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ScienceDirect
Journal of Functional Analysis 258 (2010) 3818–3840
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

BMO estimates for the $H^\infty(\mathbb{B}_n)$ Corona problem

Şerban Costea [a], Eric T. Sawyer [a,1], Brett D. Wick [b,*,2]

[a] *McMaster University, Department of Mathematics and Statistics, 1280 Main Street West, Hamilton, Ontario L8S 4K1, Canada*

[b] *Georgia Institute of Technology, School of Mathematics, 686 Cherry Street, Atlanta, GA 30332-1060, USA*

Received 20 October 2009; accepted 24 December 2009

Available online 13 January 2010

Communicated by J. Bourgain

Abstract

We study the $H^\infty(\mathbb{B}_n)$ Corona problem $\sum_{j=1}^N f_j g_j = h$ and show it is always possible to find solutions f that belong to $BMOA(\mathbb{B}_n)$ for any $n > 1$, including infinitely many generators N. This theorem improves upon both a 2000 result of Andersson and Carlsson and the classical 1977 result of Varopoulos. The former result obtains solutions for strictly pseudoconvex domains in the larger space $H^\infty \cdot BMOA$ with $N = \infty$, while the latter result obtains $BMOA(\mathbb{B}_n)$ solutions for just $N = 2$ generators with $h = 1$. Our method of proof is to solve $\bar\partial$-problems and to exploit the connection between BMO functions and Carleson measures for $H^2(\mathbb{B}_n)$. Key to this is the exact structure of the kernels that solve the $\bar\partial$ equation for $(0, q)$ forms, as well as new estimates for iterates of these operators. A generalization to multiplier algebras of Besov–Sobolev spaces is also given.
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Keywords: Corona problem; Carleson measures; BMO; Besov–Sobolev spaces

1. Introduction

In 1962 Lennart Carleson demonstrated in [3] the absence of a corona in the maximal ideal space of $H^\infty(\mathbb{D})$ by showing that if $\{g_j\}_{j=1}^N$ is a finite set of functions in $H^\infty(\mathbb{D})$ satisfying

* Corresponding author.
E-mail addresses: secostea@math.mcmaster.ca (Ş. Costea), sawyer@mcmaster.ca (E.T. Sawyer), wick@math.gatech.edu (B.D. Wick).
1 Research supported in part by a grant from the National Science and Engineering Research Council of Canada.
2 Research supported in part by National Science Foundation DMS Grant # 0752703.

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doi:10.1016/j.jfa.2009.12.015

$$\sum_{j=1}^{N} |g_j(z)| \geqslant \delta > 0, \quad z \in \mathbb{D}, \tag{1.1}$$

then there are functions $\{f_j\}_{j=1}^N$ in $H^\infty(\mathbb{D})$ with

$$\sum_{j=1}^{N} f_j(z) g_j(z) = 1, \quad z \in \mathbb{D} \quad \text{and} \quad \sum_{j=1}^{N} \|f_j\|_\infty \leqslant C. \tag{1.2}$$

Later, Hörmander noted a connection between the Corona problem and the Koszul complex, and in the late 1970's Tom Wolff gave a simplified proof using the theory of the $\bar{\partial}$ equation and Green's theorem (see [6]). This proof has since served as a model for proving corona type theorems for other Banach algebras. While there is a large literature on such corona theorems in one complex dimension (see e.g. [8]), progress in higher dimensions has been limited. Indeed, apart from the simple cases in which the maximal ideal space of the algebra can be identified with a compact subset of $\mathbb{C}^n$, no corona theorem has been proved in higher dimensions until the recent work of the authors [5] on the Drury-Arveson Hardy space multipliers. Instead, partial results have been obtained, which we will discuss more below.

We of course have the analogous question in several complex variables when we consider $H^\infty(\mathbb{B}_n)$. The Corona problem for the Banach algebra $H^\infty(\mathbb{B}_n)$ is to show that if $g_1, \ldots, g_N \in H^\infty(\mathbb{B}_n)$ satisfy

$$\sum_{j=1}^{N} |g_j(z)| \geqslant 1, \quad \forall z \in \mathbb{B}_n,$$

then the ideal generated by $\{g_j\}_{j=1}^N$ is all of $H^\infty(\mathbb{B}_n)$, equivalently $\sum_{j=1}^N f_j(z) g_j(z) = 1$ for all $z \in \mathbb{B}_n$ for some $f_1, \ldots, f_N \in H^\infty(\mathbb{B}_n)$. This famous problem has remained open for $n > 1$ since Lennart Carleson proved the $n = 1$ dimensional case in 1962, but there are some partial results.

Most notably, there is the classical result of Varopoulos where $BMOA(\mathbb{B}_n)$ estimates were obtained for solutions f to the Bézout equation $f_1 g_1 + f_2 g_2 = 1$ [11]. The restriction to just $N = 2$ generators provides some algebraic simplifications to the problem. Note also that the more general equation

$$f_1 g_1 + f_2 g_2 = h, \quad h \in H^\infty,$$

can then be solved for $f \in H^\infty \cdot BMOA$.

Over two decades later, the case $2 \leqslant N \leqslant \infty$ was studied by Andersson and Carlsson [1] in 2000 who obtained $H^\infty \cdot BMOA$ solutions f to the infinite Bézout equation $\sum_{i=1}^\infty f_i g_i = 1$, and hence also to the more general equation

$$\sum_{i=1}^{\infty} f_i g_i = h, \quad h \in H^\infty. \tag{1.3}$$

To see that $H^\infty \cdot BMOA$ is strictly larger than $BMOA$, recall that the multiplier algebra of $BMOA$ is a *proper* subspace of H^∞ satisfying a vanishing Carleson condition (see e.g. Theorem 6.2 in [1]).

Our proof uses the methods of [5], that in turn generalize the integration by parts and estimates of Ortega and Fabrega [9]. Key to these new estimates are the almost invariant holomorphic derivatives from Arcozzi, Rochberg and Sawyer [2]. Consequently our proof can be used to handle any number of generators N with no additional difficulty and always yields $BMOA(\mathbb{B}_n)$ solutions f to (1.3). See [1] for further references to related material.

This leads to the main result of this paper in which we obtain $BMOA(\mathbb{B}_n)$ solutions to the $H^\infty(\mathbb{B}_n)$ Corona Problem (1.3) with infinitely many generators.

Theorem 1. *There is a constant $C_{n,\delta}$ such that given $g = (g_i)_{i=1}^\infty \in H^\infty(\mathbb{B}_n; \ell^2)$ satisfying*

$$1 \geqslant \sum_{j=1}^\infty |g_j(z)|^2 \geqslant \delta^2 > 0, \quad z \in \mathbb{B}_n, \tag{1.4}$$

there is for each $h \in H^\infty(\mathbb{B}_n)$ a vector-valued function $f \in BMOA(\mathbb{B}_n; \ell^2)$ satisfying

$$\begin{gathered} \|f\|_{BMOA(\mathbb{B}_n;\ell^2)} \leqslant C_{n,\delta} \|h\|_{H^\infty(\mathbb{B}_n)}, \\ \sum_{j=1}^\infty f_j(z) g_j(z) = h(z), \quad z \in \mathbb{B}_n. \end{gathered} \tag{1.5}$$

This theorem can be generalized to hold for the multiplier algebras $M_{B_p^\sigma(\mathbb{B}_n)}$ of the Besov–Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$ in place of the multiplier algebra $H^\infty(\mathbb{B}_n)$ of the classical Hardy space $H^2(\mathbb{B}_n) = B_2^{\frac{n}{2}}(\mathbb{B}_n)$. See Theorem 5 in the final section below.

Our method of proof uses the notation and techniques from [5]. However, for the convenience of the reader, this paper is written so that it is mostly self-contained.

2. Preliminaries

We begin by collecting all the relevant facts that will be necessary to prove Theorem 1. While many of these facts may be known to experts, we collect them all in one location for convenience.

2.1. BMO and Carleson measures

In this subsection we recall the well-known connection between BMO functions on the boundary $\partial\mathbb{B}_n$ of the ball and Carleson measures for $H^2(\mathbb{B}_n)$.

First, we define the space $\mathcal{CM}(\mathbb{B}_n)$. This is the collection of functions on the unit ball $\mathbb{B}_n$ such that

$$\|h\|_{\mathcal{CM}(\mathbb{B}_n)} \equiv \sup_{\zeta \in \mathbb{B}_n} \sqrt{\frac{\int_{S_\zeta} |h(z)|^2 \, d\lambda_n(z)}{(1 - |\zeta|)^n}} < \infty,$$

and for $\zeta \in \mathbb{B}_n \setminus \{0\}$ the *Carleson tent* S_ζ is defined by

$$S_\zeta = \left\{ z \in \mathbb{B}_n \colon \frac{1 - |\zeta|}{|1 - \bar{\zeta} z|} > \frac{1}{2} \right\}. \tag{2.1}$$

In classical language $h \in \mathcal{CM}(\mathbb{B}_n)$ if and only if the measure $d\mu_h(z) = |h(z)|^2\, d\lambda_n(z)$ is a *Carleson measure* for $H^2(\mathbb{B}_n)$; i.e. $H^2(\mathbb{B}_n) \subset L^2(d\mu_h)$.

Also, recall the space $BMO(\partial\mathbb{B}_n)$, which is the collection of functions that are in $L^2(\partial\mathbb{B}_n)$ such that

$$\|b\|^2_{BMO(\partial\mathbb{B}_n)} \equiv \sup_{Q_\delta(\eta)\subset\partial\mathbb{B}_n} \frac{1}{|Q_\delta(\eta)|} \int\limits_{Q_\delta(\eta)} |b - b_{Q_\delta(\eta)}|^2\, d\sigma(\zeta) < \infty,$$

where $Q_\delta(\eta)$ is the non-isotropic ball of radius $\delta > 0$ and center η in $\partial\mathbb{B}_n$ and

$$b_{Q_\delta(\eta)} = \frac{1}{|Q_\delta(\eta)|} \int\limits_{Q_\delta(\eta)} b(\xi)\, d\sigma(\xi).$$

One then defines $BMOA(\mathbb{B}_n) = BMO(\partial\mathbb{B}_n) \cap H^2(\mathbb{B}_n)$. Finally, we define vector-valued versions $BMO(\partial\mathbb{B}_n;\ell^2)$, $H^2(\mathbb{B}_n;\ell^2)$, $BMOA(\mathbb{B}_n;\ell^2)$ and $\mathcal{CM}(\mathbb{B}_n;\ell^2)$ in the usual way.

A well-known fact connecting the spaces $BMOA(\mathbb{B}_n;\ell^2)$ and $\mathcal{CM}(\mathbb{B}_n)$ is the following:

Lemma 1. *For $g \in H^2(\mathbb{B}_n;\ell^2)$ we have*

$$c\|g\|_{BMOA(\mathbb{B}_n;\ell^2)} \leqslant \big\|\big(1-|z|^2\big)^{\frac{n}{2}+1} g'(z)\big\|_{\mathcal{CM}(\mathbb{B}_n;\ell^2)} \leqslant C\|g\|_{BMOA(\mathbb{B}_n;\ell^2)}. \tag{2.2}$$

Proof. The scalar version of Lemma 1 is proved in Theorem 5.14 of [12]. The proof given there extends to the ℓ^2-valued case in a routine manner with a couple of possible exceptions which we now address. A key step in proving the first inequality in (2.2) is:

$$\begin{aligned}
&\bigg|\int\limits_{\partial\mathbb{B}_n} f(\zeta)\overline{g(\zeta)}\, d\sigma(\zeta)\bigg| \\
&\quad = \bigg|\int\limits_{\mathbb{B}_n} Rf(z)\overline{Rg(z)}|z|^{-2n} \log\frac{1}{|z|}\, dV(z)\bigg| \\
&\quad \leqslant \bigg(\int\limits_{\mathbb{B}_n} \frac{|Rf(z)|^2}{|f(z)|}|z|^{-2n}\log\frac{1}{|z|}\, dV(z)\bigg)^{\frac12} \bigg(\int\limits_{\mathbb{B}_n} \big|f(z)\big|\big|Rg(z)\big|^2|z|^{-2n}\log\frac{1}{|z|}\, dV(z)\bigg)^{\frac12},
\end{aligned}$$

followed by the two inequalities

$$\int\limits_{\mathbb{B}_n} \frac{|Rf(z)|^2}{|f(z)|}|z|^{-2n}\log\frac{1}{|z|}\, dV(z) \leqslant C\|f\|_{H^1(\mathbb{B}_n;\ell^2)}, \tag{2.3}$$

and

$$\int\limits_{\mathbb{B}_n} \big|f(z)\big|\big|Rg(z)\big|^2|z|^{-2n}\log\frac{1}{|z|}\, dV \leqslant \big\|\big(1-|z|^2\big)^{\frac{n}{2}+1} g'(z)\big\|_{\mathcal{CM}(\mathbb{B}_n;\ell^2)}\|f\|_{H^1(\mathbb{B}_n;\ell^2)}. \tag{2.4}$$

A crucial equality used in the proof of (2.3) in the scalar case when $n = 1$ is

$$\|f\|_{H^1(\mathbb{D})} = \int_{\mathbb{D}} \frac{|f'(z)|^2}{|f(z)|} \log \frac{1}{|z|}\, dV(z),$$

which in turn follows from Green's theorem and the identity

$$\Delta\big|f(z)\big| = \frac{|f'(z)|^2}{|f(z)|}.$$

When $f = \{f_k\}_{k=1}^{\infty}$ is ℓ^2-valued, this identity becomes

$$\begin{aligned}
\Delta\big|f(z)\big| &= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\big(f(z)\overline{f(z)}\big)^{\frac{1}{2}} = 4\frac{\partial}{\partial z}\frac{1}{2}\big|f(z)\big|^{-1} f(z)\cdot\overline{f'(z)} \\
&= 2\left\{\big|f(z)\big|^{-1} f'(z)\cdot\overline{f'(z)} - \frac{1}{2}\big|f(z)\big|^{-3}\big(f'(z)\cdot\overline{f(z)}\big)\big(f(z)\cdot\overline{f'(z)}\big)\right\} \\
&= \frac{2}{|f(z)|^3}\left\{\big|f(z)\big|^2\big|f'(z)\big|^2 - \frac{1}{2}\big|\langle f(z), f'(z)\rangle\big|^2\right\},
\end{aligned}$$

which by the Cauchy–Schwarz inequality yields the approximation

$$\Delta\big|f(z)\big| \approx \frac{|f'(z)|^2}{|f(z)|}.$$

This approximation and Green's theorem lead to

$$\frac{1}{2\pi}\int_{\mathbb{T}} \big|f\big(e^{i\theta}\big)\big|\, d\theta = \|f\|_{H^1(\mathbb{D})} \approx \int_{\mathbb{D}} \frac{|f'(z)|^2}{|f(z)|} \log\frac{1}{|z|}\, dV(z) \tag{2.5}$$

in the ℓ^2-valued case when $n = 1$. Now we consider the case of dimension $n > 1$ and apply (2.5) to each slice function $f_\zeta(z)$ with $\zeta \in \partial\mathbb{B}_n$. The result when $f(0) = 0$ is

$$\begin{aligned}
\|f\|_{H^1(\mathbb{B}_n)} &= c\int_{\partial\mathbb{B}_n}\int_{\mathbb{T}} \big|f_\zeta\big(e^{i\theta}\big)\big|\, d\theta\, d\sigma(\zeta) \\
&\approx \int_{\partial\mathbb{B}_n}\int_{\mathbb{D}} \frac{|f_\zeta'(z)|^2}{|f_\zeta(z)|}\log\frac{1}{|z|}\, dV(z)\, d\sigma(\zeta) \\
&= C\int_{\mathbb{B}_n} \frac{|Rf(w)|^2}{|f(w)|}|w|^{-2n}\log\frac{1}{|w|}\, dV(w),
\end{aligned}$$

since $f_\zeta'(z) = z^{-1}Rf(z\zeta)$ and $dV(z)\, d\sigma(\zeta) = r\, dr\, d\theta\, d\sigma(\zeta) = Cr^{2-2n}\, dV(r\zeta)\, d\theta$. This proves (2.3).

The inequality (2.4) is the ℓ^2-valued version of the Hörmander–Carleson Theorem when $p=1$. The scalar case is proved in Theorem 5.9 of [12] using the theory of the invariant Poisson integral $\mathbb{P}$ together with the subharmonic inequality (Corollary 4.5 in [12]):

$$\left|f(z)\right|^{\frac{p}{2}} \leqslant \mathbb{P}\left[|f|^{\frac{p}{2}}\right](z), \quad z \in \mathbb{B}_n.$$

The subharmonic inequality extends to ℓ^2-valued $f(z)$ by noting that

$$\left|f(z)\right|^{\frac{p}{2}} = \sup_{|v|\leqslant 1}\left|\left\langle f(z), v\right\rangle\right|^{\frac{p}{2}} \leqslant \sup_{|v|\leqslant 1}\mathbb{P}\left[\left|\langle f, v\rangle\right|^{\frac{p}{2}}\right](z) \leqslant \mathbb{P}\left[|f|^{\frac{p}{2}}\right](z),$$

and then the proof of (2.4) is completed using the scalar theory of $\mathbb{P}$ as in [12].

The arguments in [12] now complete the proof of (2.2). □

We will also need the following slight generalization of the special case $p=2$ and $\sigma=\frac{n}{2}$ of the multilinear estimate in Proposition 3 of [5] (the scalar case is Theorem 3.5 in [9]). Note that $B_2^{\frac{n}{2}}(\mathbb{B}_n) = H^2(\mathbb{B}_n)$ and $\|\mathbb{M}_g\|_{B_2^{\frac{n}{2}}(\mathbb{B}_n)\to B_2^{\frac{n}{2}}(\mathbb{B}_n;\ell^2)} = \|g\|_{H^\infty(\mathbb{B}_n;\ell^2)}$.

Lemma 2. *Suppose that $M, m \geqslant 1$ and $\alpha = (\alpha_0, \dots, \alpha_M) \in \mathbb{Z}_+^{M+1}$ with $|\alpha| = m$. For $g_1, \dots, g_M \in H^\infty(\mathbb{B}_n; \ell^2)$ and $h \in H^2(\mathbb{B}_n)$ we have*

$$\begin{aligned}&\int_{\mathbb{B}_n} \left(1-|z|^2\right)^n \left|\left(\mathcal{Y}^{\alpha_1} g_1\right)(z)\right|^2 \dots \left|\left(\mathcal{Y}^{\alpha_M} g_M\right)(z)\right|^2 \left|\left(\mathcal{Y}^{\alpha_0} h\right)(z)\right|^2 d\lambda_n(z) \\ &\quad \leqslant C_{n,M,m}\left(\prod_{j=1}^M \|g_j\|^2_{H^\infty(\mathbb{B}_n;\ell^2)}\right)\|h\|^2_{H^2(\mathbb{B}_n)}. \end{aligned} \tag{2.6}$$

Here $\mathcal{Y}^m$ is the vector of all differential operators of the form $X_1X_2\dots X_m$ where each X_i is either $(1-|z|^2)I$, $(1-|z|^2)R$ or D. The operator I is the identity, the operator R is the radial derivative, and the operator

$$D = \left(1-|z|^2\right)P_z\nabla + \sqrt{1-|z|^2}\,Q_z\nabla$$

is an almost invariant derivative defined in [5]. The iteration $X_1X_2\dots X_m$ is *not* a composition of operators, but as in [5] one fixes the coefficients, then composes the frozen operators, and then unfreezes the coefficients.

Finally, the generalization in (2.6) is that the multiplier functions g_j need not be the same function, as they were in Proposition 3 of [5]. However, the proof given in [5] applies to different g_j as well (the scalar case in [9] is for different g_j). We observe that the proof is actually simplified due to the fact that for $s \geqslant \frac{n}{2}$, $\|\mathbb{M}_g\|_{B_2^s(\mathbb{B}_n)\to B_2^s(\mathbb{B}_n;\ell^2)} = \|g\|_{H^\infty(\mathbb{B}_n;\ell^2)}$.

Using the geometric characterization of Carleson measures for $H^2(\mathbb{B}_n;\ell^2)$, Lemma 1 says that $g \in BMOA(\mathbb{B}_n;\ell^2)$ if and only if the measure μ_g^m associated to g by

$$d\mu_g^m(z) \equiv \left|\left(1-|z|^2\right)^{\frac{n}{2}}\mathcal{Y}^m g(z)\right|^2 d\lambda_n(z),$$

is a *Carleson measure* for $H^2(\mathbb{B}_n;\ell^2)$; i.e. $H^2(\mathbb{B}_n;\ell^2)\subset L^2(\mu_g^m;\ell^2)$, for some (equivalently all) $m\geqslant 1$. On the other hand, $g\in H^2(\mathbb{B}_n;\ell^2)$ if and only if g is holomorphic and the measure μ_g^m is finite.

Remark 1. We note in passing that $\mathcal{CM}(\mathbb{B}_n)\subset L^2(\lambda_n)$. Moreover, $L^2(\lambda_n)$ is Möbius-invariant and so the functions g in $L^2(\lambda_n)$ satisfy nothing better than the growth estimate $\int_{S_\zeta}|g|^2\,d\lambda_n\leqslant \|g\|^2_{L^2(\lambda_n)}$, while the functions g in $\mathcal{CM}(\mathbb{B}_n)$ satisfy the restrictive growth estimate

$$\int\limits_{S_\zeta}|g|^2\,d\lambda_n\leqslant \|g\|^2_{\mathcal{CM}(\mathbb{B}_n)}\big(1-|\zeta|\big)^n.$$

2.2. The Koszul complex

Here we briefly review the algebra behind the Koszul complex as presented for example in [7] in the finite dimensional setting. A more detailed treatment in that setting can be found in Section 5.5.3 of [10]. Fix h holomorphic as in (1.5). Now if $g=(g_j)_{j=1}^\infty$ satisfies $|g|^2=\sum_{j=1}^\infty|g_j|^2\geqslant\delta^2>0$, let

$$\Omega_0^1=\frac{\overline{g}}{|g|^2}=\left(\frac{\overline{g_j}}{|g|^2}\right)_{j=1}^\infty=\big(\Omega_0^1(j)\big)_{j=1}^\infty,$$

which we view as a 1-tensor (in $\ell^2=\mathbb{C}^\infty$) of $(0,0)$-forms with components $\Omega_0^1(j)=\frac{\overline{g_j}}{|g|^2}$. Then $f=\Omega_0^1h$ satisfies $\mathcal{M}_gf=f\cdot g=h$, but in general fails to be holomorphic. The Koszul complex provides a scheme which we now recall for solving a sequence of $\overline{\partial}$ equations that result in a correction term $\Lambda_g\Gamma_0^2$ that when subtracted from f above yields a holomorphic solution to the second line in (1.5). See below.

The 1-tensor of $(0,1)$-forms $\overline{\partial}\Omega_0=(\overline{\partial}\frac{\overline{g_j}}{|g|^2})_{j=1}^\infty=(\overline{\partial}\Omega_0^1(j))_{j=1}^\infty$ is given by

$$\overline{\partial}\Omega_0^1(j)=\overline{\partial}\frac{\overline{g_j}}{|g|^2}=\frac{|g|^2\overline{\partial g_j}-\overline{g_j}\overline{\partial}|g|^2}{|g|^4}=\frac{1}{|g|^4}\sum_{k=1}^\infty g_k\overline{\{g_k\partial g_j-g_j\partial g_k\}}$$

and can be written as

$$\overline{\partial}\Omega_0^1=\Lambda_g\Omega_1^2\equiv\left[\sum_{k=1}^\infty\Omega_1^2(j,k)g_k\right]_{j=1}^\infty,$$

where the antisymmetric 2-tensor Ω_1^2 of $(0,1)$-forms is given by

$$\Omega_1^2=\big[\Omega_1^2(j,k)\big]_{j,k=1}^\infty=\left[\frac{\overline{\{g_k\partial g_j-g_j\partial g_k\}}}{|g|^4}\right]_{j,k=1}^\infty$$

and $\Lambda_g\Omega_1^2$ denotes its contraction by the vector g in the final variable.

We can repeat this process and by induction we have

$$\bar{\partial}\Omega_q^{q+1} = \Lambda_g \Omega_{q+1}^{q+2}, \quad 0 \leqslant q \leqslant n, \tag{2.7}$$

where Ω_q^{q+1} is an alternating $(q+1)$-tensor of $(0,q)$-forms. Recall that h is holomorphic. When $q = n$ we have that $\Omega_n^{n+1} h$ is $\bar{\partial}$-closed and this allows us to solve a chain of $\bar{\partial}$ equations

$$\bar{\partial}\Gamma_{q-2}^{q} = \Omega_{q-1}^{q} h - \Lambda_g \Gamma_{q-1}^{q+1},$$

for alternating q-tensors Γ_{q-2}^{q} of $(0,q-2)$-forms, using the ameliorated Charpentier solution operators $\mathcal{C}_{n,s}^{0,q}$ defined in Theorem 4 below (note that our notation suppresses the dependence of Γ on h). With the convention that $\Gamma_n^{n+2} \equiv 0$ we have

$$\begin{aligned} \bar{\partial}\big(\Omega_q^{q+1} h - \Lambda_g \Gamma_q^{q+2}\big) &= 0, \quad 0 \leqslant q \leqslant n, \\ \bar{\partial}\Gamma_{q-1}^{q+1} &= \Omega_q^{q+1} h - \Lambda_g \Gamma_q^{q+2}, \quad 1 \leqslant q \leqslant n. \end{aligned} \tag{2.8}$$

Now

$$f \equiv \Omega_0^1 h - \Lambda_g \Gamma_0^2$$

is holomorphic by the first line in (2.8) with $q = 0$, and since Γ_0^2 is antisymmetric, we compute that $\Lambda_g \Gamma_0^2 \cdot g = \Gamma_0^2(g,g) = 0$ and

$$\mathcal{M}_g f = f \cdot g = \Omega_0^1 h \cdot g - \Lambda_g \Gamma_0^2 \cdot g = h - 0 = h.$$

Thus $f = (f_i)_{i=1}^{\infty}$ is a vector of holomorphic functions satisfying the second line in (1.5). The first line in (1.5) is the subject of the remaining sections of the paper.

2.2.1. Wedge products and factorization of the Koszul complex

Here we record the remarkable factorization of the Koszul complex in Andersson and Carlsson [1]. To describe the factorization we introduce an exterior algebra structure on $\ell^2 = \mathbb{C}^\infty$. Let $\{e_1, e_2, \ldots\}$ be the usual basis in $\mathbb{C}^\infty$, and for an increasing multi-index $I = (i_1, \ldots, i_\ell)$ of integers in $\mathbb{N}$, define

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell},$$

where we use $\wedge$ to denote the wedge product in the exterior algebra $\Lambda^*(\mathbb{C}^\infty)$ of $\mathbb{C}^\infty$, as well as for the wedge product on forms in $\mathbb{C}^n$. Note that $\{e_I\colon |I| = r\}$ is a basis for the alternating r-tensors on $\mathbb{C}^\infty$.

If $f = \sum_{|I|=r} f_I e_I$ is an alternating r-tensor on $\mathbb{C}^\infty$ with values that are $(0,k)$-forms in $\mathbb{C}^n$, which may be viewed as a member of the exterior algebra of $\mathbb{C}^\infty \otimes \mathbb{C}^n$, and if $g = \sum_{|J|=s} g_J e_J$ is an alternating s-tensor on $\mathbb{C}^\infty$ with values that are $(0,\ell)$-forms in $\mathbb{C}^n$, then as in [1] we define the wedge product $f \wedge g$ in the exterior algebra of $\mathbb{C}^\infty \otimes \mathbb{C}^n$ to be the alternating $(r+s)$-tensor on $\mathbb{C}^\infty$ with values that are $(0, k+\ell)$-forms in $\mathbb{C}^n$ given by

$$\begin{aligned} f \wedge g &= \left(\sum_{|I|=r} f_I e_I \right) \wedge \left(\sum_{|J|=s} g_J e_J \right) \\ &= \sum_{|I|=r, |J|=s} (f_I \wedge g_J)(e_I \wedge e_J) \\ &= \sum_{|K|=r+s} \left(\pm \sum_{I+J=K} f_I \wedge g_J \right) e_K. \end{aligned} \tag{2.9}$$

Note that we simply write the exterior product of an element from $\Lambda^*(\mathbb{C}^\infty)$ with an element from $\Lambda^*(\mathbb{C}^n)$ as juxtaposition, without writing an explicit wedge symbol. This should cause no confusion since the basis we use in $\Lambda^*(\mathbb{C}^\infty)$ is $\{e_i\}_{i=1}^\infty$, while the basis we use in $\Lambda^*(\mathbb{C}^n)$ is $\{dz_j, d\widehat{z_j}\}_{j=1}^n$, quite different in both appearance and interpretation.

In terms of this notation we then have the following factorization in Theorem 3.1 of Andersson and Carlsson [1]:

$$\Omega_0^1 \wedge \bigwedge_{i=1}^{\ell} \widetilde{\Omega_0^1} = \left(\sum_{k_0=1}^{\infty} \frac{\overline{g_{k_0}}}{|g|^2} e_{k_0} \right) \wedge \bigwedge_{i=1}^{\ell} \left(\sum_{k_i=1}^{\infty} \frac{\overline{\partial g_{k_i}}}{|g|^2} e_{k_i} \right) = -\frac{1}{\ell+1} \Omega_\ell^{\ell+1}, \tag{2.10}$$

where

$$\Omega_0^1 = \left(\frac{\overline{g_i}}{|g|^2} \right)_{i=1}^{\infty} \quad \text{and} \quad \widetilde{\Omega_0^1} = \left(\frac{\overline{\partial g_i}}{|g|^2} \right)_{i=1}^{\infty}.$$

The factorization in [1] is proved in the finite dimensional case, but this extends to the infinite dimensional case by continuity. Since the ℓ^2 norm is quasi-multiplicative on wedge products by Lemma 5.1 in [1] we have

$$\left| \Omega_\ell^{\ell+1} \right|^2 \leqslant C_\ell \left| \Omega_0^1 \right|^2 \left| \widetilde{\Omega_0^1} \right|^{2\ell}, \quad 0 \leqslant \ell \leqslant n, \tag{2.11}$$

where the constant C_ℓ depends only on the number of factors ℓ in the wedge product, and *not* on the underlying dimension of the vector space (which is infinite for $\ell^2 = \mathbb{C}^\infty$).

2.3. Charpentier's solution operators

In Theorem I.1 on p. 127 of [4], Charpentier proves the following formula for $(0, q)$-forms:

Theorem 2. *For $q \geqslant 0$ and all forms $f(\xi) \in C^1(\overline{\mathbb{B}_n})$ of degree $(0, q+1)$, we have for $z \in \mathbb{B}_n$:*

$$f(z) = C_q \int_{\mathbb{B}_n} \overline{\partial} f(\xi) \wedge \mathcal{C}_n^{0,q+1}(\xi, z) + c_q \overline{\partial}_z \left\{ \int_{\mathbb{B}_n} f(\xi) \wedge \mathcal{C}_n^{0,q}(\xi, z) \right\}.$$

Here $\mathcal{C}_n^{0,q}(\xi, z)$ is an $(n, n-q-1)$-form in ξ on the ball and a $(0, q)$-form in z on the ball that will be recalled below. Using Theorem 2, we can solve $\overline{\partial}_z u = f$ for a $\overline{\partial}$-closed $(0, q+1)$-form f

as follows. Set

$$u(z) \equiv c_q \int_{\mathbb{B}_n} f(\xi) \wedge \mathcal{C}_n^{0,q}(\xi, z).$$

Taking $\bar{\partial}_z$ of this we see from Theorem 2 and $\bar{\partial} f = 0$ that

$$\bar{\partial}_z u = c_q \bar{\partial}_z \left(\int_{\mathbb{B}_n} f(\xi) \wedge \mathcal{C}_n^{0,q}(\xi, z) \right) = f(z).$$

The actual structure of the kernels $\mathcal{C}_n^{0,q}(\xi, z)$ is very important for our proof. The case of $q = 0$ is given in [4], and additional properties of the kernels of general (p, q) were illustrated. In [5] we explicitly compute the kernels $\mathcal{C}_n^{0,q}(\xi, z)$. Before we give the structure of the kernels, first we introduce some notation.

Notation 1. Let $\omega_n(z) = \bigwedge_{j=1}^n dz_j$. For n a positive integer and $0 \leqslant q \leqslant n-1$ let P_n^q denote the collection of all permutations ν on $\{1, \dots, n\}$ that map to $\{i_\nu, J_\nu, L_\nu\}$ where J_ν is an increasing multi-index with $\operatorname{card}(J_\nu) = n - q - 1$ and $\operatorname{card}(L_\nu) = q$. Let $\epsilon_\nu \equiv \operatorname{sgn}(\nu) \in \{-1, 1\}$ denote the signature of the permutation ν.

Note that the number of increasing multi-indices of length $n - q - 1$ is $\frac{n!}{(q+1)!(n-q-1)!}$, while the number of increasing multi-indices of length q are $\frac{n!}{q!(n-q)!}$. Since we are only allowed certain combinations of J_ν and L_ν (they must have disjoint intersection and they must be increasing multi-indices), it is straightforward to see that the total number of permutations in P_n^q that we are considering is $\frac{n!}{(n-q-1)!q!}$.

Denote by $\Delta : \mathbb{C}^n \times \mathbb{C}^n \to [0, \infty)$ the map

$$\Delta(w, z) \equiv |1 - w\bar{z}|^2 - \left(1 - |w|^2\right)\left(1 - |z|^2\right).$$

We remark that it is possible to view $\Delta(z, w)$ in many other ways due to the symmetry of the unit ball $\mathbb{B}_n$. For example we will later use

$$\Delta(w, z) = |1 - w\bar{z}|^2 \left|\varphi_w(z)\right|^2. \tag{2.12}$$

See [5] for the additional representations of this function. It is convenient to isolate the following factor common to all summands in the formula:

$$\Phi_n^q(w, z) \equiv \frac{(1 - w\bar{z})^{n-1-q}(1 - |w|^2)^q}{\Delta(w, z)^n}, \quad 0 \leqslant q \leqslant n-1. \tag{2.13}$$

Theorem 3. *Let n be a positive integer and suppose that $0 \leqslant q \leqslant n-1$. Then*

$$\mathcal{C}_n^{0,q}(w, z) = \sum_{\nu \in P_n^q} (-1)^q \Phi_n^q(w, z) \operatorname{sgn}(\nu)(\overline{w_{i_\nu}} - \overline{z_{i_\nu}}) \bigwedge_{j \in J_\nu} d\overline{w_j} \bigwedge_{l \in L_\nu} d\overline{z_l} \bigwedge \omega_n(w). \tag{2.14}$$

The proof of this theorem is a long computation that can be found in [5].

We also need the following ameliorations of the Charpentier solution operators. These are obtained by treating the solution operators $\mathcal{C}_n^{0,q}(w,z)$ with $w, z \in \mathbb{C}^n$ as actually being a function with $w, z \in \mathbb{C}^s$ with $s > n$. One then can integrate out the extra variables to obtain the following result.

Theorem 4. *Suppose that $s > n$ and $0 \leqslant q \leqslant n-1$. Then we have*

$$\begin{aligned}\mathcal{C}_{n,s}^{0,q}(w,z) &= \mathcal{C}_n^{0,q}(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1} c_{j,n,s}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^j \\ &= \Phi_{n,s}^q(w,z)\sum_{|J|=q}\sum_{k\notin J}(-1)^{\mu(k,J)}(\overline{z_k}-\overline{w_k})\,d\bar{z}^J\wedge d\overline{w}^{(J\cup\{k\})^c}\wedge\omega_n(w).\end{aligned}$$

The interested reader can find this theorem in [5].

In order to establish appropriate inequalities for the Charpentier solution operators, we will need to control terms of the form $(\overline{z-w})^\alpha \frac{\partial^m}{\partial\overline{w}^\alpha}F(w)$, $D_z^m\Delta(w,z)$ and $D\{(1-\overline{w}z)^k\}$ inside the integral for T as given in the integration by parts formula in Lemma 4 below. We collect the necessary estimates in the following proposition.

Proposition 1. *For $z, w \in \mathbb{B}_n$ and $m \in \mathbb{N}$, we have the following three crucial estimates*:

$$\left|(\overline{z-w})^\alpha \frac{\partial^m}{\partial\overline{w}^\alpha}F(w)\right| \leqslant C\left(\frac{\sqrt{\Delta(w,z)}}{1-|w|^2}\right)^m \left|\overline{D}^m F(w)\right|, \quad m = |\alpha|. \tag{2.15}$$

$$\begin{aligned}\left|D_z\Delta(w,z)\right| &\leqslant C\left\{\left(1-|z|^2\right)\Delta(w,z)^{\frac{1}{2}} + \Delta(w,z)\right\}, \\ \left|\left(1-|z|^2\right)R\Delta(w,z)\right| &\leqslant C\left(1-|z|^2\right)\sqrt{\Delta(w,z)},\end{aligned} \tag{2.16}$$

$$\begin{aligned}\left|D_z^m\left\{(1-\overline{w}z)^k\right\}\right| &\leqslant C|1-\overline{w}z|^k\left(\frac{1-|z|^2}{|1-\overline{w}z|}\right)^{\frac{m}{2}}, \\ \left|\left(1-|z|^2\right)^m R^m\left\{(1-\overline{w}z)^k\right\}\right| &\leqslant C|1-\overline{w}z|^k\left(\frac{1-|z|^2}{|1-\overline{w}z|}\right)^m.\end{aligned} \tag{2.17}$$

Lemma 3. *Let $b > -1$. For $\Psi \in C(\overline{\mathbb{B}_n}) \cap C^\infty(\mathbb{B}_n)$ we have*

$$\int_{\mathbb{B}_n}\left(1-|w|^2\right)^b\Psi(w)\,dV(w) = \int_{\mathbb{B}_n}\left(1-|w|^2\right)^{b+m}R_b^m\Psi(w)\,dV(w).$$

When estimating the solution operators in the space $\mathcal{CM}(\mathbb{B}_n)$ the following lemma will play an important role.

Lemma 4. *Suppose that $s > n$ and $0 \leqslant q \leqslant n-1$. For all $m \geqslant 0$ and smooth $(0, q+1)$-forms η in $\overline{\mathbb{B}_n}$ we have the formula*

$$\mathcal{C}^{0,q}_{n,s}\eta(z)=\sum_{k=0}^{m-1}c'_{k,n,s}\mathcal{S}_{n,s}\big(\overline{\mathcal{D}}^k\eta\big)[\overline{\mathcal{Z}}](z)+\sum_{\ell=0}^{q}c_{\ell,n,s}\Phi^{\ell}_{n,s}\big(\overline{\mathcal{D}}^m\eta\big)(z), \tag{2.18}$$

where the ameliorated operators $\mathcal{S}_{n,s}$ and $\Phi^{\ell}_{n,s}$ have kernels given by

$$\mathcal{S}_{n,s}(w,z)=c_{n,s}\frac{(1-|w|^2)^{s-n-1}}{(1-\overline{w}z)^s}=c_{n,s}\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n-1}\frac{1}{(1-\overline{w}z)^{n+1}},$$

$$\Phi^{\ell}_{n,s}(w,z)=\Phi^{\ell}_{n}(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-\ell-1}c_{j,n,s}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\overline{z}|^2}\right)^{j}.$$

We have included these theorems so that this paper would be mostly self-contained.

Remark 2. The proof of Lemma 4 can be found in [5, pp. 18 and 19], and the proof for the case of the nonameliorated operators $\mathcal{S}_n$ and Φ^{ℓ}_n can be found on pp. 64–66. However, in the latter proof we only considered the two cases $m=0$ and 1. The reader can find the cases $m\geqslant 2$ treated in the *first* version of the paper [5] on the *arXiv* website.

3. Carleson measures and Schur's lemma

Key to the proof of Theorem 1 will be the knowledge that certain positive operators are bounded on $\mathcal{CM}(\mathbb{B}_n)$. In particular, these operators will be connected with the Charpentier solution operators.

Lemma 5. *Let $a,b,c\in\mathbb{R}$. Then the operator*

$$T_{a,b,c}h(z)=\int_{\mathbb{B}_n}\frac{(1-|z|^2)^a(1-|w|^2)^b(\sqrt{\Delta(w,z)})^c}{|1-w\overline{z}|^{n+1+a+b+c}}h(w)\,dV(w) \tag{3.1}$$

is bounded on $\mathcal{CM}(\mathbb{B}_n)$ if

$$c>-2n \quad \textit{and} \quad -2a<-n<2(b+1). \tag{3.2}$$

We remind the reader that in [5] it is shown that the operator $T_{a,b,c}$ is bounded on $L^2(\lambda_n)$ if and only if (3.2) holds. Note that since $T_{a,b,c}$ is a positive operator, Minkowski's inequality yields

$$\big|\big\{T_{a,b,c}h_i(z)\big\}_{i=1}^{\infty}\big|\leqslant T_{a,b,c}\big|\{h_i\}_{i=1}^{\infty}\big|(z),\quad z\in\mathbb{B}_n, \tag{3.3}$$

and it follows that the extension $T_{a,b,c}\{h_i\}_{i=1}^{\infty}=\{T_{a,b,c}h_i\}_{i=1}^{\infty}$ is bounded on

$$\mathcal{CM}\big(\mathbb{B}_n;\ell^2\big)=\big\{h\in L^2\big(\lambda_n;\ell^2\big)\colon |h|\in\mathcal{CM}(\mathbb{B}_n)\big\}$$

if and only if $T_{a,b,c}$ is bounded on $\mathcal{CM}(\mathbb{B}_n)$.

Proof. Fix $\zeta \in \mathbb{B}_n \setminus \{0\}$ and let

$$\delta = 1 - |\zeta| \quad \text{and} \quad N = \log_2\left(\frac{1}{1-|\zeta|}\right).$$

For $0 \leqslant k \leqslant N$ set

$$\zeta_k = \{1 - 2^k \delta\} \frac{\zeta}{|\zeta|} = \left\{\frac{1 - 2^k\delta}{1-\delta}\right\}\zeta.$$

Then $\zeta_0 = \zeta$ and ζ_k lies on the real line through ζ and is 2^k times as far from the boundary as is ζ: $1 - |\zeta_k| = 2^k \delta$. For a positive function $h \in \mathcal{CM}(\mathbb{B}_n)$ define

$$\begin{aligned} h_1 &\equiv \chi_{S_{\zeta_1}} h, \\ h_k &\equiv \chi_{S_{\zeta_k} \setminus S_{\zeta_{k-1}}} h, \quad 2 \leqslant k \leqslant N. \end{aligned}$$

Then

$$\left(\int\limits_{S_\zeta} |T_{a,b,c} h|^2 \, d\lambda_n\right)^{\frac{1}{2}} \leqslant C \|h\|_{\mathcal{CM}(\mathbb{B}_n)} + \sum_{k=1}^{N} \left(\int\limits_{S_\zeta} |T_{a,b,c} h_k|^2 \, d\lambda_n\right)^{\frac{1}{2}}.$$

Since $T_{a,b,c}$ is bounded on $L^2(\lambda_n)$ by [5], we have

$$\begin{aligned} \int\limits_{S_\zeta} |T_{a,b,c} h_k|^2 \, d\lambda_n &\leqslant C \int\limits_{\mathbb{B}_n} |h_k|^2 \, d\lambda_n = C \int\limits_{S_{\zeta_k}} |h|^2 \, d\lambda \\ &\leqslant C \|h\|^2_{\mathcal{CM}(\mathbb{B}_n)} \big(1 - |\zeta_k|\big)^n \\ &= C 2^{kn} \|h\|^2_{\mathcal{CM}(\mathbb{B}_n)} \big(1 - |\zeta|\big)^n, \end{aligned}$$

which is an adequate estimate for k bounded. For k large we claim that $z \in S_\zeta$ and $w \in S_{\zeta_k} \setminus S_{\zeta_{k-1}}$ imply

$$|1 - w\bar{z}| \approx 2^k \delta \quad \text{and} \quad \big|\varphi_w(z)\big| \approx 1 \quad \text{and} \quad \sqrt{\Delta(w,z)} \approx 2^k \delta. \tag{3.4}$$

The second equivalence is obvious by Möbius invariance, and the third equivalence follows from the first two and the formula for $\Delta(w, z)$ in (2.12).

We now prove the first equivalence in (3.4), for which it suffices to prove that

$$c 2^k \delta \leqslant |1 - w\bar{\zeta}| \leqslant C 2^k \delta. \tag{3.5}$$

Indeed, since $d(w, z) = |1 - w\bar{z}|^{\frac{1}{2}}$ satisfies the triangle inequality on the ball $\mathbb{B}_n$, and since $|1 - z\bar{\zeta}| < 2\delta$ by (2.1), we have from (3.5) that

$$|1 - w\bar{z}|^{\frac{1}{2}} \leqslant \sqrt{2\delta} + \sqrt{C2^k\delta} \leqslant \sqrt{C2^{k+2}\delta},$$
$$|1 - w\bar{z}|^{\frac{1}{2}} \geqslant \sqrt{c2^k\delta} - \sqrt{2\delta} \geqslant \sqrt{c2^{k-2}\delta},$$

for k large enough.

So to complete the proof of the claim (3.4), we must demonstrate (3.5). However, (3.5) clearly holds for ζ bounded away from the boundary $\partial\mathbb{B}_n$, and so we may suppose that $0 < \delta \leqslant \frac{1}{4}$. Now from (2.1) we have

$$\frac{1 - |\zeta_k|}{|1 - \overline{\zeta_k} w|} > \frac{1}{2} \quad \text{and} \quad \frac{1 - |\zeta_{k-1}|}{|1 - \overline{\zeta_{k-1}} w|} \leqslant \frac{1}{2},$$

which yields

$$\left|1 - \frac{1 - 2^k\delta}{1 - \delta}\bar{\zeta} w\right| < 2^{k+1}\delta \quad \text{and} \quad \left|1 - \frac{1 - 2^{k-1}\delta}{1 - \delta}\bar{\zeta} w\right| \geqslant 2^k\delta.$$

Now we conclude from the Euclidean triangle inequality that

$$\begin{aligned} |1 - \bar{\zeta} w| &\leqslant \left|1 - \frac{1 - 2^k\delta}{1 - \delta}\bar{\zeta} w\right| + \left(1 - \frac{1 - 2^k\delta}{1 - \delta}\right)|\bar{\zeta} w| \\ &\leqslant 2^{k+1}\delta + \frac{(2^k - 1)\delta}{1 - \delta} \leqslant \left(2^{k+1} + \frac{(2^k - 1)}{\frac{3}{4}}\right)\delta \leqslant 2^{k+2}\delta, \end{aligned}$$

as well as

$$\begin{aligned} |1 - \bar{\zeta} w| &\geqslant \left|1 - \frac{1 - 2^{k-1}\delta}{1 - \delta}\bar{\zeta} w\right| - \left(1 - \frac{1 - 2^{k-1}\delta}{1 - \delta}\right)|\bar{\zeta} w| \\ &\geqslant 2^k\delta - \frac{(2^{k-1} - 1)\delta}{1 - \delta} \geqslant \left(2^k - \frac{(2^{k-1} - 1)}{\frac{3}{4}}\right)\delta \geqslant 2^{k-2}\delta, \end{aligned}$$

provided $0 < \delta \leqslant \frac{1}{4}$. This completes the proof of (3.5), and hence (3.4).

We thus have for k large enough, say $k \geqslant K$,

$$\begin{aligned} &\int\limits_{S_\zeta} |T_{a,b,c} h_k|^2 \, d\lambda_n \\ &\quad = \int\limits_{S_\zeta} \left| \int\limits_{S_{\zeta_k} \setminus S_{\zeta_{k-1}}} \frac{(1 - |z|^2)^a (1 - |w|^2)^b \Delta(w, z)^{\frac{c}{2}}}{|1 - w\bar{z}|^{n+1+a+b} + c} h(w) \, dV(w) \right|^2 d\lambda_n(z) \\ &\quad \leqslant C \int\limits_{S_\zeta} \left| \int\limits_{S_{\zeta_k} \setminus S_{\zeta_{k-1}}} \frac{(1 - |z|^2)^a (1 - |w|^2)^b (2^k\delta)^c}{(2^k\delta)^{n+1+a+b+c}} h(w) \, dV(w) \right|^2 d\lambda_n(z) \end{aligned}$$

$$\leqslant C\left(2^k\delta\right)^{-2(n+1+a+b)}\left\{\int\limits_{S_\zeta}\left(1-|z|^2\right)^{2a-n-1}dV(z)\right\}$$
$$\times\left(\int\limits_{S_{\zeta_k}\setminus S_{\zeta_{k-1}}}\left(1-|w|^2\right)^{b+n+1}h(w)\,d\lambda_n(w)\right)^2.$$

Now by Hölder's inequality,

$$\left(\int\limits_{S_{\zeta_k}\setminus S_{\zeta_{k-1}}}\left(1-|w|^2\right)^{b+n+1}h(w)\,d\lambda_n(w)\right)^2$$
$$\leqslant\left(\int\limits_{S_{\zeta_k}\setminus S_{\zeta_{k-1}}}\left(1-|w|^2\right)^{2(b+n+1)}d\lambda_n(w)\right)\left(\int\limits_{S_{\zeta_k}\setminus S_{\zeta_{k-1}}}\left|h(w)\right|^2d\lambda_n(w)\right)$$
$$\leqslant\left(\int\limits_{S_{\zeta_k}\setminus S_{\zeta_{k-1}}}\left(1-|w|^2\right)^{2(b+n+1)}d\lambda_n(w)\right)\|h\|^2_{\mathcal{CM}(\mathbb{B}_n)}\left(2^k\delta\right)^n$$
$$\leqslant C\left(2^k\delta\right)^{2(b+n+1)}\|h\|^2_{\mathcal{CM}(\mathbb{B}_n)}\left(2^k\delta\right)^n$$
$$=C\left(2^k\delta\right)^{2b+3n+2}\|h\|^2_{\mathcal{CM}(\mathbb{B}_n)},$$

provided $2(b+n+1)>n$, i.e. $2(b+1)>-n$. Indeed, to obtain the estimate

$$\int\limits_{S_{\zeta_k}\setminus S_{\zeta_{k-1}}}\left(1-|w|^2\right)^{2(b+n+1)}d\lambda_n(w)\leqslant C\left(2^k\delta\right)^{2(b+n+1)},$$

we decompose the annulus $S_{\zeta_k}\setminus S_{\zeta_{k-1}}$ into a union of unit radius Bergman balls B_j^ℓ whose Euclidean distance from the boundary is approximately $2^{-\ell}2^k\delta$:

$$S_{\zeta_k}\setminus S_{\zeta_{k-1}}=\bigcup_{\ell=0}^{\infty}\bigcup_{j=1}^{A_\ell}B_j^\ell.$$

Since

$$A_\ell\left(2^{-\ell}2^k\delta\right)^n=\sum_{j=1}^{A_\ell}\frac{|B_j^\ell|}{2^{-\ell}2^k\delta}\leqslant\frac{|S_{\zeta_k}|}{2^k\delta}=\left(2^k\delta\right)^n,$$

we have the estimate $A_\ell\leqslant 2^{\ell n}$. Thus we compute that

$$\int_{S_{\zeta_k}\setminus S_{\zeta_{k-1}}} \left(1-|w|^2\right)^{2(b+n+1)} d\lambda_n(w)$$

$$= \sum_{\ell=0}^{\infty}\sum_{j=1}^{A_\ell}\int_{B_j^\ell}\left(1-|w|^2\right)^{2(b+n+1)} d\lambda_n(w) \approx \sum_{\ell=0}^{\infty}\sum_{j=1}^{A_\ell}\left(2^{-\ell}2^k\delta\right)^{2(b+n+1)}$$

$$\leqslant \left(2^k\delta\right)^{2(b+n+1)}\sum_{\ell=0}^{\infty}\left(2^{-\ell}\right)^{2(b+n+1)}2^{\ell n} \leqslant C\left(2^k\delta\right)^{2(b+n+1)},$$

since $2(b+n+1) > n$.

We also compute that

$$\int_{S_\zeta}\left(1-|z|^2\right)^{2a-n-1} dV(z) \leqslant C\delta^n\int_0^\delta t^{2a-n-1}\,dt \leqslant C\delta^n\delta^{2a-n} = C\delta^{2a},$$

so that altogether we have

$$\int_{S_\zeta}|T_{a,b,c}h_k|^2\,d\lambda_n \leqslant C\left(2^k\delta\right)^{-2(n+1+a+b)}\left\{\delta^{2a}\right\}\left(2^k\delta\right)^{2b+3n+2}\|h\|^2_{\mathcal{CM}(\mathbb{B}_n)}$$

$$= C\|h\|^2_{\mathcal{CM}(\mathbb{B}_n)}2^{k(n-2a)}\delta^n.$$

Summing we obtain

$$\left(\int_{S_\zeta}|T_{a,b,c}h|^2\,d\lambda_n\right)^{\frac{1}{2}} \leqslant \sum_{k=1}^{\infty}\left(\int_{S_\zeta}|T_{a,b,c}h_k|^2\,d\lambda_n\right)^{\frac{1}{2}}$$

$$\leqslant C2^{Kn}\|h\|_{\mathcal{CM}(\mathbb{B}_n)}\left(1-|\zeta|\right)^{\frac{n}{2}} + C\|h\|_{\mathcal{CM}(\mathbb{B}_n)}\sum_{k=K}^{\infty}2^{k(\frac{n}{2}-a)}\delta^{\frac{n}{2}}$$

$$\leqslant C\|h\|_{\mathcal{CM}(\mathbb{B}_n)}\left(1-|\zeta|\right)^{\frac{n}{2}}$$

provided $a > \frac{n}{2}$. The two requirements $2(b+1) > -n$ and $a > \frac{n}{2}$ are precisely the conditions on a and b in (3.2), and this completes the proof of Lemma 5. □

4. Proof of Theorem 1

To prove Theorem 1 we follow the argument in [5]. We obtain from the Koszul complex a function $f = \Omega_0^1 h - \Lambda_g\Gamma_0^2 \in H(\mathbb{B}_n)$ that solves (1.5) where Γ_0^2 is an antisymmetric 2-tensor of $(0,0)$-forms that solves

$$\overline{\partial}\Gamma_0^2 = \Omega_1^2 h - \Lambda_g\Gamma_1^3,$$

and inductively where Γ_q^{q+2} is an alternating $(q+2)$-tensor of $(0,q)$-forms that solves

$$\bar\partial \Gamma_q^{q+2} = \Omega_{q+1}^{q+2} h - \Lambda_g \Gamma_{q+1}^{q+3},$$

up to $q = n-1$ (since $\Gamma_n^{n+2} = 0$ and the $(0,n)$-form Ω_n^{n+1} is $\bar\partial$-closed). Using the Charpentier solution operators $\mathcal{C}_{n,s}^{0,q}$ on $(0,q+1)$-forms we have

$$\begin{aligned} f &= \Omega_0^1 h - \Lambda_g \Gamma_0^2 \\ &= \Omega_0^1 h - \Lambda_g \mathcal{C}_{n,s_1}^{0,0} \Omega_1^2 h + \Lambda_g \mathcal{C}_{n,s_1}^{0,0} \Lambda_g \mathcal{C}_{n,s_2}^{0,1} \Omega_2^3 h - \Lambda_g \mathcal{C}_{n,s_1}^{0,0} \Lambda_g \mathcal{C}_{n,s_2}^{0,1} \Lambda_g \mathcal{C}_{n,s_3}^{0,2} \Omega_3^4 h + \cdots \\ &\quad + (-1)^n \Lambda_g \mathcal{C}_{n,s_1}^{0,0} \ldots \Lambda_g \mathcal{C}_{n,s_n}^{0,n-1} \Omega_n^{n+1} h \\ &\equiv \mathcal{F}^0 + \mathcal{F}^1 + \cdots + \mathcal{F}^n. \end{aligned} \tag{4.1}$$

The goal is then to establish

$$f = \Omega_0^1 h - \Lambda_g \Gamma_0^2 \in BMOA(\mathbb{B}_n; \ell^2),$$

which we accomplish, through an application of Lemmas 1 and 2, by showing that for $m = 1$,

$$\left\| \left(1-|z|^2\right)^{\frac{n}{2}} \left(\left(1-|z|^2\right) \frac{\partial}{\partial z}\right) \mathcal{F}^\mu(z) \right\|_{\mathcal{CM}(\mathbb{B}_n;\ell^2)} \leqslant C_{n,\delta}(g) \|h\|_{H^\infty(\mathbb{B}_n)}, \quad 0 \leqslant \mu \leqslant n. \tag{4.2}$$

It is useful at this point to recall the analogous inequality from [5] with $L^2(\lambda_n;\ell^2)$ and $H^2(\mathbb{B}_n)$ in place of $\mathcal{CM}(\mathbb{B}_n;\ell^2)$ and $H^\infty(\mathbb{B}_n)$ respectively:

$$\left\| \left(1-|z|^2\right)^{\frac{n}{2}} \left(\left(1-|z|^2\right) \frac{\partial}{\partial z}\right) \mathcal{F}^\mu(z) \right\|_{L^2(\lambda_n;\ell^2)} \leqslant C_{n,\delta}(g) \|h\|_{H^2(\mathbb{B}_n)}, \quad 0 \leqslant \mu \leqslant n.$$

In [5] we constructed integers $1 = m_0 < m_1 < m_2 < \cdots < m_n$ and used (3.3) and the boundedness of the operators $T_{a,b,c}$ on $L^2(\lambda_n)$ for a, b, c satisfying (3.2) in order to prove

$$\left\| \left(1-|z|^2\right)^{\frac{n}{2}+1} \frac{\partial}{\partial z} \mathcal{F}^\mu(z) \right\|_{L^2(\lambda_n;\ell^2)} \leqslant C_{n,\delta} \left\| \left(1-|z|^2\right)^{\frac{n}{2}} \mathcal{X}^{m_\mu} \left(\widehat{\Omega_\mu^{\mu+1}} h\right)(z) \right\|_{L^2(\lambda_n;\ell^2)}, \tag{4.3}$$

where $\widehat{\Omega_\mu^{\mu+1}} = \Omega_0^1 \wedge \bigwedge_{i=1}^\mu \widehat{\Omega_0^1}$ and $\widehat{\Omega_0^1} = (\frac{\overline{Dg_i}}{|g|^2})_{i=1}^\infty$. Recall from (2.10) that $\Omega_\mu^{\mu+1} = \Omega_0^1 \wedge \bigwedge_{i=1}^\mu \widetilde{\Omega_0^1}$ where $\widetilde{\Omega_0^1} = (\frac{\overline{\partial g_i}}{|g|^2})_{i=1}^\infty$, and so the form $\widehat{\Omega_\mu^{\mu+1}}$ is obtained from $\Omega_\mu^{\mu+1}$ by replacing each occurrence of ∂ with D. We then went on to prove in (8.10) of [5] that

$$\left\| \left(1-|z|^2\right)^{\frac{n}{2}} \mathcal{X}^{m_\mu} \left(\widehat{\Omega_\mu^{\mu+1}} h\right)(z) \right\|_{L^2(\lambda_n;\ell^2)} \leqslant C_{n,\delta} \|g\|_{H^\infty(\mathbb{B}_n;\ell^2)}^{m_\mu+\mu} \|h\|_{H^2(\mathbb{B}_n)}, \tag{4.4}$$

using the multilinear inequality in Lemma 2.

We can now prove

$$\left\| \left(1-|z|^2\right)^{\frac{n}{2}+1} \frac{\partial}{\partial z} \mathcal{X}^{m_0} \mathcal{F}^{\mu}(z) \right\|_{\mathcal{CM}(\mathbb{B}_n;\ell^2)} \leqslant C_{n,\delta} \left\| \left(1-|z|^2\right)^{\frac{n}{2}} \mathcal{X}^{m_\mu} \left(\widehat{\Omega_\mu^{\mu+1}} h\right)(z) \right\|_{\mathcal{CM}(\mathbb{B}_n;\ell^2)} \tag{4.5}$$

by following *verbatim* the argument in [5] used to prove (4.3), but using the boundedness of $T_{a,b,c}$ on $\mathcal{CM}(\mathbb{B}_n;\ell^2)$ rather than on $L^2(\lambda_n;\ell^2)$. Recall from (3.3) that the boundedness of $T_{a,b,c}$ on $\mathcal{CM}(\mathbb{B}_n;\ell^2)$ is equivalent to the boundedness of $T_{a,b,c}$ on the scalar space $\mathcal{CM}(\mathbb{B}_n)$. The routine verification of these assertions are left to the reader.

Finally, we prove

$$\left\| \left(1-|z|^2\right)^{\frac{n}{2}} \mathcal{X}^{m_\mu} \left(\widehat{\Omega_\mu^{\mu+1}} h\right)(z) \right\|_{\mathcal{CM}(\mathbb{B}_n;\ell^2)} \leqslant C_{n,\delta} \|g\|_{H^\infty(\mathbb{B}_n;\ell^2)}^{m_\mu+\mu} \|h\|_{H^\infty(\mathbb{B}_n)}, \tag{4.6}$$

using Lemma 2 and a slight variant of the argument used to prove (8.10) in [5]. Following the argument at the top of page 41 in [5], the Liebniz formula yields

$$\begin{aligned} \mathcal{X}^m \left(\widehat{\Omega_\mu^{\mu+1}} h\right) &= \mathcal{X}^m \left(\Omega_0^1 \wedge \left(\widehat{\Omega_0^1}\right)^\mu h\right) \\ &= \sum_{\alpha \in \mathbb{Z}_+^{\mu+2}:\ |\alpha|=m} \left(\mathcal{X}^{\alpha_0} \Omega_0^1\right) \wedge \bigwedge_{j=1}^{\mu} \left(\mathcal{X}^{\alpha_j} \widehat{\Omega_0^1}\right) \left(\mathcal{X}^{\alpha_{\mu+1}} h\right). \end{aligned}$$

Since $d\nu$ is a Carleson measure if and only if

$$\int_{\mathbb{B}_n} \left|\varphi(z)\right|^2 d\nu(z) \leqslant C \|\varphi\|_{H^2(\mathbb{B}_n)}^2, \quad \varphi \in H^2(\mathbb{B}_n),$$

it thus suffices to show that

$$\begin{aligned} &\int_{\mathbb{B}_n} \left(1-|z|^2\right)^n \left| \left(\mathcal{X}^{\alpha_0} \Omega_0^1\right) \wedge \bigwedge_{j=1}^{\mu} \left(\mathcal{X}^{\alpha_j+1} \Omega_0^1\right) \right|^2 \left|\mathcal{X}^{\alpha_{\mu+1}} h\right|^2 \left|\varphi(z)\right|^2 \\ &\quad \leqslant C_{n,\delta} \|g\|_{H^\infty(\mathbb{B}_n;\ell^2)}^{2(m_\mu+\mu)} \|h\|_{H^\infty(\mathbb{B}_n)}^2 \|\varphi\|_{H^2(\mathbb{B}_n)}^2, \end{aligned}$$

for all $\varphi \in H^2(\mathbb{B}_n)$.

Now we recall (8.12) and (8.14) from [5]:

$$\mathcal{X}^k\left(\Omega_0^1\right) = \mathcal{X}^k\left(\frac{\overline{g}}{|g|^2}\right) = \sum_{\ell=0}^{k} c_\ell \left(\mathcal{X}^{k-\ell} \overline{g}\right)\left(\mathcal{X}^\ell |g|^{-2}\right), \tag{4.7}$$

and

$$\left|\mathcal{X}^{\ell}|g|^{-2}\right|^2 \leqslant \sum_{1\leqslant\alpha_1\leqslant\alpha_2\leqslant\cdots\leqslant\alpha_M:\ \alpha_1+\alpha_2+\cdots+\alpha_M=\ell} c_\alpha |g|^{-4-2\ell} \prod_{m=1}^{M}\left(\sum_{i=1}^{\infty}\left|\mathcal{X}^{\alpha_m}\overline{g_i}\right|^2\right). \tag{4.8}$$

Thus we see that it suffices to prove

$$\int_{\mathbb{B}_n}\left(1-|z|^2\right)^n\left|\left(\mathcal{Y}^{\alpha_1}g\right)(z)\right|^2\ldots\left|\left(\mathcal{Y}^{\alpha_M}g\right)(z)\right|^2\left|\left(\mathcal{Y}^{\alpha_0}h\right)(z)\right|^2\left|\varphi(z)\right|^2 d\lambda_n(z)$$
$$\leqslant C_{n,\delta}\|g\|^{2M}_{H^\infty(\mathbb{B}_n;\ell^2)}\|h\|^2_{H^\infty(\mathbb{B}_n)}\|\varphi\|^2_{H^2(\mathbb{B}_n)},$$

and this latter inequality follows easily from Lemma 2 with appropriate choices of function. Altogether this yields (4.2) and completes the proof of Theorem 1.

Remark 3. We comment briefly on how we obtain *BMO*, as opposed to $H^\infty \cdot BMOA$, estimates for solutions to the Bezout equation (1.3). In [1] Andersson and Carlsson obtain $H^\infty \cdot BMOA$ estimates for solutions f to (1.2) with constants independent of dimension by establishing inequalities of the form

$$\left\|K\left\{(\Lambda_g K)^{\mu-1}\Omega_\mu^{\mu+1}h\right\}\right\|_{BMOA} \leqslant C\left\|\left(1-|z|^2\right)^{-\frac{1}{2}}(\Lambda_g K)^{\mu-1}\Omega_n^{n+1}h\right\|_{\mathcal{CM}}, \tag{4.9}$$

and

$$\left\|\left(1-|z|^2\right)^{\frac{k-1}{2}}(\Lambda_g K)^{\mu-1-k}\Omega_n^{n+1}h\right\|_{\mathcal{CM}} \leqslant C\left\|\left(1-|z|^2\right)^{\frac{k}{2}}(\Lambda_g K)^{\mu-2-k}\Omega_n^{n+1}h\right\|_{\mathcal{CM}}, \tag{4.10}$$

and finally

$$\left\|\left(1-|z|^2\right)^{\frac{\mu}{2}-1}\Omega_n^{n+1}h\right\|_{\mathcal{CM}} \leqslant C(g)\|h\|^2_{H^\infty}, \tag{4.11}$$

where K is a certain solution operator to the $\overline{\partial}$-equation on strictly pseudoconvex domains (see (4.6), (4.5) and (5.3) in [1]). Iterating these inequalities yields

$$\left\|K\left\{(\Lambda_g K)^{\mu-1}\Omega_\mu^{\mu+1}h\right\}\right\|_{BMOA} \leqslant C(g)\|h\|^2_{H^\infty},$$

and then applying the final contraction Λ_g results in the $H^\infty \cdot BMOA$ estimate. These methods yield the best known estimates in terms of the positive parameter δ in (1.4), and yield estimates independent of the number of generators N.

On the other hand, after using Lemma 1 to reduce matters to

$$\left\|\left(1-|z|^2\right)^{\frac{n}{2}+1}\frac{\partial}{\partial z}\left\{\Lambda_g\mathcal{C}^{0,0}_{n,s_1}\ldots\Lambda_g\mathcal{C}^{0,\mu-1}_{\mu,s_\mu}\Omega_\mu^{\mu+1}h\right\}(z)\right\|_{\mathcal{CM}} \leqslant C(g)\|h\|^2_{H^\infty},$$

we follow [5] in using the integration by parts in Lemma 4 together with the estimates in Proposition 1 to reduce matters to an inequality of the form

$$\left\|T_1T_2\ldots T_\mu\left\{\left(1-|z|^2\right)^{\frac{n}{2}}\mathcal{X}^m\left(\widehat{\Omega_\mu^{\mu+1}h}\right)\right\}\right\|_{\mathcal{CM}} \leqslant C(g)\|h\|^2_{H^\infty},$$

where the T_i are operators of the type $T_{a,b,c}$ in (3.1), and where $\widehat{\Omega_\mu^{\mu+1}}$ is the form obtained from $\Omega_\mu^{\mu+1}$ by replacing each occurrence of ∂ with the "larger" D. Then Lemma 5 reduces matters to proving

$$\left\| \left(1-|z|^2\right)^{\frac{n}{2}} \mathcal{X}^m \left(\widehat{\Omega_\mu^{\mu+1}} h\right) \right\|_{\mathcal{CM}} \leqslant C(g) \|h\|_{H^\infty}^2,$$

which finally follows from the multilinear inequality in Lemma 2. It is in this way that we avoid having to multiply a *BMOA* solution by a bounded holomorphic function.

5. A generalization

In [5] the corona theorem (including the semi-infinite matrix case) was established for the multiplier algebras $M_{B_p^\sigma(\mathbb{B}_n)}$ of $B_p^\sigma(\mathbb{B}_n)$ when $p=2$ and $0 \leqslant \sigma \leqslant \frac{1}{2}$. However, the Corona problem remains open for all of the remaining multiplier algebras $M_{B_p^\sigma(\mathbb{B}_n)}$. In this section we consider two weaker assertions and prove the weakest one. Recall that $B_p^\sigma(\mathbb{B}_n;\ell^2)$ can be characterized as consisting of those $f \in H(\mathbb{B}_n;\ell^2)$ such that

$$d\mu_f(z) \equiv \left|\left(1-|z|^2\right)^\sigma \mathcal{Y}^m f(z)\right|^p d\lambda_n(z)$$

is a finite measure. Moreover, Proposition 3 in [5] shows that the multiplier space $M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ satisfies the containment

$$\begin{aligned} &M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)} \\ &\quad \subset \left\{ f \in H^\infty\left(\mathbb{B}_n;\ell^2\right):\ \mu_f \text{ is a Carleson measure for } B_p^\sigma(\mathbb{B}_n)\right\}. \end{aligned} \tag{5.1}$$

Remark 4. Theorem 3.7 of [9] shows that equality actually holds in the scalar-valued version of (5.1). The argument there can be extended to prove equality in (5.1) itself, but as we do not need this result, we do not pursue it further here.

We can rewrite (5.1) in the more convenient form

$$M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)} \subset H^\infty\left(\mathbb{B}_n;\ell^2\right) \cap X_p^\sigma\left(\mathbb{B}_n;\ell^2\right),$$

where

$$X_p^\sigma\left(\mathbb{B}_n;\ell^2\right) = \left\{ f \in B_p^\sigma\left(\mathbb{B}_n;\ell^2\right):\ \mu_f \text{ is a Carleson measure for } B_p^\sigma(\mathbb{B}_n)\right\}$$

is normed by

$$\|f\|_{X_p^\sigma(\mathbb{B}_n;\ell^2)}^p = \sup_{\varphi \in B_p^\sigma(\mathbb{B}_n)} \frac{\int_{\mathbb{B}_n} |\varphi(z)|^p \, d\mu_f(z)}{\|\varphi\|_{B_p^\sigma(\mathbb{B}_n)}^p}.$$

Conjecture 1. *Given* $g \in M_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ *satisfying*

$$\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)} \leqslant 1,$$
$$\sum_{j=1}^{\infty} \left|g_j(z)\right|^2 \geqslant \delta^2 > 0, \quad z \in \mathbb{B}_n,$$

and $h \in M_{B_p^\sigma(\mathbb{B}_n)}$, *there is a vector-valued function* $f \in X_p^\sigma(\mathbb{B}_n;\ell^2)$ *such that*

$$\|f\|_{X_p^\sigma(\mathbb{B}_n;\ell^2)} \leqslant C_{n,\sigma,p,\delta} \|h\|_{M_{B_p^\sigma(\mathbb{B}_n)}},$$
$$\sum_{j=1}^{N} f_j(z) g_j(z) = h(z), \quad z \in \mathbb{B}_n.$$

While we are unable to settle this conjecture here, we can prove the weaker conjecture obtained by relaxing the Carleson measure condition slightly in the definition of the space X_p^σ. We say that a positive measure μ is a *weak* Carleson measure for $B_p^\sigma(\mathbb{B}_n)$ if

$$\sup_{\zeta \in \mathbb{B}_n} \frac{\int_{S_\zeta} d\mu(z)}{(1-|\zeta|^2)^{p\sigma}} < \infty.$$

Note that a Carleson measure μ for $B_p^\sigma(\mathbb{B}_n)$ is automatically a weak Carleson measure for $B_p^\sigma(\mathbb{B}_n)$. This can be seen by testing the embedding for μ over reproducing kernels. Let

$$WX_p^\sigma(\mathbb{B}_n;\ell^2) = \left\{ f \in B_p^\sigma(\mathbb{B}_n;\ell^2) : \mu_f \text{ is a weak Carleson measure for } B_p^\sigma(\mathbb{B}_n) \right\}$$

be normed by

$$\|f\|^p_{WX_p^\sigma(\mathbb{B}_n;\ell^2)} = \sup_{\zeta \in \mathbb{B}_n} \frac{\int_{S_\zeta} d\mu_f(z)}{(1-|\zeta|^2)^{p\sigma}}.$$

Theorem 5. *Given* $g \in M_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ *satisfying*

$$\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)} \leqslant 1,$$
$$\sum_{j=1}^{\infty} \left|g_j(z)\right|^2 \geqslant \delta^2 > 0, \quad z \in \mathbb{B}_n,$$

and $h \in M_{B_p^\sigma(\mathbb{B}_n)}$, *there is a vector-valued function* $f \in WX_p^\sigma(\mathbb{B}_n;\ell^2)$ *such that*

$$\|f\|_{WX_p^\sigma(\mathbb{B}_n;\ell^2)} \leqslant C_{n,\sigma,p,\delta} \|h\|_{M_{B_p^\sigma(\mathbb{B}_n)}},$$
$$\sum_{j=1}^{N} f_j(z) g_j(z) = h(z), \quad z \in \mathbb{B}_n.$$

In order to prove Theorem 5 we introduce, in analogy with $\mathcal{CM}(\mathbb{B}_n)$, a Banach space $\mathcal{WCM}_p^\sigma(\mathbb{B}_n)$ of measurable functions h on the ball $\mathbb{B}_n$ such that $|h(z)|^p \, d\lambda_n(z)$ is a weak Carleson measure for $B_p^\sigma(\mathbb{B}_n)$;

$$\mathcal{WCM}_p^\sigma(\mathbb{B}_n) = \left\{ h\colon \sup_{\zeta \in \mathbb{B}_n} \frac{\int_{S_\zeta} |h(z)|^p \, d\lambda_n(z)}{(1-|\zeta|^2)^{p\sigma}} < \infty \right\}.$$

Note that

$$WX_p^\sigma\big(\mathbb{B}_n; \ell^2\big) = \big\{ f \in H\big(\mathbb{B}_n; \ell^2\big)\colon \big(1-|z|^2\big)^\sigma \big|\mathcal{Y}^m f(z)\big| \in \mathcal{WCM}_p^\sigma(\mathbb{B}_n) \big\}.$$

Using the argument above we will see below that Theorem 1 follows from:

Lemma 6. *Let $a, b, c \in \mathbb{R}$, $1 < p < \infty$ and $\sigma \geqslant 0$. Then the operator $T_{a,b,c}$ defined by*

$$T_{a,b,c} h(z) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^a (1-|w|^2)^b (\sqrt{\Delta(w,z)})^c}{|1-w\bar{z}|^{n+1+a+b+c}} h(w) \, dV(w)$$

is bounded on $\mathcal{WCM}_p^\sigma(\mathbb{B}_n)$ if

$$c > -2n \quad \text{and} \quad -pa < -n < p(b+1).$$

In Lemma 5 above we proved that Lemma 6 holds in the special case $\sigma = \frac{n}{2}$ and $p = 2$ by exploiting the characterization of Carleson measures for $H^2(\mathbb{B}_n)$ as being precisely the weak Carleson measures. Thus the proof of Lemma 5 above also applies to prove Lemma 6. The straightforward verification is left to the reader.

We will also need the following slight generalization of Proposition 3 in [5].

Proposition 2. *Suppose that $1 < p < \infty$, $0 \leqslant \sigma < \infty$, $M \geqslant 1$, $m > 2(\frac{n}{p} - \sigma)$ and $\alpha = (\alpha_0, \dots, \alpha_M) \in \mathbb{Z}_+^{M+1}$ with $|\alpha| = m$. For $g_1, \dots, g_M \in M_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n; \ell^2)}$ and $h \in B_p^\sigma(\mathbb{B}_n)$ we have*

$$\int_{\mathbb{B}_n} \big(1-|z|^2\big)^{p\sigma} \big|\big(\mathcal{Y}^{\alpha_1} g_1\big)(z)\big|^p \dots \big|\big(\mathcal{Y}^{\alpha_M} g_M\big)(z)\big|^p \big|\big(\mathcal{Y}^{\alpha_0} h\big)(z)\big|^p \, d\lambda_n(z)$$

$$\leqslant C_{n,M,\sigma,p} \left(\prod_{j=1}^M \|\mathbb{M}_{g_j}\|^p_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n; \ell^2)} \right) \|h\|^p_{B_p^\sigma(\mathbb{B}_n)}.$$

Proposition 3 in [5] proves the case $g_1 = \cdots = g_M$ and the proof given there carries over immediately to the case of different g_j here.

Now we combine Lemma 6 and Proposition 2 to obtain Theorem 5. Note that (3.3) again shows that boundedness of $T_{a,b,c}$ on the vector-valued version $\mathcal{CM}_p^\sigma(\mathbb{B}_n; \ell^2)$ is equivalent to boundedness on $\mathcal{CM}_p^\sigma(\mathbb{B}_n)$.

Proof of Theorem 5. We must establish the following two inequalities:

$$\begin{aligned}&\left\|\left(1-|z|^2\right)^{\sigma+m_0}\left(\frac{\partial}{\partial z}\right)^{m_0}\mathcal{F}^{\mu}(z)\right\|_{\mathcal{WCM}_p^{\sigma}(\mathbb{B}_n;\ell^2)}\\&\quad\leqslant C_{n,\delta}\left\|\left(1-|z|^2\right)^{\sigma}\mathcal{X}^{m_\mu}\left(\widehat{\Omega_\mu^{\mu+1}h}\right)(z)\right\|_{\mathcal{WCM}_p^{\sigma}(\mathbb{B}_n;\ell^2)},\end{aligned}\tag{5.2}$$

and

$$\begin{aligned}&\left\|\left(1-|z|^2\right)^{\sigma}\mathcal{X}^{m_\mu}\left(\widehat{\Omega_\mu^{\mu+1}h}\right)(z)\right\|_{\mathcal{WCM}_p^{\sigma}(\mathbb{B}_n;\ell^2)}\\&\quad\leqslant C_{n,\delta}\|\mathbb{M}_g\|^{m_\mu+\mu}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\|h\|_{M_{B_p^\sigma(\mathbb{B}_n)}}.\end{aligned}\tag{5.3}$$

Just as for (4.5) in the previous section, inequality (5.2) here follows *verbatim* the argument in [5] but with the boundedness of $T_{a,b,c}$ on $\mathcal{WCM}_p^{\sigma}(\mathbb{B}_n)$ used in place of boundedness on $L^p(\lambda_n)$. To establish (5.3), and even the stronger inequality with the larger $\mathcal{CM}_p^{\sigma}(\mathbb{B}_n;\ell^2)$ norm on the left side, it suffices to show

$$\begin{aligned}&\int\limits_{\mathbb{B}_n}\left|\varphi(z)\right|^p\left|\left(1-|z|^2\right)^{\sigma}\mathcal{X}^{m_\mu}\left(\widehat{\Omega_\mu^{\mu+1}h}\right)(z)\right|^p d\lambda_n(z)\\&\quad\leqslant C_{n,\delta}\|\mathbb{M}_g\|^{(m_\mu+\mu)p}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\|h\|^p_{M_{B_p^\sigma(\mathbb{B}_n)}}\|\varphi\|^p_{B_p^\sigma(\mathbb{B}_n)}.\end{aligned}$$

But this follows using Proposition 2 together with the argument used to prove (4.6) at the end of the previous section. □

References

[1] M. Andersson, H. Carlsson, Estimates of the solutions of the H^p and *BMOA* corona problem, Math. Ann. 316 (2000) 83–102.
[2] N. Arcozzi, R. Rochberg, E. Sawyer, Carleson measures and interpolating sequences for Besov spaces on complex balls, Mem. Amer. Math. Soc. 859 (2006), 163 pp.
[3] L. Carleson, Interpolations by bounded analytic functions and the corona problem, Ann. of Math. 76 (1962) 547–559.
[4] P. Charpentier, Solutions minimales de l'é quation $\overline{\partial}u = f$ dans la boule et le polydisque, Ann. Inst. Fourier (Grenoble) 30 (1980) 121–153.
[5] Ş. Costea, E.T. Sawyer, B.D. Wick, The corona theorem for the Drury-Arveson Hardy space and other holomorphic Besov–Sobolev spaces on the unit ball in $\mathbb{C}^n$, arXiv:0811.0627v7, 8 April 2009, 82 pp.
[6] J. Garnett, Bounded Analytic Functions, Pure Appl. Math., vol. 96, Academic Press, 1981.
[7] K.C. Lin, The H^p corona theorem for the polydisc, Trans. Amer. Math. Soc. 341 (1994) 371–375.
[8] N.K. Nikol'skiĭ, Operators, Functions, and Systems: an Easy Reading, vol. 1: Hardy, Hankel, and Toeplitz, translated from the French by Andreas Hartmann, Math. Surveys Monogr., vol. 92, Amer. Math. Soc., Providence, RI, 2002.
[9] J.M. Ortega, J. Fabrega, Pointwise multipliers and decomposition theorems in analytic Besov spaces, Math. Z. 235 (2000) 53–81.
[10] E. Sawyer, Function Theory: Interpolation and Corona Problems, Fields Inst. Monogr., vol. 25, Amer. Math. Soc., Providence, RI, 2009.
[11] N.Th. Varopoulos, BMO functions and the $\overline{\partial}$ equation, Pacific J. Math. 71 (1977) 221–273.
[12] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Springer-Verlag, 2004.