,
4. when \( |\kappa| < \sqrt{β} \), it has distinct limit points.
For a smooth curve $\gamma$ on a product manifold $M = M_1 \times M_2$ we set $\gamma = (\gamma_1, \gamma_2)$, where $\gamma_i$ is a smooth curve on $M_i$. When both $M_1$ and $M_2$ are Kähler manifolds and $\gamma$ is a trajectory for $B_\kappa$ on $M$, we have $\nabla_{\dot{\gamma}_i} \dot{\gamma}_i = \kappa J \dot{\gamma}_i$ for $i = 1, 2$, and $\gamma_i$ has constant speed.

By putting $\lambda_i = \|\dot{\gamma}_i(0)\|$ and $\dot{\gamma}_i(t) = \gamma_i(t/\lambda_i)$ in case $\lambda_i \neq 0$, we find $\dot{\gamma}_i$ is a trajectory for $B_{\kappa/\lambda_i}$ on $M_i$. Since $\gamma$ is bounded if and only if both $\gamma_1$ and $\gamma_2$ are bounded, we obtain the following result on trajectories for a Kähler magnetic field on a product of complex space forms.

**Proposition 1.** We consider a Kähler magnetic field $B_\kappa$ on a product of complex space forms

$$M = \mathbb{C}P^{n_1}(\alpha_1) \times \cdots \times \mathbb{C}P^{n_p}(\alpha_p) \times \mathbb{C}' \times \mathbb{C}H^{n_1}(\beta_1) \times \cdots \times \mathbb{C}H^{n_q}(\beta_q).$$

1. If $|\kappa| > \max\{\sqrt{\beta_1}, \ldots, \sqrt{\beta_q}, 0\}$, every trajectory for $B_\kappa$ is bounded. Here, we read the condition to be $\kappa \neq 0$ if $M$ has a Euclidean factor but does not have complex hyperbolic factors.

2. When $M$ has neither a Euclidean factor nor complex projective factors, we put $\beta = 1/(\sum_{j=1}^q 1/\beta_j)$.
   1. If $|\kappa| \leq \sqrt{\beta}$, then every trajectory for $B_\kappa$ is unbounded in both directions and has limit points at infinity.
   2. When $|\kappa| < \sqrt{\beta}$, every trajectory has distinct limit points.
   3. When $|\kappa| = \sqrt{\beta}$, only trajectories $\gamma = (\gamma_1, \ldots, \gamma_q)$ with $\|\dot{\gamma}_j(0)\| = \sqrt{\beta_j}$, $j = 1, \ldots, q$, is horocycle. Other trajectories have distinct limit points.
   4. If $\sqrt{\beta} < |\kappa| \leq \max\{\sqrt{\beta_1}, \ldots, \sqrt{\beta_q}\}$, we have both bounded and unbounded trajectories for $B_\kappa$.

**Proof.** We just make mention on the fourth assertion of (2). It is clear we have unbounded trajectories for $B_\kappa$ which lies on some $\mathbb{C}H^{n_j}(\beta_j)$. We here give an example of a bounded trajectory. We put $\varepsilon = (|\kappa|^2/\beta - 1)/2q$ and set $\lambda_j = \sqrt{|\kappa|^2/\beta_j - \varepsilon}$, $j = 1, \ldots, q - 1$, and $\lambda_q = \sqrt{1 - \sum_{j=1}^q \lambda_j^2}$. As we can see $|\kappa|/\lambda_j > \sqrt{\beta_j}$ we find a trajectory $\gamma$ for $B_\kappa$ on $M$ with $\|\dot{\gamma}_j(0)\| = \lambda_j$ is bounded. □

**Remark 2.** A bounded trajectory $\gamma = (\gamma_1, \ldots, \gamma_p, \gamma_0, \gamma_{p+1}, \ldots, \gamma_{p+q})$ for $B_\kappa$ with $\|\gamma_0\| \neq 0$ on $M$ in Proposition 1 is closed if and only if the $\mathbb{Z}$-module generated by

$$A = \left\{ \frac{2\pi}{\sqrt{\kappa^2 + \alpha_i \|\dot{\gamma}_i\|^2}} \left| \|\dot{\gamma}_i\| \neq 0, \ i = 1, \ldots, p \right\} \cup \left\{ \frac{2\pi}{|\kappa|} \right\} \cup \left\{ \frac{2\pi}{\sqrt{\kappa^2 - \beta_j \|\dot{\gamma}_{p+j}\|^2}} \left| \|\dot{\gamma}_{p+j}\| \neq 0, \ j = 1, \ldots, q \right\}$$

is simply generated. In this case, its length is the least common multiple of the elements of the set $A$. When $\gamma_0 = 0$, the same hold if we omit the element $2\pi/|K|$ from $A$. 


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3. Semi-conjugacy of Kähler magnetic flows for a product manifold

We call two flows \( \phi_t, \psi_t \) on a smooth manifold \( X \) are **smoothly conjugate** in the strong sense if there exist an diffeomorphism \( f \) and a non-zero constant \( \lambda \) with \( f \circ \phi_t = \psi_{\lambda t} \circ f \).

According to the preceding paper [1], we have the following conjugacy on Kähler magnetic flows for complex space forms. For a complex projective space \( \mathbb{C}P^m(\alpha) \), all Kähler magnetic flows are smoothly conjugate each other in the strong sense. More precisely, for each \( \kappa \) there exists a diffeomorphism \( f_\kappa \) of the unit tangent bundle \( U\mathbb{C}P^m(\alpha) \) which satisfies

\[
(f_\kappa \circ B_\kappa \phi_t) = B_\sqrt{\frac{(\kappa^2 + \alpha)}{\kappa^2 + \alpha}} \circ f_\kappa,
\]

where \( \phi_t \) denotes the geodesic flow for \( \mathbb{C}P^m(\alpha) \).

By putting \( f_{\nu,\kappa} = f_{\nu}^{-1} \circ f_\kappa \) for arbitrary constants \( \kappa \) and \( \nu \), we find

\[
f_{\nu,\kappa} \circ B_\kappa \phi_t = B_\sqrt{\frac{(\kappa^2 + \alpha) \nu^2}{\kappa^2 + \alpha} \nu^{-1}} \circ f_{\nu,\kappa}.
\]

For a complex Euclidean space \( \mathbb{C}r \), all Kähler magnetic flows with positive strength are smoothly conjugate each other in the strong sense: For non-zero \( \kappa \) and \( \nu \), a diffeomorphism \( f_{\nu,\kappa} \) of \( U\mathbb{C}r \cong \mathbb{C}r \times \{ u \in \mathbb{C}r | \|u\| = 1 \} \) defined by

\[
(z, u) \mapsto \left( z + \left( \frac{1}{\kappa} - \frac{1}{|\nu|} \right) \sqrt{-1} u, \text{sgn}(\nu)u \right),
\]

where \( \text{sgn}(\nu) \) denotes the signature of a real number \( \nu \), satisfies

\[
f_{\nu,\kappa} \circ B_\kappa \phi_t = B_\sqrt{\frac{(\kappa^2 - \alpha) \nu^2}{\kappa^2 - \alpha} \nu^{-1}} \circ f_{\nu,\kappa}.
\]

The geodesic flow is not conjugate to other Kähler magnetic flows. For a complex hyperbolic space \( \mathbb{C}H^n(-\beta) \), Kähler magnetic flows are classified into three smoothly conjugacy classes according to their strength:

1. If \( |\kappa| < \sqrt{\beta} \), then there exists a diffeomorphism \( f_\kappa \) of \( U\mathbb{C}H^n(-\beta) \) which satisfies

\[
f_{\nu,\kappa} \circ B_\kappa \phi_t = B_\sqrt{\frac{(\kappa^2 - \beta) \nu^2}{\kappa^2 - \beta} \nu^{-1}} \circ f_{\nu,\kappa},
\]

by the diffeomorphism \( f_{\nu,\kappa} = f_{\nu}^{-1} \circ f_\kappa \) for \( \kappa, \nu \) with \( |\kappa|, |\nu| < \sqrt{\beta} \).

2. If \( |\kappa|, |\nu| > \sqrt{\beta} \), then there exists a diffeomorphism \( f_{\nu,\kappa} \) of \( U\mathbb{C}H^n(-\beta) \) which satisfies

\[
f_{\nu,\kappa} \circ B_\kappa \phi_t = B_\sqrt{\frac{\kappa^2 - \beta}{(\kappa^2 - \beta) \nu^2}} \circ f_{\nu,\kappa}.
\]

3. When \( |\kappa| = \sqrt{\beta} \), the Kähler magnetic flow \( B_\kappa \phi_t \) is so-called a horocycle flow. They are not conjugate to other Kähler magnetic flows.

In concerned with splitting theorem, we are interested in Kähler magnetic flows on a product manifold. Let \( \phi_t, \psi_t \) be smooth flows on a smooth manifold \( X \). We shall say that \( \phi_t \) is **smoothly semi-conjugate** to \( \psi_t \) if there exist a dense open subset \( Y \) of \( X \), an injective smooth open map \( f \) and a non-zero constant \( \lambda \) such that

1. \( Y \) is invariant under \( \phi_t \),
2. \( f(Y) \) is invariant under \( \psi_t \),

where \( f \circ \phi_t = \psi_{\lambda t} \circ f \) for all \( t \).
We suppose a Kähler manifold $M$ for $(iii)$ $F_{\nu,\kappa}$.  

Proof. Let $M$ be a Kähler manifold which has a complex Euclidean factor. We put $S = \{v = (v_e, v_n) \in UM \mid v_e \in T \mathbb{C}^r, v_n \in T N, v_e \neq 0, v_n \neq 0\}$.  

There exists an injective smooth open map $F_{\nu,\kappa} : S \to S$ which satisfies the following conditions:  

(i) $F_{\nu,\kappa}(S)$ is invariant under $\mathbb{B}_v \psi\kappa$,  
(ii) $F_{\nu,\kappa}^{-1} : F_{\nu,\kappa}(S) \to S$ is also smooth,  
(iii) $F_{\nu,\kappa} \circ \mathbb{B}_v \psi\kappa = \mathbb{B}_v \psi\kappa^{(e)} \circ F_{\nu,\kappa}$ on $S$,  
(iv) $F_{\nu,\kappa}$ is the identity map,  
(v) $F_{\nu,\kappa}^{(\kappa_1,\kappa_2)} = F_{\nu,\kappa_2,\kappa_1}$ for every $\kappa_1, \kappa_2, \kappa_3$ with $|\kappa_1| > |\kappa_2| > |\kappa_3| > 0$.  

Proof. Let $f_{\nu,\kappa}$ be a diffeomorphism of $U \mathbb{C}^r$ which satisfies $f_{\nu,\kappa}^{(e)} \circ f_{\nu,\kappa}^{(\kappa_1)} = f_{\nu,\kappa_2,\kappa_1}$ on Kähler magnetic flows $f_{\nu,\kappa}^{(e)}$ for $C'$. By using a map $\mu : S \to (0, 1)$ which is given by $\mu(v) = \sqrt{1 - \nu^2 ||v_e||^2/k^2}$, we define an injective smooth map $F_{\nu,\kappa} : S \to S$ by  

$$F_{\nu,\kappa}(v) = \left(\mu(v) \frac{v_e}{||v_e||}, \frac{v_e}{k} v_n\right).$$  

Since smooth curves $\gamma_e$ on $C'$ and $\gamma_n$ on $N$ with $\nabla_{\gamma_e} \gamma_e = v \gamma_e, \nabla_{\gamma_n} \gamma_n = v \gamma_n$ have constant speed, we see the image  

$$F_{\nu,\kappa}(S) = \left\{u = (u_e, u_n) \in UM \mid \sqrt{1 - \nu^2/k^2} < ||u_e|| < 1, 0 < ||u_n|| < \frac{||v||}{k}\right\}$$  

is invariant under the action of $\mathbb{B}_v \psi\kappa$.  

What we have to show is the third property of $F_{\nu,\kappa}$. For a unit tangent vector $v = (v_e, v_n) \in S$ we denote by $\gamma_e$ the trajectory for $B_{\kappa,|v_e|}$ on $C'$ with $\dot{\gamma}_e(0) = v_e/||v_e||$ and by $\gamma_n$ the trajectory for $B_{\kappa,|v_n|}$ on $N$ with $\dot{\gamma}_n(0) = v_n/||v_n||$. If we define $\gamma_e$ and $\gamma_n$ by $\gamma_e(t) = \gamma_e(\|v_e\|t)$ and $\gamma_n(t) = \gamma_n(\|v_n\|t)$, respectively, we see $\mathbb{B}_v \psi\kappa(v) = (\gamma'_e(t), \gamma'_n(t))$. Similarly, we denote by $\tilde{\rho}_e$ the trajectory for $B_{\nu/|\mu(v)|}$ on $C'$ with $\tilde{\rho}_e(0) = f_{\mu(v)}(v_e/||v_e||)$, and define $\rho_e$ and $\rho_n$ by $\rho_e(t) = \tilde{\rho}_e(\mu(v)t)$ and $\rho_n(t) = \bar{\gamma}_n(v_n/\|v_n\|t/k)$, respectively. We then see $\mathbb{B}_v \psi\kappa(F_{\nu,\kappa}(v)) = (\rho'_e(t), \rho'_n(t))$. Since we have
$\rho_\kappa^t \left( \frac{\kappa}{v} \right) = \frac{v}{\kappa} \rho_\kappa \left( \left\| v_\kappa \right\| t \right) = \frac{v}{\kappa} \gamma_\kappa^t (t),$

$\rho_\nu^t \left( \frac{\nu}{v} \right) = \mu(v) \frac{\mu(v)\kappa}{v} t = \mu(v) \mathbb{B}_v/\mu(v) \varphi_{\mu(v)\kappa}^{(e)} \circ f_{\mu(v)\kappa} \left( \frac{v}{\left\| v_\nu \right\|} \right)$

$= \mu(v) f_{\mu(v)\kappa} \left( \frac{v}{\left\| v_\nu \right\|} \right) \circ \mathbb{B}_v/\mu(v) ||v_\nu|| \left( \frac{v}{\left\| v_\nu \right\|} \right)$

$= \mu(v) f_{\mu(v)\kappa} \left( \frac{v}{\left\| v_\nu \right\|} \right) \left( \frac{\gamma_\nu^t (t)}{\left\| \gamma_\nu^t (t) \right\|} \right),$

we obtain $F_{\nu,\kappa} \circ \mathbb{B}_v \varphi_{\nu} = \mathbb{B}_v \varphi_{\mu(v)\kappa} \circ F_{\nu,\kappa}$ on $S$. \square

Such a theorem does not hold for a Kähler manifold which has complex projective factors or complex hyperbolic factors. In treating trajectories the difficulty lies on the fact that changing parameter homothetically causes changing the strength of a magnetic field.

This is different from treating geodesics. In order to carry our argument to the next stage we here restrict ourselves to a product of complex projective spaces.

**Theorem 4.** Let $M = \mathbb{C}P^{m_1}(\alpha_1) \times \cdots \times \mathbb{C}P^{m_p}(\alpha_p)$ be a product of complex projective spaces. Every Kähler magnetic flow $\mathbb{B}_v \varphi_{\nu}$ is smoothly semiconjugate to the geodesic flow $\varphi_{\nu}$. To say more precisely, we put

$$S = S \left( \mathbb{C}P^{m_1}, \ldots, \mathbb{C}P^{m_p} \right) = \left\{ v = (v_1, \ldots, v_p) \in UM \mid v_i \in T\mathbb{C}P^{m_i}, v_i \neq 0, 1 \leq i \leq p \right\}$$

and $\alpha = 1/\left( \sum_{i=1}^p 1/\alpha_i \right)$. For every $\kappa$ there is an injective smooth open map $F_\kappa : S \to S$ which satisfies the following conditions:

(i) $F_\kappa (S)$ is invariant under the geodesic flow $\varphi_{\nu}$ of $M$,
(ii) $F_\kappa^{-1} : F_\kappa (S) \to S$ is also smooth,
(iii) $F_\kappa \circ \mathbb{B}_v \varphi_{\nu} = \varphi_{\gamma_{\nu}^{F_\kappa (S)}} \circ F_\kappa$ on $S$,
(iv) if $|\kappa| \geq |v|$, then $F_\kappa (S) \subset F_\nu (S)$.

**Proof.** We define a smooth map $\mu = (\mu_1, \ldots, \mu_p) : S \to (0, 1) \times \cdots \times (0, 1)$ by

$$\mu_i (v) = \sqrt{\left( \kappa^2 / \alpha_i \right) + \left\| v_1 \right\|^2 / \left( \kappa^2 / \alpha_i \right) + 1}, \quad i = 1, \ldots, p.$$

By using the diffeomorphism $f_{\nu}^{(i)}$ of $U \mathbb{C}P^{m_i}$, which gives the conjugacy of a Kähler magnetic flow $\mathbb{B}_v \varphi_{\nu}$ and the geodesic flow $\varphi_{\nu}$ for $\mathbb{C}P^{m_i}$ we define $F_\kappa : S \to S$ by

$$F_\kappa (v) = \left( \mu_1 (v) f_{\kappa}^{(1)} \left( \frac{v_1}{\left\| v_1 \right\|} \right), \ldots, \mu_p (v) f_{\kappa}^{(p)} \left( \frac{v_p}{\left\| v_p \right\|} \right) \right).$$

The condition $F_\kappa (v) = F_\kappa (w)$ implies $\left\| v_i \right\| = \left\| w_i \right\|$ for every $i$, which guarantees that $F_\kappa$ is injective. The image of $F_\kappa$ is an open set.
\[ F_\kappa(S) = \left\{ u = (u_1, \ldots, u_p) \in UM \left\vert \frac{\kappa}{\sqrt{\alpha_i (\kappa^2 / \alpha_i) + 1}} < \|u_i\| \right\} \leq \sqrt{\frac{(\kappa^2 / \alpha_i) + 1}{\alpha_i (\kappa^2 / \alpha_i) + 1}}, \quad i = 1, \ldots, p \right\}, \]

in particular, \( F_\kappa(S) \subset F_\nu(S) \) if \( |\kappa| \geq |\nu| \). Since every geodesic has constant speed, we find this image is invariant under \( \varphi_t \). If we define \( \lambda = (\lambda_1, \ldots, \lambda_p): F_\kappa(S) \to (0, 1) \times \cdots \times (0, 1) \) by

\[ \lambda_i(u) = \sqrt{\left(\frac{\kappa^2}{\alpha_i} + 1\right)\|u_i\|^2 - \left(\frac{\kappa^2}{\alpha_i}\right)}, \quad i = 1, \ldots, p, \]

then we have

\[ F_\kappa^{-1}(u) = \left(\lambda_1(u)g_{\lambda_1}(u)\left(\frac{u_1}{\|u_1\|}\right), \ldots, \lambda_p(v)g_{\lambda_p}(v)\left(\frac{u_p}{\|u_p\|}\right)\right), \]

where \( g_{\nu}^{(i)} \) is the inverse of \( f_{\nu}^{(i)} \). Hence \( F_\kappa^{-1} \) is also smooth. Since we can get the third property along the same lines as in the proof of Theorem 3, we get our conclusion.

\[ \square \]

**Remark 5.**

1. For each \( \kappa, \nu \) with \( |\kappa| \geq |\nu| \), we define a smooth map \( F_{\nu,\kappa}: S \to S \) by \( F_{\nu,\kappa} = F_\nu \circ F_\kappa^{-1} \circ F_\kappa \). Then it satisfies
   (i) \( F_{\nu,\kappa} \circ B_{\kappa} \varphi_t = B_\nu \varphi_{\sqrt{1 + (\kappa^2 \alpha^{-1})}} \circ F_{\nu,\kappa} \),
   (ii) \( F_{\kappa,\kappa} \) is the identity map,
   (iii) \( F_{\kappa_1,\kappa_2} \circ F_{\kappa_2,\kappa_1} \) for every \( \kappa_1, \kappa_2, \kappa_3 \) with \( |\kappa_1| \geq |\kappa_2| \geq |\kappa_3| \).

2. As we have
   \[ UM = \bigcup_i UCP_{P_{mi}} \bigcup_{i_1 < i_2} S(CP_{m_{i_1}}, CP_{m_{i_2}}) \cup \cdots \cup S(CP_{m_1}, \ldots, CP_{m_\nu}), \]
   we have a corresponding smooth map of each component \( S(CP_{m_{i_1}}, \ldots, CP_{m_{i_\nu}}) \) which gives semi-conjugacy between two Kähler magnetic flows. But we cannot define a corresponding smooth map of \( UM \) in general. We may say we have a “building structure” of conjugacy on Kähler magnetic flows.

Next we treat a product of complex hyperbolic spaces. For a product \( M = CH^{n_1} \times \cdots \times CH^{n_\nu} \) of complex hyperbolic spaces we set

\[ S = S(CH^{n_1}, \ldots, CH^{n_\nu}) = \left\{ v = (v_1, \ldots, v_\nu) \in UM \mid v_j \in TCH^{n_j}, \quad v_j \neq 0, \quad 1 \leq j \leq \nu \right\}. \]

**Theorem 6.** Let \( M = CH^{n_1}(-\beta_1) \times \cdots \times CH^{n_\nu}(-\beta_\nu) \) be a product of complex hyperbolic spaces. We put \( \beta = 1/(\sum_{j=1}^\nu 1/\beta_j) \).

1. For each \( \kappa \) with \( |\kappa| < \sqrt{\beta} \), the geodesic flow is smoothly semi-conjugate to the Kähler magnetic flow \( B_\kappa \varphi_\kappa \). More precisely, for each \( \kappa \) with \( |\kappa| < \sqrt{\beta} \) there is an injective smooth open map \( G_\kappa: S \to S \) which satisfies the following conditions:
(i) $G_{\kappa}(S)$ is invariant under $B_{\kappa}\varphi_t$.
(ii) $G_{\kappa}^{-1} : G_{\kappa}(S) \to S$ is also smooth.
(iii) $G_{\kappa} \circ \varphi_{\sqrt{-(\kappa^2/\beta)u}} = B_{\kappa}\varphi_t \circ G_{\kappa}$ on $S$.
(iv) if $|\kappa| \geq |v|$, then $G_{\kappa}(S) \subset G_{\kappa}(S)$.

(2) For each pair $\kappa, v$ with $|\kappa| \geq |v| > \max\{\sqrt{\beta_1}, \ldots, \sqrt{\beta_q}\}$, a Kähler magnetic flow $B_{\kappa}\varphi_t$ is smoothly semi-conjugate to a Kähler magnetic flow $B_{\nu}\varphi_t$. More precisely, there is an injective smooth open map $F_{\nu, \kappa} : S \to S$ which satisfies the following conditions:
(i) $F_{\nu, \kappa}(S)$ is invariant under $B_{\nu}\varphi_t$.
(ii) $F_{\nu, \kappa}^{-1} : F_{\nu, \kappa}(S) \to S$ is also smooth.
(iii) $F_{\nu, \kappa} \circ B_{\nu}\varphi_t = B_{\nu}\varphi_{\alpha_{\nu}(\kappa,v)}\sqrt{-(\kappa^2/\beta)u} \circ F_{\nu, \kappa}$ on $S$.
(iv) if $|\kappa| \geq |\nu| > \max\{\sqrt{\beta_1}, \ldots, \sqrt{\beta_q}\}$, then $F_{\nu, \kappa} \circ F_{\nu, \kappa} = F_{\nu, \kappa} \circ F_{\nu, \kappa}$.

Proof. 1) We define a smooth map $\lambda = (\lambda_1, \ldots, \lambda_p) : S \to (0, 1) \times \cdots \times (0, 1)$ by

$$\lambda_j(u) = \left\{ v = (v_1, \ldots, v_q) \in UM \mid \frac{|\kappa|}{\sqrt{\beta_j}} \leq \|v_j\| < \sqrt{1 - \kappa^2 \left(\frac{1}{\beta_j} - 1\right)}, \quad j = 1, \ldots, q \right\},$$

which is invariant under a Kähler magnetic flow $B_{\kappa}\varphi_t$. If we define $\mu = (\mu_1, \ldots, \mu_q) : G_{\kappa}(S) \to (0, 1) \times \cdots \times (0, 1)$ by

$$\mu_j(v) = \left\{ v = (v_1, \ldots, v_q) \in UM \mid \frac{|\kappa|}{\sqrt{\beta_j}} \leq \|v_j\| < \sqrt{1 - \kappa^2 \left(\frac{1}{\beta_j} - 1\right)}, \quad j = 1, \ldots, q \right\},$$

we then have

$$G_{\kappa}^{-1}(v) = \left( \mu_1^{-1}(v) f_{\kappa/\mu_1}^{11}(\frac{v_1}{\|v_1\|}) \cdots \mu_q^{-1}(v) f_{\kappa/\mu_q}^{q1}(\frac{v_q}{\|v_q\|}) \right).$$

One can easily get the conclusion along the same lines as in the proof of Theorem 3.

(2) We define $\mu = (\mu_1, \ldots, \mu_q) : S \to (0, 1) \times \cdots \times (0, 1)$ by

$$\mu_j(v) = \left\{ v = (v_1, \ldots, v_q) \in UM \mid \frac{1}{\beta_j} \left( v_j^2 - \mu_j^{-1}(v) (v_j^2 - \beta_j \|v_j\|^2) \right) < \frac{1}{\beta_j} \left( \frac{v^2 - \beta_j (v_j^2 - \beta_j \|v_j\|^2)}{\kappa^2 - \beta} \right), \quad j = 1, \ldots, q, \right\},$$

and a smooth map $F_{\nu, \kappa} : S \to S$ by

$$F_{\nu, \kappa}(v) = \left( \mu_1^{-1}(v) \frac{v_1}{\|v_1\|} \cdots \mu_q^{-1}(v) \frac{v_q}{\|v_q\|} \right).$$
where \( f^{(j)}_{\kappa_1,\kappa_2} \) is the diffeomorphism of \( U \cap H^n \) which gives the conjugacy of Kähler magnetic flows \( \mathbb{B}_{\kappa_1} \varphi_t \), \( i = 1, 2 \). We then have

\[
F_{\nu,\kappa}(S) = \left\{ u = (u_1, \ldots, u_q) \in U M \left| \frac{\beta(k^2 - v^2)}{\beta_j(k^2 - \beta)} < \|u_j\|^2 < 1 - \frac{(\beta_j - \beta)(k^2 - v^2)}{\beta_j(k^2 - \beta)}, \quad j = 1, \ldots, q \right. \right\}.
\]

As we find

\[
F^{-1}_{\nu,\kappa}(u) = \left( \lambda_1(u)g^{(1)}_{\nu,\kappa}(u_1 \|u_1\|), \ldots, \lambda_q(u)g^{(q)}_{\nu,\kappa}(u_q \|u_q\|) \right),
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_p) : F_{\nu,\kappa}(S) \rightarrow (0, 1) \times \cdots \times (0, 1) \) is given by

\[
\lambda_j(u) = \sqrt{\frac{\kappa^2 - \beta}{\beta_j(k^2 - \beta)}} \left( \frac{\kappa^2(v^2 - \beta)}{\kappa^2 - \beta} - v^2 + \beta_j\|u_j\|^2 \right), \quad j = 1, \ldots, q,
\]

and \( g^{(j)}_{\kappa_1,\kappa_2} = (f^{(j)}_{\kappa_1,\kappa_2})^{-1} \), we can get our conclusion by same argument as in the proof of Theorem 3.

**Remark 7.** For a product \( M \) in Theorem 6, we define a smooth map \( G_{\kappa,\nu} : S \rightarrow S \) by \( G^{-1}_{\kappa,\nu} \circ G_{\nu,\kappa} \) for each \( \kappa, \nu \) with \( |\nu| \leq |\kappa| < \sqrt{\beta} \). Then it satisfies

(i) \( G_{\kappa,\nu} \circ \mathbb{B}_{\nu} \varphi_t = \mathbb{B}_{\kappa} \varphi_{\kappa(\nu - \nu^2)/(\beta - \nu^2)} \circ G_{\kappa,\nu} \).

(ii) \( G_{\kappa,\kappa} \) is the identity map,

(iii) \( G_{\kappa_3,\kappa_1} = G_{\kappa_3,\kappa_2} \circ G_{\kappa_2,\kappa_1} \) for every \( \kappa_1, \kappa_2, \kappa_3 \) with \( |\kappa_1| \leq |\kappa_2| \leq |\kappa_3| < \sqrt{\beta} \).

We should note that semi-conjugacy for Kähler magnetic flows does not hold for a product of mixed type. Each Kähler magnetic flow for a product manifold \( \mathbb{C}P^m(\alpha) \times \mathbb{C}H^n(-\beta) \) is not semi-conjugate to other Kähler magnetic flow for this manifold. The reader should compare this with the first statement of Proposition 1.

**Remark 8.** Since we have corresponding diffeomorphisms \( f_{\nu,\kappa} \) of the unit tangent bundle \( U N \) of a Kähler manifold \( N \) of constant holomorphic sectional curvature (see [1]), our results can be extend to results for a product of Kähler manifolds of constant holomorphic sectional curvature.

At the last stage of this paper we pose a problem: Is the converse of Theorem 3 true? More precisely, our problem is the following. Suppose there exists an injective map \( F_{\nu,\kappa} \) of a dense subset \( S \) of \( U M \) for each \( \kappa, \nu \) with \( |\kappa| \geq |\nu| > 0 \) which gives semi-conjugacy of Kähler magnetic flows in the following relation: \( F_{\nu,\kappa} \circ \mathbb{B}_{\nu} \varphi_{t_0} = \mathbb{B}_{\kappa} \varphi_{\kappa t_0} \circ F_{\nu,\kappa} \). Is it true that the universal covering \( \tilde{M} \) of \( M \) has a complex Euclidean factor? Of course the relation formula is important. Without this we cannot exclude products of complex projective spaces. When we attack this problem, one of difficulties lies on the fact that the map \( F_{\nu,\kappa} \) is not unique. For example, for an arbitrary constant \( t_0 \) the map \( \mathbb{B}_{\nu} \varphi_{t_0} \circ F_{\nu,\kappa} \) also satisfies the relation.
References