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Forced oscillation of second-order nonlinear differential equations with positive and negative coefficients

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ABSTRACT

In this paper we give new oscillation criteria for forced super- and sub-linear differential equations by means of nonprincipal solutions.

1. Introduction

The concept of the principal solution was introduced in 1936 by Leighton and Morse [1] in studying positiveness of certain quadratic functional associated with

$$(r(t)x')' + v(t)x = 0, \quad t \ge t_0. \tag{1.1}$$

Since then the principal and nonprincipal solutions have been used successfully in connection with oscillation and asymptotic theory of (1.1) and related equations, see for instance [1–8] and the references cited therein. For some extensions to Hamiltonian systems, half-linear differential equations, dynamic equations and impulsive differential equations, we refer in particular to [4,9–12].

We recall that a nontrivial solution u of (1.1) is said to be principal if for every solution v of (1.1) such that $u \neq cv$, $c \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = 0.$$

It is well known that a principal solution u of (1.1) exists uniquely up to a multiplication by a nonzero constant if and only if (1.1) is nonoscillatory. A solution v that is linearly independent of u is called a nonprincipal solution. Roughly speaking, the words "principal" and "nonprincipal" may be replaced by "small" and "large" or "recessive" and "dominant". For other characterizations of principal and nonprincipal solutions of (1.1), see [5, Theorem 6.4], [13, Theorem 5.59].

In 1999, Wong [8], by employing a nonprincipal solution of (1.1), obtained the following oscillation criterion for

$$(r(t)x')' + v(t)x = f(t). \tag{1.2}$$

For extensions of the theorem to impulsive differential equations and dynamic equations on time scales, see [11,12].

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Theorem 1.1 (*Wong's Theorem*)**.** *Suppose that (1.1) is nonoscillatory. Let z be a positive solution of (1.1) satisfying*

$$\int_a^\infty \frac{1}{r(s)z^2(s)}\mathrm{d}s < \infty \tag{1.3}$$

for some a sufficiently large, i.e., a nonprincipal solution. If

$$\overline{\lim_{t\to\infty}}\, H(t) = -\underline{\lim_{t\to\infty}}\, H(t) = \infty, \tag{1.4}$$

where

$$H(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s z(\tau)f(\tau)\mathrm{d}\tau \right) \mathrm{d}s, \tag{1.5}$$

then Eq. (1.2) is oscillatory.

The aim of our work is to extend the above theorem to nonlinear equations of the form

$$(r(t)x')' + p(t)|x|^{\beta-1}x - q(t)|x|^{\gamma-1}x = f(t), \quad t \geq t_0, \tag{1.6}$$

where

(i) $0 < \gamma < 1 < \beta$;
(ii) $r \in C([t_0, \infty), (0, \infty))$, $p, q \in C([t_0, \infty), [0, \infty))$, $f \in C([t_0, \infty), \mathbb{R})$.

It is clear that the two special cases of (1.6) are the Emden–Fowler super-linear equation

$$(r(t)x')' + p(t)|x|^{\beta-1}x = f(t), \quad \beta > 1 \tag{1.7}$$

and the Emden–Fowler sub-linear equation

$$(r(t)x')' - q(t)|x|^{\gamma-1}x = f(t), \quad 0 < \gamma < 1. \tag{1.8}$$

Typically, nonlinear results require the coefficient in an Emden–Fowler equation

$$x'' + a(t)|x|^{\alpha-1}x = f(t)$$

to be non-negative, see [14]. Fortunately, we are able to take $-q$ to be negative in (1.8). On the other hand, letting $\beta \to 1^+$ and $\gamma \to 1^-$ in (1.6) results in (1.1) with $\nu(t) = p(t) - q(t)$, i.e.,

$$(r(t)x')' + [p(t) - q(t)]x = f(t), \tag{1.9}$$

and thus our result extends Theorem 1.1 by a limiting process $\beta, \gamma \to 1$ in (1.6).

We remark that the oscillation of the solutions of (1.7) and (1.8) has been studied by many authors, see for instance [8,14–24], but to the best of our knowledge there is no result in the literature similar to Theorem 1.1 for such nonlinear equations, especially for (1.6).

Consider a slightly more general equation than (1.6)

$$(r(t)x')' + p(t)F(x) - q(t)G(x) = f(t), \quad t \geq t_0, \tag{1.10}$$

where r, p, q, f are as in (ii), $F, G \in C(\mathbb{R}, \mathbb{R})$.

By a solution of (1.10) defined on an interval $[T, \infty)$, $T \geq t_0$, we mean a function x, x', $(rx')' \in \mathrm{C}[T, \infty)$, satisfying (1.10). We note that the assumption of r, p, f being continuous is not sufficient to ensure the existence of extendable solutions of (1.7) on $[T, \infty)$, see [23]. However, as usual in the oscillation theory we only consider solutions of (1.10) which are extendable to $[T, \infty)$ and nontrivial in the neighborhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Eq. (1.10) is called oscillatory (nonoscillatory) if all solutions are oscillatory (nonoscillatory).

We shall assume that

(C_1) $xF(x) > 0$ and $xG(x) > 0$ for $x \neq 0$;
(C_2) (a) $\lim_{|x|\to\infty} x^{-1}F(x) > 1$, $\lim_{|x|\to 0} x^{-1}F(x) < 1$,
(b) $\lim_{|x|\to\infty} x^{-1}G(x) < 1$, $\lim_{|x|\to 0} x^{-1}G(x) > 1$.

Using (C_1) and (C_2), it is easy to find positive constants $\alpha_0, \beta_0, \gamma_0, \delta_0$ such that

$$\begin{aligned} &\max_{x\geq 0} \Phi(x) = \beta_0, \quad \min_{x\leq 0} \Phi(x) = -\alpha_0; \\ &\max_{x\leq 0} \Psi(x) = \delta_0, \quad \min_{x\geq 0} \Psi(x) = -\gamma_0, \end{aligned} \tag{1.11}$$

where $\Phi(x) = x - F(x)$ and $\Psi(x) = x - G(x)$.

In what follows we define

$$\mathcal{N}_1(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s [\beta_0 p(\tau) + \gamma_0 q(\tau)] z(\tau) \mathrm{d}\tau \right) \mathrm{d}s, \tag{1.12}$$

and

$$\mathcal{N}_2(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s [\alpha_0 p(\tau) + \delta_0 q(\tau)] z(\tau) \mathrm{d}\tau \right) \mathrm{d}s. \tag{1.13}$$

2. Main results

Associated with Eq. (1.10) we assume that the linear equation

$$(r(t)x')' + [p(t) - q(t)]x = 0 \tag{2.1}$$

is nonoscillatory. Denote by $z(t)$ a positive nonprincipal solution of (2.1) which is defined on an interval $[a, \infty)$. Noting that

$$\int_a^\infty \frac{\mathrm{d}s}{r(s)z^2(s)} < \infty, \tag{2.2}$$

we define

$$\mathcal{H}(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s z(\tau) f(\tau) \mathrm{d}\tau \right) \mathrm{d}s. \tag{2.3}$$

The main result of this paper is the following theorem.

Theorem 2.1. *Suppose that (2.1) is nonoscillatory and let $z(t)$ be a positive solution of it satisfying (2.2), i.e. a nonprincipal solution. If*

$$\overline{\lim_{t\to\infty}} \{\mathcal{H}(t) - \mathcal{N}_2(t)\} = - \underline{\lim_{t\to\infty}} \{\mathcal{H}(t) + \mathcal{N}_1(t)\} = \infty, \tag{2.4}$$

where $\mathcal{H}$ is given by (2.3), and $\mathcal{N}_1$ and $\mathcal{N}_2$ are as defined by (1.12) and (1.13), respectively, then Eq. (1.10) is oscillatory.

Proof. Suppose that there is a nonoscillatory solution $x(t)$ of (1.10). We may assume that $x(t) \neq 0$ on $[a, \infty)$ for some $a \geq t_0$ sufficiently large. The change of variables $x = z(t)w$, where $z(t)$ is a positive nonprincipal solution of (2.1), transforms (1.10) into

$$(r(t)z^2 w')' = \{f(t) + p(t)\Phi(x) - q(t)\Psi(x)\}z, \quad t \geq a. \tag{2.5}$$

Integration of (2.5) leads to

$$w(t) = c_1 + c_2 \int_a^t \frac{\mathrm{d}s}{r(s)z^2(s)} + \mathcal{H}(t) + \int_a^t \frac{1}{r(s)z^2(s)} \int_a^s \{p(\tau)\Phi(x(\tau)) - q(\tau)\Psi(x(\tau))\} z(\tau) \mathrm{d}\tau \mathrm{d}s \tag{2.6}$$

where $c_1 = w(a)$ and $c_2 = r(a)z^2(a)w'(a)$ are constants.

If $x(t) > 0$ on $[a, \infty)$, then using (1.11) we obtain

$$w(t) \leq c_1 + c_2 \int_a^t \frac{\mathrm{d}s}{r(s)z^2(s)} + \mathcal{H}(t) + \mathcal{N}_1(t). \tag{2.7}$$

Similarly, if $x(t) < 0$ on $[a, \infty)$, then again using (1.11) we obtain

$$w(t) \geq c_1 + c_2 \int_a^t \frac{\mathrm{d}s}{r(s)z^2(s)} + \mathcal{H}(t) - \mathcal{N}_2(t). \tag{2.8}$$

Note that (2.2), (2.4), (2.7) and (2.8) imply that

$$\overline{\lim_{t\to\infty}}\, w(t) = - \underline{\lim_{t\to\infty}}\, w(t) = +\infty. \tag{2.9}$$

Because $z(t)$ is positive, (2.9) implies that $x(t)$ cannot have a definite sign on $[a, \infty)$, a contradiction. □

When $F(x) = |x|^{\beta-1}x$ and $G(x) = |x|^{\gamma-1}x$, $0 < \gamma < 1 < \beta$, then

$$\alpha_0 = \beta_0 = (\beta - 1)\beta^{\beta/(1-\beta)} > 0, \qquad \delta_0 = \gamma_0 = (1-\gamma)\gamma^{\gamma/(1-\gamma)} > 0,$$

and we obtain the following oscillation criterion for Eq. (1.6).

Theorem 2.2. *Suppose that (2.1) is nonoscillatory and let $z(t)$ be a positive solution of it satisfying (2.2), i.e. a nonprincipal solution. If*

$$\overline{\lim_{t\to\infty}}\{\mathcal{H}(t)-\mathcal{N}_0(t)\}=-\underline{\lim_{t\to\infty}}\{\mathcal{H}(t)+\mathcal{N}_0(t)\}=\infty, \tag{2.10}$$

where

$$\mathcal{N}_0(t)=\int_a^t\frac{1}{r(s)z^2(s)}\left(\int_a^s[\alpha_0p(\tau)+\delta_0q(\tau)]z(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \tag{2.11}$$

then Eq. (1.6) is oscillatory.

Corollary 2.3. *Suppose that (2.1) with $q(t)\equiv 0$ is nonoscillatory and let $z(t)$ be a positive solution of it satisfying (2.2), i.e. a nonprincipal solution. If*

$$\overline{\lim_{t\to\infty}}\{\mathcal{H}(t)-\mathcal{N}_{01}(t)\}=-\underline{\lim_{t\to\infty}}\{\mathcal{H}(t)+\mathcal{N}_{01}(t)\}=\infty, \tag{2.12}$$

where

$$\mathcal{N}_{01}(t):=\int_a^t\frac{\alpha_0}{r(s)z^2(s)}\left(\int_a^s p(\tau)z(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \tag{2.13}$$

then Eq. (1.7) is oscillatory.

Corollary 2.4. *Let $z(t)$ be a positive solution of (2.1) with $p(t)\equiv 0$ satisfying (2.2), i.e. a nonprincipal solution. If*

$$\overline{\lim_{t\to\infty}}\{\mathcal{H}(t)-\mathcal{N}_{02}(t)\}=-\underline{\lim_{t\to\infty}}\{\mathcal{H}(t)+\mathcal{N}_{02}(t)\}=\infty, \tag{2.14}$$

where

$$\mathcal{N}_{02}(t):=\int_a^t\frac{\delta_0}{r(s)z^2(s)}\left(\int_a^s q(\tau)z(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \tag{2.15}$$

then Eq. (1.8) is oscillatory.

Remark 1. Theorem 2.2 is interesting because it reduces to Theorem 1.1 for the linear equation (1.9) with $\nu(t)=p(t)-q(t)$ by letting $\beta,\gamma\to 1$ in (1.6).

Remark 2. Corollary 2.4 is of particular interest where the coefficient $-q(t)$ is non-positive and Eq. (1.8) can still be oscillatory by the forcing condition (2.14).

Remark 3. It will be interesting to improve Corollary 2.3 for Eq. (1.7) by relaxing the assumption that $p(t)$ is non-negative. In case when $\Phi(x)$ is bounded, say $|\Phi(x)|\le M$ for some $M>0$ and for all $x\in\mathbb{R}$, then we can show similarly to Corollary 2.3 the following:

Proposition 1. *Under the assumption of Corollary 2.3 and that the coefficient $p(t)$ is not assumed to be non-negative, if*

$$\overline{\lim_{t\to\infty}}\{\mathcal{H}(t)-\mathcal{N}_{03}(t)\}=-\underline{\lim_{t\to\infty}}\{\mathcal{H}(t)+\mathcal{N}_{03}(t)\}=\infty,$$

where

$$\mathcal{N}_{03}(t):=\int_a^t\frac{M}{r(s)z^2(s)}\left(\int_a^s |p(\tau)|z(\tau)\mathrm{d}\tau\right)\mathrm{d}s,$$

then Eq. (1.7) is oscillatory.

3. Examples

Example 3.1. Consider the forced super-linear equation

$$(t^{-2}x')'+2t^{-4}|x|^{\beta-1}x=(3-t^2)\sin t+5t\cos t,\quad \beta>1. \tag{3.1}$$

The corresponding linear equation

$$(t^{-2}z')'+2t^{-4}z=0$$

is the nonoscillatory with a nonprincipal solution $z(t) = t^2$. Then, the functions $\mathscr{H}$ and $\mathscr{N}_{01}$ become

$$\mathscr{H}(t) = \int_a^t \frac{1}{s^2} \left(\int_a^s \{(3 - \tau^2) \sin \tau + 5\tau \cos \tau\} \tau^2 \mathrm{d}\tau \right) \mathrm{d}s, \quad a > 0$$

and

$$\mathscr{N}_{01}(t) = 2(\beta - 1)\beta^{\beta/(1-\beta)} \int_a^t \frac{1}{s^2} \left(\int_a^s \frac{1}{\tau^2} \mathrm{d}\tau \right) \mathrm{d}s, \quad a > 0.$$

After some simple calculations, we obtain

$$\mathscr{H}(t) = t^2 \sin t + t \cos t - \sin t + c_1 t^{-1} + c_2,$$

where $c_1 = a^3(\sin a + a \cos a)$ and $c_2 = (1 - 2a^2) \sin a - a(a^2 + 1) \cos a$, and

$$\mathscr{N}_{01}(t) = (\beta - 1)\beta^{\beta/(1-\beta)}(t^{-2} + c_3 t^{-1} + c_4)$$

where $c_3 = -2/a$ and $c_4 = 1/a^2$. Clearly, the condition (2.12) is satisfied and hence Eq. (3.1) is oscillatory for any choice of $\beta > 1$ by Corollary 2.3.

Example 3.2. Consider the forced sub-linear equation

$$x'' - |x|^{\gamma-1}x = \mathrm{e}^{\mu t} \sin(\zeta t), \quad 0 < \gamma < 1 \tag{3.2}$$

where $\mu > 1$ and $\zeta \neq 0$ are real constants. The corresponding linear equation

$$z'' - z = 0$$

is nonoscillatory with a nonprincipal solution $z(t) = \mathrm{e}^t$. Then, the functions $\mathscr{H}(t)$ and $\mathscr{N}_{02}(t)$ read as

$$\mathscr{H}(t) = \int_a^t \mathrm{e}^{-2s} \left(\int_a^s \mathrm{e}^{(\mu+1)\tau} \sin(\zeta\tau) \mathrm{d}\tau \right) \mathrm{d}s, \quad a > 0$$

and

$$\mathscr{N}_{02}(t) = (1 - \gamma)\gamma^{\gamma/(1-\gamma)} \int_a^t \mathrm{e}^{-2s} \left(\int_a^s \mathrm{e}^{\tau} \mathrm{d}\tau \right) \mathrm{d}s, \quad a > 0.$$

A straightforward calculation gives

$$\int_a^s \mathrm{e}^{(\mu+1)\tau} \sin(\zeta\tau) \mathrm{d}\tau = k_1 \mathrm{e}^{(\mu+1)s} \{(\mu + 1) \sin(\zeta s) - \zeta \cos(\zeta s)\} + k_2 \tag{3.3}$$

where $k_1 = \{\zeta^2 + (\mu + 1)^2\}^{-1}$ and $k_2 = -k_1 \mathrm{e}^{(\mu+1)a}\{(\mu + 1) \sin(\zeta a) - \zeta \cos(\zeta a)\}$.

Using (3.3), we see that

$$\mathscr{H}(t) = k_3 \mathrm{e}^{(\mu-1)t}\{(\mu^2 - \zeta^2 - 1) \sin(\zeta t) - 2\mu\zeta \cos(\zeta t)\} + k_4 \mathrm{e}^{-2t} + k_5$$

where $k_3 = k_1\{\zeta^2 + (\mu - 1)^2\}^{-1}$, $k_4 = -k_2/2$ and

$$k_5 = (k_2/2)\mathrm{e}^{-2a} - k_3 \mathrm{e}^{(\mu-1)a}\{(\mu^2 - \zeta^2 - 1) \sin(\zeta a) - 2\mu\zeta \cos(\zeta a)\},$$

and that

$$\mathscr{H}(t) \pm \mathscr{N}_{02}(t) = k_3 \mathrm{e}^{(\mu-1)t}\{(\mu^2 - \zeta^2 - 1) \sin(\zeta t) - 2\mu\zeta \cos(\zeta t)\} \pm \sigma \mathrm{e}^{-t} + (k_4 \mp \sigma \mathrm{e}^{-a}/2)\mathrm{e}^{-2t} + k_5 \mp \sigma \mathrm{e}^{-a}/2,$$

where $\sigma = -\delta_0 = (\gamma - 1)\gamma^{\gamma/(1-\gamma)}$. Therefore, the condition (2.14) holds and hence, we conclude that Eq. (3.2) is oscillatory for any choice of $\gamma \in (0, 1)$, $\mu > 1$ and $\zeta \neq 0$, by Corollary 2.4.

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