JOURNAL OF COMBINATORIAL THEORY (B) 16, 197–214 (1974)

A Certain Polynomial of a Graph and Graphs with an Extremal Number of Trees

A. K. KELMANS AND V. M. CHELNOKOV

Institute of Control Sciences, Moscow, USSR

Communicated by A. A. Zykov

Received May 26, 1972

The polynomial we consider here is the characteristic polynomial of a certain (not adjacency) matrix associated with a graph. This polynomial was introduced in connection with the problem of counting spanning trees in graphs [8]. In the present paper the properties of this polynomial are used to construct some classes of graphs with an extremal numbers of spanning trees.

INTRODUCTION

Most papers on trees in a graph are devoted to determination of the number of spanning trees. One approach to this problem proceeds from the fact that the number of trees of a graph equals a principal minor of a certain matrix related to the graph [1–3]. In many papers this fact was used to derive formulas for the number of trees in graphs having special structure (e.g., [4–7]).

Investigation of the characteristic polynomial of the above matrix (this polynomial was called the characteristic polynomial of a graph) has led to an algorithm for obtaining the polynomials and the numbers of trees of graphs decomposable with respect to certain operations into graphs with known polynomials [8, 9].

Many formulas which were derived elsewhere (see, e.g., [4–7, 10–15]) may be easily obtained by the method described in [8] since the graphs for which these formulas were deduced may be constructed by using the above operations from graphs whose polynomials are known or may be easily found.

Another way to find the number of spanning trees of a graph is to use the principle of inclusion and exclusion [15]. By this method formulas for the number of trees in some special cases were also obtained. As will be shown below, the basic formula for the number of trees of a graph derived in [15] from the principle of inclusion and exclusion is one of the relations between the coefficients of the characteristic polynomials of a graph and its complement stated in [8, 9]. This shows the relationship between these approaches.

Note that the problem of counting trees in a graph is a part of a more general problem of determination of the probability that a graph with randomly removed edges is connected. For certain classes of graphs various probabilistic characteristics, for example, the probability of the connectivity (and in particular, the number of trees), satisfy linear difference equations [16]. These graphs are constructed, roughly speaking, by a certain glueing of several copies of the same graph. An even simple cycle with diameters belongs to this type of graphs.

Much fewer papers are devoted to construction of graphs with an extremal number of trees. A more general problem consists in construction of graphs with a maximal (or minimal) probability of the connectivity when edges are randomly removed. It was proved that a complete graph has more trees than any other graph with the same numbers of vertices and edges [9] and a cycle in which each edge is replaced by k multiple edges has less trees than any other $2k$-connected graph with the same numbers of vertices and edges [17].

In this paper, classes of graphs with extremal numbers of trees are constructed. Basic definitions are introduced in Section 1. In Section 2 we list some properties of the characteristic polynomial of a graph which were stated in previous works and then deduce several new properties of this polynomial. All these properties are used for the construction of our extremal graphs in Section 3. Finally, in Section 4, we derive several estimates for the number of spanning trees and consider some general principles of such estimation.

1. Basic Notation

Non-oriented graphs will be considered. For a graph G, having n vertices and m edges, the notations G_n, G^m, or $G_n{}^m$ will be used. We shall write $\Gamma = L$, if graphs Γ and L are isomorphic, and $\Gamma \neq L$ otherwise. The graph $\bar{G}$ is called the complementary graph of G if it is obtained from G by replacement of adjacent pairs of vertices by non-adjacent pairs and vice versa.

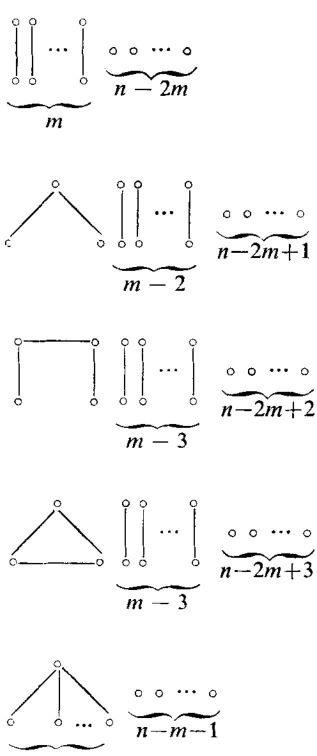
A graph without cycles will be called a forest and denoted by F. It is obvious that $F_n{}^m$ has $n - m$ components of connectivity. A forest with k components will be referred to as a k-forest. A 1-forest is called a tree. $\gamma(F)$ denotes the product of the numbers of vertices in components of the forest F. A k-forest spanning a given graph G (i.e., being a subgraph of G with the same number of vertices as G) is called a k-forest of G and two such forests are different if they have different sets of edges. $T(G)$ denotes the number of different trees of a given graph G.

For any two graphs Γ and L let $\Gamma + L = \Gamma \cup L$ and $\Gamma \cdot L$ be the graph which consists of all elements of Γ and L and such that every vertex of Γ is joined by one edge with every vertex of L [18]. Below these operations of addition and multiplication will be applied only to graphs without common elements. In particular, if g is a graph with one vertex, then

$$\overbrace{g \cdot g \cdot \cdots \cdot g}^{n} = g^n = K_n$$

is the complete graph with n vertices,

$$\overbrace{g + g + \cdots + g}^{n} = ng = \overline{g^n}.$$



$$N_n^m = mg^2 + (n - 2m)g$$

$$H_n^m = g(2g) + (m - 2)g^2 + (n - 2m + 1)g$$

$$\Pi_n^m = \Pi_4^3 + (m - 3)g^2 + (n - 2m + 2)g$$

$$\Delta_n^m = g^3 + (m - 3)g^2 + (n - 2m + 3)g$$

$$\Lambda_n^m = g(mg) + (n - m - 1)g$$

FIGURE 1

Let us introduce the special notation for some types of graphs (see Fig. 1):

$$N_n^m = mg^2 + (n - 2m)\, g,\ n \geqslant 2m;$$

$$H_n^m = g(2g) + (m - 2)\, g^2 + (n - 2m + 1)\, g,\ n \geqslant 2m - 1;$$

$$\Lambda_n^m = g(mg) + (n - m + 1)\, g,\ n \geqslant m + 1;$$

$$\Delta_n^m = g^3 + (m - 3)\, g^2 + (n - 2m + 3)\, g,\ n \geqslant 2m - 3.$$

If Π is a simple chain of three edges, then let

$$\Pi_n^m = \Pi + (m - 3)\, g^2 + (n - 2m + 2)\, g,\ n \geqslant 2m - 2.$$

In this notation we shall omit the lower index if the corresponding graphs have no isolated vertices.

We shall need the following formulas, which are easily obtainable by the methods developed in [8] or [15]:

$$T(\overline{N_n^m}) = n^{n-m-2}(n - 2)^m \text{ [4]}; \tag{1.1}$$

$$T(\overline{H_n^m}) = n^{n-m-2}(n - 2)^{m-2}\, (n - 1)(n - 3); \tag{1.2}$$

$$T(\overline{\Lambda_n^m}) = n^{n-m-2}(n - 1)^{m-1}\, (n - m - 1)\text{[4]}. \tag{1.3}$$

Let $\Psi[G] = \Psi(G, \Gamma_{n_1}, \ldots, \Gamma_{n_k})$ denote the graph obtained from non-empty graphs $G, \Gamma_{n_1}, \ldots, \Gamma_{n_k}$ $(k = 1, 2, \ldots)$ by additions and multiplications applied in the fashion defined by the operator Ψ. Since

$$\overline{\Gamma \cdot L} = \bar{\Gamma} + \bar{L},$$

we have

$$\overline{\Psi(G, \Gamma_{n_1}, \ldots, \Gamma_{n_k})} = \bar{\Psi}(\bar{G}, \bar{\Gamma}_{n_1}, \ldots, \bar{\Gamma}_{n_k}), \tag{1.4}$$

where $\bar{\Psi}$ means that in Ψ the operations of addition are replaced by multiplications and vice versa. Since for any non-empty graphs their product is a connected graph and their sum is disconnected, then the graphs $\Psi[\Gamma]$ and $\Psi[L]$ are both connected or both disconnected depending on which of two operations is the last in the operator Ψ. For any G_n the graphs $\Psi[G_n]$ have the same number of vertices, which we shall denote by p_n.

For a graph G with vertices $x_1, \ldots, x_n$, let $C(G) = \{c_{ij}\}$ be the $n \times n$ matrix in which c_{ii} is the degree of the vertex x_i in G and $-c_{ij}$ equals the number of multiple edges joining the vertices x_i and x_j.

2. Characteristic Polynomial of a Graph

This paper is based essentially on the properties of the polynomial $\det\{\lambda E - C(G)\}$ introduced and investigated in [8, 9] and which has been called (with an allowance for differences in notation) the characteristic polynomial of a graph.[1] Consider the properties of this polynomial which we shall need in further discussion:

1. Since $\det C(G) = 0$, then

$$\begin{aligned} P(\lambda, G_n) &= \frac{1}{\lambda} \det\{\lambda E - C(G_n)\} \\ &= \prod_{i=1}^{n-1} (\lambda - \lambda_i) \\ &= \lambda^{n-1} - b_1\lambda^{n-2} + \cdots + (-1)^i\, b_i\lambda^{n-1-i} + \cdots + (-1)^{n-1}\, b_{n-1} \end{aligned} \tag{2.1}$$

where

$$b_i = b_i(G_n) = \sigma_i(\lambda_1, \ldots, \lambda_{n-1}) \tag{2.2}$$

is the symmetric polynomial of the order $n-1$.

2. Following [8],

$$P(\lambda, \Gamma + L) = \lambda P(\lambda, \Gamma)\, P(\lambda, L), \tag{2.3}$$

$$P(\lambda, G_{n_1} \cdot G_{n_2}) = (\lambda - n_1 - n_2)\, P(\lambda - n_2, G_{n_1})\, P(\lambda - n_1, G_{n_2}). \tag{2.4}$$

3. From (2.3) and (2.4), it follows [8] that

$$P(\lambda, \Psi[G_n]) = P(\lambda, \Psi(G_n, \Gamma_{n_1}, \ldots, \Gamma_{n_k})) = R(\lambda)\, P(\lambda - \alpha, G_n), \tag{2.5}$$

where $R(\lambda)$ is uniquely determined by the operator Ψ and the polynomials $P(\lambda, \Gamma_{n_i})$ and α is uniquely determined by the operator Ψ and the numbers of vertices n_i of the graphs Γ_{n_i}, $i = 1, \ldots, k$. In the expression

$$0 \leqslant \alpha \leqslant p_n - n, \tag{2.6}$$

$\alpha = 0$ is equivalent to $\Psi[G_n] = G_n + \Gamma$ and $\alpha = p_n - n$ is equivalent to $\Psi[G_n] = (G_n + \Gamma) \cdot L$, where Γ may be an empty graph.

[1] The same term is used for the characteristic polynomial of the adjacency matrix of a graph.

4. Since $C(G)$ is a non-negative definite matrix, then it has the eigenvalues

$$\lambda_i = \lambda_i(G) \geqslant 0, \qquad i = 1,..., n-1. \tag{2.7}$$

5. If G_n has no multiple edges, then [8]

$$\lambda_i(\bar{G}_n) = n - \lambda_i(G_n),\ i = 1,..., n-1. \tag{2.8}$$

6. Let

$$B(\lambda, G_n) = \frac{1}{\lambda} \det\{\lambda E + C(G_n)\} = \sum_{i=0}^{n-1} b_i(G_n)\, \lambda^{n-1-i}.$$

From (2.8),

$$P(\lambda, \bar{G}_n) = B(\lambda - n, G_n). \tag{2.9}$$

7. From (2.7) and (2.8),

$$\lambda_i(G_n) \leqslant n, \tag{2.10}$$

and, from (2.3) and (2.4),

$$\max_i \lambda_i(G_n) = \lambda_{\max}(G_n) = n$$

if and only if $\overline{G_n}$ is a disconnected graph (see also (2.18)) [9].

8. From (2.10),

$$P(\lambda, G_n) > 0 \qquad \text{at} \quad \lambda > n \tag{2.11}$$

and $P(n, G_n) = 0$ if and only if $\bar{G}_n$ is a disconnected graph (see also (2.18)).

9. From (2.11) and (2.6) it follows that, in (2.5),

$$R(\lambda) > 0 \qquad \text{at} \quad \lambda > p_n \tag{2.12}$$

and $R(p_n) > 0$ if $\overline{\Psi[G_n]}$ is connected.

10. Let G be a disconnected graph with the components $G_{n_1},..., G_{n_k}$. Then, from (2.10),

$$\lambda_i(G) \leqslant \max_s n_s. \tag{2.13}$$

11. If d_x is the degree of the vertex x in G, then [9]

$$\lambda_i(G) \leqslant \max_{x,y \in G, x \neq y} (d_x + d_y).$$

12. Let G_Y be the graph obtained from G by identifying the vertices of the set Y with subsequent removal of the loops. Then, from [9],

$$b_i = b_i(G_n) = \sum_{Y \subset G, |Y| = n-i} T(G_Y), \qquad i = 0, 1, \ldots, n-1.$$

Every k-forest ($k = n - i$) of a graph G_n with exactly one vertex of $Y(|Y| = k)$ in every component is associated with the tree of G_Y obtained from this k-forest by identifying all vertices of Y. This correspondence between k-forests of the above type and trees of G_Y is obviously one-to-one. Since, for any k-forest F^i from G_n there are exactly $\gamma(F^i)$ different sets Y with $k = n - i$ vertices which have exactly one common vertex with every component of F^i, then

$$b_i(G_n) = \sum_{F^i \subset G_n} \gamma(F^i), \qquad i = 0, 1, \ldots, n-1. \tag{2.14}$$

In particular,

$$b_0(G) = 1,\ b_1(G^m) = 2m,\ b_2(G^m) = 2m^2 - m - \tfrac{1}{2}\delta_2(G^m), \tag{2.15}$$

$$b_{n-1}(G_n) = nT(G_n), \tag{2.16}$$

where

$$\delta_k(G) = \sum_{x \in G} d_x^{\,k}.$$

13. From (2.8) it follows [9] that

$$\begin{aligned} b_k(\bar{G}_n) &= \sum_{i=0}^{k} (-1)^i \binom{n-i-1}{n-k-1} n^{k-i} b_i(G_n) \\ &= \frac{1}{(n-k-1)!} P_\lambda^{(n-k-1)}(n, G_n). \end{aligned} \tag{2.17}$$

In particular, for $k = n - 1$,

$$T(\bar{G}_n) = \frac{1}{n} b_{n-1}(\bar{G}_n) = \sum_{i=0}^{n-1} (-1)^i\, b_i(G_n)\, n^{n-2-i} = \frac{1}{n} P(n, G_n). \tag{2.18}$$

It should be noted that the latter relation may also be obtained by the principle of inclusion and exclusion [15]. Indeed, let $t_n(G^k)$ be the number of different trees of the complete graph K_n containing a given graph G^k; $S_k = \sum_{G^k \subset G} t_n(G^k)$, where $G = G_n{}^m$ and the summation is over all $\binom{m}{k}$ subgraphs G^k of G. By the method of inclusion and exclusion [19]:

$$T(\bar{G}) = S_0 - S_1 + \cdots + (-1)^k S_k + \cdots + (-1)^m S_m.$$

As stated in [15],

$$t_n(G^k) = \begin{cases} 0, & \text{if } G^k \text{ has a cycle,} \\ \gamma(F^k)\, n^{n-2-k}, & \text{if } G^k = F^k \text{ has no cycles.} \end{cases}$$

Thus, for $k \geqslant n$, $S_k = 0$ and

$$T(\bar{G}_n) = \sum_{i=0}^{n-1} (-1)^i\, n^{n-2-i} \sum_{F^i \subset G} \gamma(F^i).$$

By (2.14), this relation coincides with (2.18).

Now we shall derive some properties of the characteristic polynomial which we shall use below.

LEMMA 2.1.

$$b_3(G^m) = \tfrac{4}{3}m^3 - 2m^2 - (m-1)\,\delta_2 + \tfrac{1}{3}\delta_3 - 2\Delta, \tag{2.19}$$

where Δ is the number of triangles in G and

$$\delta_k = \delta_k(G) = \sum_{x \in G} d_x^{\,k}.$$

Proof. Let $\{\Lambda^3\}, \{\Delta^3\}, \{\Pi\}, \{H^3\}, \{N^3\}$ be the numbers of selections of the corresponding types in the set of all $\binom{m}{3}$ selections of three edges from G so that

$$\{\Lambda^3\} + \{\Delta^3\} + \{\Pi\} + \{H^3\} + \{N^3\} = \binom{m}{3}.$$

At the same time,

$$\sum_{x \in G} \binom{d_x}{3} = \{\Lambda^3\}, \quad \sum_{x \in G} (m - d_x) \binom{d_x}{2} = 3\{\Delta^3\} + 2\{\Pi\} + \{H^3\}.$$

According to (2.14), $b_3(G) = 4\{\Lambda^3\} + 4\{\Pi\} + 6\{H^3\} + 8\{N^3\}$. From these four relations one can easily obtain the required statement.

Let $G \backslash u$ denote the graph obtained from G by removal of the edge u. We shall assume from now on that

$$\lambda_{\max}(G_n) = \lambda_1(G_n) \geqslant \lambda_2(G_n) \geqslant \cdots \geqslant \lambda_{n-1}(G_n) = \lambda_{\min}(G_n).$$

LEMMA 2.2. *For any edge u of a graph G*

$$\lambda_k(G_n) \geqslant \lambda_k(G_n \backslash u),\ k = 1, 2, \ldots, n-1.$$

Proof. Let $C_u = C(G_n) - C(G_n \backslash u)$. The matrix C_u is non-negative definite since $\det(\lambda E - C_u) = \lambda^{n-2}(\lambda - 2)$. By virtue of the Courant-Fischer Theorem [20],

$$\lambda_k(G_n) = \lambda_k(C(G \backslash u) + C_u) \geqslant \lambda_k(G_n \backslash u),\ k = 1, \ldots, n-1.$$

Let $d_{\max}(G) = d_{\max}$ and $d_{\min}(G)$ be the maximal and minimal degrees of vertices in G, respectively.

LEMMA 2.3. *If* $m \geqslant 1$, *then*

$$\lambda_{\max}(G^m) = \lambda_1(G^m) \geqslant d_{\max}(G^m) + 1,\ \lambda_k(G^m) \geqslant 1 \tag{2.20}$$

for $k = 2, 3, \ldots, d_{\max}$. *If* $G \neq K$, *then*

$$\lambda_{\min}(G) \leqslant d_{\min}(G).$$

Proof. Let L be obtained from G by removal of all edges which are non-incident to the vertex of the maximal degree. By Lemma 2.2,

$$\lambda_k(G) \geqslant \lambda_k(L)$$

and

$$P(\lambda, L) = (\lambda - d_{\max} - 1)(\lambda - 1)^{d_{\max}-1}\, \lambda^{n-1-d_{\max}}$$

from whence follows the truth of (2.20). The inequality for $\lambda_{\min}(G)$ is obtained from the inequality for $\lambda_{\max}(G)$ by using (2.8).

Let u be an edge in $G = G^m$ and

$$\Phi(\lambda, G_n{}^m) = \lambda^{m-n+1} P(\lambda, G_n{}^m). \tag{2.21}$$

LEMMA 2.4.

$$\Phi(\lambda, G) = \Phi(a, G) + \sum_{u \in G} \int_a^\lambda \Phi(x, G \backslash u)\, dx. \tag{2.22}$$

Proof. From (2.14), $(m - i)\, b_i(G) = \sum_{u \in G} b_i(G \backslash u)$ for $i = 0, 1, \ldots, n-1$. Therefore

$$\sum_{u \in G} \Phi(\lambda, G \backslash u) = \sum_{i=0}^{n-1} (-1)^i\, (m - i)\, b_i(G)\, \lambda^{m-i-1} = \Phi_\lambda{}'(\lambda, G),$$

which is equivalent to required statement.

Let p_n be the number of vertices of the graph $\Psi[G_n]$.

LEMMA 2.5. *Let* $P(\lambda, \Gamma_n) > P(\lambda, L_n)$ for $\lambda \geqslant n$. *Then*

(1) $P(\lambda, \Psi[\Gamma_n]) > P(\lambda, \Psi[L_n])$ *for* $\lambda > p_n$ (*for* $\lambda \geqslant p_n$ *if* $\overline{\Psi[\Gamma_n]}$ *is a connected graph*);

(2) $T(\bar{\Gamma}_n) > T(\bar{L}_n)$;

(3) $T(\overline{\Psi[\Gamma_n]}) > T(\overline{\Psi[L_n]})$ *for a connected* $\overline{\Psi[\Gamma_n]}$.

Proof. Consider the relation (2.5) for the graphs $\Psi[\Gamma_n]$ and $\Psi[L_n]$. Since, from (2.6), it follows that $\lambda - \alpha \geqslant n$ for $\lambda \geqslant p_n$, then by the lemma's condition $P(\lambda - \alpha, \Gamma_n) > P(\lambda - \alpha, L_n)$ for $\lambda \geqslant p_n$. At the same time, by (2.12), $R(\lambda) > 0$ for $\lambda > p_n$ (for $\lambda \geqslant p_n$ if $\overline{\Psi[\Gamma_n]}$ is a connected graph). Therefore Inequality (1) follows from (2.5) for $\Psi[\Gamma_n]$ and $\Psi[L_n]$. Inequality (2) may be obtained from the lemma's condition by using (2.18) with $\lambda = n$. Inequality (3) follows from (2.18) and Inequality (1) with $\lambda = p_n$.

Remark 2.1. In view of (1.4), Inequality (3) in Lemma 2.5 may be replaced by Inequality (3′):

$$T(\Psi[\bar{\Gamma}_n]) > T(\Psi[\bar{L}_n])$$

for a connected $\Psi[\bar{\Gamma}_n]$.

Remark 2.2. From (2.5), (2.6), and (2.12), it follows that Inequalities (1) and (3) of the lemma hold also when the lemma's conditions are satisfied only at $\lambda > n$.

Tables I and II illustrate the characteristic polynomials of the graphs G_4^2 and G_6^3.

Since, by (2.3), $P(\lambda, G + kg) = \lambda^k P(\lambda, G)$, then for graphs from the tables the relationships between the polynomials $P(\lambda, G + kg)$ are the same as between the polynomials $P(\lambda, G)$ for $\lambda > 0$.

TABLE I

G_4^2	$P(\lambda, G)$	$\lambda_{\max}(G)$	$T(\bar{G})$	Remarks
N_4^2	$\lambda^3 - 4\lambda^2 + 4\lambda = \lambda(\lambda - 2)^2$	2	4	$P(\lambda, N_4^2) >$
Λ_4^2	$\lambda^3 - 4\lambda^2 + 3\lambda = \lambda(\lambda - 1)(\lambda - 3)$	3	3	$> P(\lambda, \Lambda_4^2)$ at $\lambda > 0$

TABLE II

G	$P(\lambda, G)$	$\lambda_{\max}(G)$	$T(\bar{G})$	Remarks
N_6^3	$\lambda^5-6\lambda^4+12\lambda^3-8\lambda^2=\lambda^2(\lambda-2)^3$	2	384	$P(\lambda, N_6^3)>P(\lambda, H_6^3)>$
H_6^3	$\lambda^5-6\lambda^4+11\lambda^3-6\lambda^2=\lambda^2(\lambda-1)(\lambda-2)(\lambda-3)$	3	360	$> P(\lambda, G_6^3)$ for $G_6^3 \neq N_6^3$,
Π_6^3	$\lambda^5-6\lambda^4+10\lambda^3-4\lambda^2=\lambda^2(\lambda-2)[(\lambda-2)^2-2]$	$2+\sqrt{2}$	336	$G_6^3 \neq H_6^3$ and $\lambda>\lambda_{\max}(G_6^3)$;
		$V!$	V	$P(\lambda, \Lambda_6^3)<P(\lambda, G_6^3)$
Δ_6^3	$\lambda^5-6\lambda^4+9\lambda^3=\lambda^3(\lambda-3)^2$	3	324	for $G_6^3 \neq \Lambda_6^3$ and $\lambda \geqslant 4$
Λ_6^3	$\lambda^5-6\lambda^4+9\lambda^3-4\lambda^2=\lambda^2(\lambda-1)^2(\lambda-4)$	4	300	

3. Graphs with Extremal Numbers of Trees

In this section the above properties of the characteristic polynomial of a graph will be used to show that the graphs of a certain type have the following extremal property: their polynomials are greater (or less) than polynomials of other graphs with the same numbers of vertices and edges for all λ exceeding a certain threshold. Thence it will follow that the complements of these graphs have the maximal (or minimal) numbers of trees over the graphs with the same numbers of vertices and edges.

Theorem 3.1. *Let $N_n^m \neq G_n^m = G$. Then*

$$P(\lambda, N_n^m) > P(\lambda, G) \qquad \text{for} \quad \lambda \geqslant \lambda_{\max}(G).$$

Proof (by induction). When $m = 2$, the inequality may be verified immediately for any $n \geqslant 4$ (see Table I). By Lemma 2.4,

$$\Phi(\lambda, N_n^m) - \Phi(\lambda, G) = \Phi(\lambda_{\max}(G), N_n^m) - \Phi(\lambda_{\max}(G), G) + \sum_{u \in G} \int_{\lambda_{\max}(G)}^{\lambda} [\Phi(x, N_n^{m-1}) - \Phi(x, G \backslash u)]\, dx. \tag{3.1}$$

By Lemma 2.2, $\lambda_{\max}(G) \geqslant \lambda_{\max}(G \backslash u)$. Therefore, by the induction assumption for all $x \in [\lambda_{\max}(G), \lambda]$, every square bracket in (3.1) is non-negative and from $G \neq N_n^m$ at least one of them is positive. At the same time, $\lambda_{\max}(N_n^m) = 2$ and, since $G \neq N_n^m$, then $d_{\max}(G) \geqslant 2$ and, by Lemma 2.3, $\lambda_{\max}(G) \geqslant 3$. Hence

$$\Phi(\lambda_{\max}(G), N_n^m) - \Phi(\lambda_{\max}(G), G) = \Phi(\lambda_{\max}(G), N_n^m) > 0$$

and therefore, for $\lambda \geqslant \lambda_{\max}(G)$,

$$\Phi(\lambda, N_n{}^m) > \Phi(\lambda, G).$$

In view of (2.21), $P(\lambda, N_n{}^m) > P(\lambda, G)$ for $\lambda \geqslant \lambda_{\max}(G)$, as required.

COROLLARY 3.1.1. *Let $N_n{}^m \neq G_n{}^m$. Then*

(1) $P(\lambda, \Psi[N_n{}^m]) > P(\lambda, \Psi[G_n{}^m])$ *for* $\lambda > p_n$ (*for* $\lambda \geqslant p_n$ *if* $\overline{\Psi[N_n{}^m]}$ *is a connected graph*);

(2) $T(\overline{N_n{}^m}) = n^{n-m-2}(n-2)^m > T(G_n{}^m)$;

(3) $T(\overline{\Psi[N_n{}^m]}) > T(\overline{\Psi[G_n{}^m]})$ *for a connected* $\overline{\Psi[N_n{}^m]}$.

By (2.10), $\lambda_{\max}(G_n) \leqslant n$, so the required inequalities follow from Theorem 3.1 and Lemma 2.5 (see also (1.1)).

THEOREM 3.2. *Let $\Lambda_n{}^m \neq G_n{}^m = G$. Then $P(\lambda, G) > P(\lambda, \Lambda_n{}^m)$ for $\lambda \geqslant m+1$.*

Proof (by induction). It is easily checked the inequality for $m = 2$ (see Table I).

From (2.22),

$$\Phi(\lambda, G) - \Phi(\lambda, \Lambda_n{}^m) = \Phi(m+1, G) - \Phi(m+1, \Lambda_n{}^m) + \sum_{u \in G} \int_{m+1}^{\lambda} [\Phi(x, G\backslash u) - \Phi(x, \Lambda_n^{m-1})]\, dx. \quad (3.2)$$

By virtue of induction assumption, every square bracket in (3.2) is nonnegative for all $x \in [m+1, \lambda]$. If, in every component of the graph G, the number of vertices is less than $m+1$, then, by (2.13), $\lambda_{\max}(G) < m+1$. Let there be a component L of G with $m+1$ vertices. Then all other components of G are isolated vertices, and so $\lambda_{\max}(G) = \lambda_{\max}(L)$. Since $G \neq \Lambda_n{}^m$, then $L \neq \Lambda^m$ and, by (2.10), $\lambda_{\max}(G) = \lambda_{\max}(L) < m+1$, i.e., $\Phi(m+1, G) > \Phi(m+1, \Lambda_n{}^m)$. This completes the proof.

COROLLARY 3.2.1. *Let $\Lambda_n{}^m \neq G_n{}^m$. Then*

(1) $P(\lambda, \Psi[G_n{}^m]) > P(\lambda, \Psi[\Lambda_n{}^m])$ *for* $\lambda > p_n$ (*for* $\lambda \geqslant p_n$ *if* $\overline{\Psi[G_n{}^m]}$ *is a connected graph*);

(2) $T(\overline{G_n{}^m}) > T(\overline{\Lambda_n{}^m}) = n^{n-m-2}(n-1)^{m-1}(n-m-1)$;

(3) $T(\overline{\Psi[G_n{}^m]}) > T(\overline{\Psi[\Lambda_n{}^m]})$ *for a connected* $\overline{\Psi[G_n{}^m]}$.

These inequalities follow from Theorem 3.2 and Lemma 2.5 by $n \geqslant m+1$ (see also (1.3)).

THEOREM 3.3. *Let $G_n^m = G \neq N_n^m$, $G_n^m \neq H_n^m$. Then*

$$P(\lambda, H_n^m) > P(\lambda, G) \qquad \textit{for} \quad \lambda > \lambda_{\max}(G).$$

Proof (by induction). For $m = 3$ the inequality may be verified by using Table II.

From Lemma 2.4,

$$\begin{aligned}\Phi(\lambda, H_n^m) &- \Phi(\lambda, G)\\ &= \Phi(\lambda_{\max}(G), H_n^m) - \Phi(\lambda_{\max}(G), G)\\ &\quad + \int_{\lambda_{\max}(G)}^{\lambda} \Big[\sum_{u\in H_n^m} \Phi(x, H_n^m\backslash u) - \sum_{u\in G} \Phi(x, G\backslash u)\Big]\, dx. \qquad (3.3)\end{aligned}$$

The graph H_n^m has two subgraphs isomorphic to N_n^{m-1} and $m-2$ subgraphs isomorphic to H_n^{m-1} over all subgraphs with $m-1$ edges. If the graph G is non-isomorphic to N_n^m and H_n^m, then it has no more than one subgraph isomorphic to N_n^{m-1}. Indeed, if such subgraph of G does exist, then the removal of any edge of this subgraph from G gives a graph non-isomorphic to N_n^{m-1}. Thence it follows that, by Theorem 3.1 and the induction assumption, the integral in (3.3) is positive for $\lambda > \lambda_{\max}(G)$. At the same time $\lambda_{\max}(H_n^m) = 3$ and, by Lemma 2.3,

$$\lambda_{\max}(G) \geqslant d_{\max}(G) + 1 \geqslant 3, \text{ i.e., } \Phi(\lambda_{\max}(G), H_n^m) \geqslant \Phi(\lambda_{\max}(G), G).$$

Hence $\Phi(\lambda, H_n^m) > \Phi(\lambda, G)$ and by (2.21), $P(\lambda, H_n^m) > P(\lambda, G)$ for $\lambda > \lambda_{\max}(G)$ as required.

COROLLARY 3.3.1. *Let $G_n^m \neq N_n^m$, $G_n^m \neq H_n^m$. Then*

(1) $P(\lambda, \Psi[H_n^m]) > P(\lambda, \Psi[G_n^m])$ *for* $\lambda > p_n$ (*for* $\lambda \geqslant p_n$ *if* $\overline{\Psi[H_n^m]}$ *is a connected graph*);

(2) $T(\overline{H_n^m}) = n^{n-m-2}(n-2)^{m-2}(n-1)(n-3) > T(\overline{G_n^m})$;

(3) $T(\overline{\Psi[H_n^m]}) > T(\overline{\Psi[G_n^m]})$ *for a connected* $\overline{\Psi[H_n^m]}$.

This follows from Theorem 3.3. and Lemma 2.5 because $\lambda_{\max}(G_n^m) \leqslant m + 1 < 2m - 1 < n$ when $m \geqslant 3$ and only for $m \geqslant 3$ do graphs G_n^m non-isomorphic to N_n^m and H_n^m exist (see also (1.2)).

THEOREM 3.4. *Let $G_n^m \neq N_n^m$ and $G_n^m \neq H_n^m$. Then*

$$P(\lambda, \overline{N_n^m}) > P(\lambda, \overline{H_n^m}) > P(\lambda, \overline{G_n^m}) \qquad \textit{for} \quad \lambda > n.$$

Proof. The inequalities $B(\lambda, N_n^m) > B(\lambda, H_n^m) > B(\lambda, G_n^m)$ for $\lambda > 0$

may be derived by the same arguments as Theorems 3.1 and 3.3. In view of (2.9) these inequalities are equivalent to the required ones.

COROLLARY 3.4.1. *Under the conditions of Theorem* 3.4.

(1) $P(\lambda, \overline{\Psi[N_n{}^m]}) > P(\lambda, \overline{\Psi[H_n{}^m]}) > P(\lambda, \overline{\Psi[G_n{}^m]})$ *for* $\lambda > p_n$ (*for* $\lambda \geqslant p_n$ *if* $\Psi[N_n{}^m]$ *is a connected graph*);

(2) $T(\Psi[N_n{}^m]) > T(\Psi[H_n{}^m]) > T(\Psi[G_n{}^m])$ *for a connected* $\Psi[N_n{}^m]$.

This corollary follows from Theorem 3.4 by Remark 2.2 to Lemma 2.5.

LEMMA 3.1. *Let* $G_n{}^m \neq \Lambda_n{}^m$ *and* k *be a natural number less than the minimal length of a cycle in* $G_n{}^m$. *Then* $b_k(\Lambda_n{}^m) \leqslant b_k(G_n{}^m)$.

Proof. The required statement follows from (2.14) because $\gamma(\Lambda^k) < \gamma(F^k)$ and $F^k = \Lambda^k$ for $F^k \subset \Lambda^m$ and any set of k edges of G with k less than the minimal length of a cycle in $G_n{}^m$ is a forest.

As the immediate consequence of Lemma 3.1 we have

LEMMA 3.2. *Let* $D_n{}^m$ *denote a graph which has* $(m+1)$*-vertices component and* $D_n{}^m \neq \Lambda_n{}^m$. *Then* $b_k(\Lambda_n{}^m) < b_k(D_n{}^m)$ *for* $k = 2, 3,\ldots, m-1$ *and* $b_m(\Lambda_n{}^m) = b_m(D_n{}^m) = m+1$.

Note that, by (2.14), $b_0(G) = 1$, $b_1(G^m) = 2m$ and $b_i(G_n{}^m) = 0$ for $i = m+1,\ldots, n-1$. So in view of (2.9) from Lemma 3.2 we have

THEOREM 3.5. *Let* $D_n{}^m \neq \Lambda_n{}^m$. *Then, for* $\lambda > n$,

$$P(\lambda, \overline{D_n{}^m}) > P(\lambda, \overline{\Lambda_n{}^m}).$$

COROLLARY 3.5.1. *Let* $D_n{}^m \neq \Lambda_n{}^m$. *Then*

(1) $P(\lambda, \overline{\Psi[D_n{}^m]}) > P(\lambda, \overline{\Psi[\Lambda_n{}^m]})$ *for* $\lambda > p_n$ (*for* $\lambda \geqslant p_n$ *if* $\Psi[\Lambda_n{}^m]$ *is a connected graph*);

(2) $T(\Psi[D_n{}^m]) > T(\Psi[\Lambda_n{}^m])$.

Remark. Generally speaking, for graphs different from $D_n{}^m$, Theorem 3.5 and Corollary 3.5.1 are not true. For example, by Theorem 3.2 for $\lambda \geqslant 4$

$$P(\lambda, \overline{\Lambda_4{}^3}) > P(\lambda, \Lambda_4{}^3) = P(\lambda, \overline{[g^3 + g]})$$

and the graph $[g^3 + g]$ has a triangle, i.e., it is not a graph of the type $D_4{}^3$.

When graphs are compared by the numbers of their trees, the following question is in order: will the relation between the numbers of trees of two given graphs be preserved after applying an operator Ψ to these graphs (provided that the graphs $\Psi[G]$ are connected)? For example, will or will not $T(\Gamma) > T(L)$ imply $T(g\Gamma) > T(gL)$? The following example illustrates that, in the general case, this inequality is not preserved. Let Q^m denote a graph which consists of a simple cycle of m edges and one isolated vertex. It is obvious that $T(\Lambda^m) = 1 > T(Q^m) = 0$. Let us prove that, for $m \geqslant 5$, $T(g\Lambda^m) < T(gQ^m)$. According to (2.16) and (2.14),

$$T(gG_n) = \sum_{i=0}^{n-1} b_i(G_n).$$

By Lemma 3.1, $b_i(\Lambda^m) < b_i(Q^m)$ for $i = 2,..., m - 1$. At the same time, from (2.14), it follows that $b_0(\Lambda^m) = b_0(Q^m) = 1$, $b_1(\Lambda^m) = 2m$, $b_3(\Lambda^m) = 4\binom{m}{3}$, $b_3(Q^m) \geqslant 6\binom{m}{3} - 2m$, $b_m(\Lambda^m) = m + 1$, $b_m(Q^m) = 0$. Therefore

$$\begin{aligned} T(gQ^m) - T(g\Lambda^m) &\geqslant [b_3(Q^m) - b_3(\Lambda^m)] + [b_m(Q^m) - b_m(\Lambda^m)] \\ &\geqslant 2\binom{m}{3} - 3m - 1 > 0 \end{aligned}$$

when $m \geqslant 5$. At the same time $T(g\Lambda^4) > T(gQ^4)$. Thence it follows that $0 = P(m + 1, \overline{Q^m}) < P(m + 1, \overline{\Lambda^m})$. However, $P(m + 2, \overline{Q^m}) > P(m + 2, \overline{\Lambda^m})$ for $m \geqslant 5$.

Thus the graphs with extremal numbers of trees have been constructed. In particular, it has been proved that, in removal of m edges ($m \leqslant (n/2)$) from the complete graph K_n, the number of retained trees is maximal when the removed edges form the graph N_n^m. The second best after N_n^m removal is the graph H_n^m. Finally, the minimal number of trees remains after removal of the graph Λ_n^m ($m \leqslant n - 1$). The next best choice after H_n^m seems to be the graph Π_n^m and the worst choice after Λ_n^m the graph Δ_n^m. However, in these cases our scheme of proof of the extremality of the graph $\overline{N_n^m}$ and others is not suitable. It may be illustrated by the example of graphs with 6 vertices and 3 edges (see Table II!). Indeed, $\lambda_{\max}(\Pi_n^m) > \lambda_{\max}(\Delta_n^m)$ and at the same time $T(\overline{\Pi_n^m}) > T(\overline{\Delta_n^m})$.

4. Estimates for the Number of Trees of a Graph

Since to find the number of trees for most graphs is very difficult, estimates of the number of trees in different terms of the graph seem to be of interest.

One way to construct these estimates is given by relation (2.18). Since this may be obtained by the principle of inclusion and exclusion, then according to Bonferroni's Inequalities [21],

$$\begin{aligned} n^{n-2} - b_1(G_n)\, n^{n-3} + \cdots + (-1)^{2i+1}\, b_{2i+1}(G_n) \\ \leqslant T(\bar{G}_n) \\ \leqslant n^{n-2} - b_1(G_n)\, n^{n-3} + \cdots + (-1)^{2j}\, b_{2j}(G_n)\, n^{n-2j-2}, \end{aligned} \tag{4.1}$$

where

$$i = 0, 1,\ldots, \left[\frac{n}{2}\right] - 1; \qquad j = 0, 1,\ldots, \left[\frac{n-1}{2}\right].$$

Thus bounds for $T(\bar{G})$ may be obtained if a certain number of the first coefficients $b_k(G)$ (or bounds for them) are known.

In particular, using (2.15), (2.19), and (4.1) with $i = j = 1$, we obtain:

$$\begin{aligned} T(\bar{G}_n) \geqslant n^{n-2} - 2mn^{n-3} + [2m^2 - m - \tfrac{1}{2}\delta_2(G_n)]\, n^{n-4} \\ - [\tfrac{4}{3}m^3 - 2m^2 - (m-1)\,\delta_2(G_n) + \tfrac{1}{3}\delta_3(G_n) - 2\Delta(G_n)]\, n^{n-5}, \end{aligned}$$

$$T(\bar{G}_n) \leqslant n^{n-2} - 2mn^{n-3} + [2m^2 - m - \tfrac{1}{2}\delta_2(G_n)]\, n^{n-4},$$

where, as above, m is the number of edges of G_n, $\Delta(G)$ is the number of triangles in G, and $\delta_k(G) = \sum_{x \in G} d_x^{\,k}$.

Another way to construct the bounds is to use the expression (2.2) of the coefficients $b_k(G)$ as symmetric polynomials. If the first s coefficients $b_i(G_n)$ are known, then bounds for $b_k(G_n)$ $s < k \leqslant n - 1$, may result from solving the following optimal problem $P_{s,k}$:

$$\sigma_k(x_1, x_2, \ldots, x_{n-1}) \to \max, \min;$$

$$\sigma_i(x_1, x_2, \ldots, x_{n-1}) = b_i,\ i = 1, 2,\ldots, s;$$

$$x_j \geqslant 0,\ j = 1, 2,\ldots, n - 1.$$

For example, the solution of the problem $P_{1,k}$ with $b_1(G^m) = 2m$ (2.15) gives

$$0 \leqslant b_k(G_n^{\,m}) \leqslant \binom{n-1}{k}\left(\frac{2m}{n-1}\right)^k, \qquad k = 2, 3,\ldots, n - 1.$$

These bounds are derived in [9]. In that paper the upper bounds have also been proved to be reached only in the case of graphs in which every two vertices are joined by one and the same number of multiple edges.

Solving the problem $P_{2,n-1}$ under the condition (2.15), we have

$$\frac{1}{n}\left(\frac{\sqrt{\quad}}{n-1}\right)^{n-1}\left(\frac{2m}{\sqrt{\quad}}+\frac{1}{\sqrt{n-2}}\right)^{n-2}\left(\frac{2m}{\sqrt{\quad}}-\sqrt{n-2}\right)$$

$$\leqslant T(G_n^{m})=\frac{1}{n}\,b_{n-1}(G_n^{m})$$

$$\leqslant \frac{1}{n}\left(\frac{\sqrt{\quad}}{n-1}\right)^{n-1}\left(\frac{2m}{\sqrt{\quad}}+\sqrt{n-2}\right)\left(\frac{2m}{\sqrt{\quad}}-\frac{1}{\sqrt{n-2}}\right)^{n-2},$$

where

$$\sqrt{\quad}=\sqrt{(n-1)\,\delta_2(G_n^{m})-4m^2}.$$

REFERENCES

1. G. KIRCHHOFF, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird, *Ann. Phys. Chem.* **72** (1847), 497–508.
2. J. LANTIERI, Méthode de détermination des arbres d'un réseau, *Ann. Télécommun.* **5** (1950), 204–208.
3. H. M. TRENT, A note on the enumeration and listing of all possible trees in a connected linear graph, *Proc. Nat. Acad. Sci. USA* **40** (1954), 1004–1007.
4. L. WEINBERG, Number of trees in a graph, *Proc. IRE* **46** (1958), 1954–1955.
5. S. D. BEDROSIAN, Formulas for the number of trees in a network, *IRE Trans. Circuit Theory* **CT-8** (1961), 363–364.
6. P. V. O'NEIL, The number of trees in a certain network, *Notices Amer. Math. Soc.* **10** (1963), 569.
7. S. D. BEDROSIAN, Generating formulas for the number of trees in a graph, *J. Franklin Inst.* **277** (1964), 313–326.
8. A. K. KELMANS, The number of trees in a graph. I, II (Russian), *Avtomat. i Telemeh.* (translated as *Automat. Remote Control*) **26** (1965), 2194–2204 and **27** (1966), 56–65.
9. A. K. KELMANS, On properties of the characteristic polynomial of a graph (Russian), "Kibernetiku na službu kommunizmu," Vol. 4, Gosènergoizdat, Moscow, 1967.
10. P. V. O'NEIL AND P. SLEPIAN, The number of trees in a network, *IEEE Trans. Circuit Theory* **CT-13** (1966), 271–281.
11. G. OLÁH, A problem on the enumeration of certain trees (Russian), *Studia Sci. Math. Hungar.* **3** (1968), 71–80.
12. P. V. O'NEIL, Enumerating of spanning trees in a certain graph, *IEEE Trans. Circuit Theory* **CT-17** (1970), 250.
13. S. D. BEDROSIAN, Formulas for the number of trees in certain incomplete graphs, *J. Franklin Inst.* **289** (1970), 67–69.
14. I. ROHLICKOVA, Poznámka a poctu koster jednoho typu grafu, in "Mat. Geometrie a teorie grafů," Karlov Universitet, Praha, 1970.
15. J. W. MOON, Enumerating labelled trees, "Graph Theory and Theoretical Physics" (F. Harary, ed.), Academic Press, New York, 1967.
16. A. K. KELMANS, On the analysis and synthesis of random graphs (Russian), Disserta-

tion for Candidate's Degree in Physical and Mathematical Sciences, Institute of Control Sciences, Moscow, 1967.
17. M. V. LOMONOSOV AND V. P. POLESSKIĬ, The lower bound for networks reliability (Russian), *Problemy Peredači Informacii* (translated as *Problems of Information Transmission*), **8** (1972), No. 2.
18. A. A. ZYKOV, Some properties of linear complexes (Russian), *Mat. Sb.* **24** (1949), No. 2, 163–168 (American Mathematical Society, Translations, Number 79, 1952).
19. M. HALL, JR., "Combinatorial Theory," Blaisdell, Waltham, Mass., 1967.
20. R. BELLMAN, "Introduction to Matrix Analysis," McGraw-Hill, New York, 1960.
21. W. FELLER, "An Introduction to Probability Theory and Its Applications," 2nd ed., Vol. I, Wiley, New York, 1957.