ON NEARLY PSEUDOCOMPACT SPACES*

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A completely regular space $X$ is called nearly pseudocompact if $\nu X - X$ is dense in $\beta X - X$, where $\beta X$ is the Stone–Čech compactification of $X$ and $\nu X$ is its Hewitt realcompactification. After characterizing nearly pseudocompact spaces in a variety of ways, we show that $X$ is nearly pseudocompact if it has a dense locally compact pseudocompact subspace, or if no point of $X$ has a closed realcompact neighborhood. Moreover, every nearly pseudocompact space $X$ is the union of two regular closed subsets $X_1, X_2$ such that $\text{Int } X_1$ is locally compact, no points of $X_2$ has a closed realcompact neighborhood, and $\text{Int}(X_1 \cap X_2) = \emptyset$. It follows that a product of two nearly pseudocompact spaces, one of which is locally compact, is also nearly pseudocompact.

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1. Introduction

Throughout, the word "space" will abbreviate the phrase "completely regular (Hausdorff) space". For any space $X$, let $\beta X$ denote its Stone–Čech compactification, $\nu X$ its Hewitt realcompactification and let $C(X)$ denote the ring of continuous real-valued functions defined on $X$. Recall from [3, Chapters 6–8] that every $f \in C(X)$ has a (unique) continuous extension $f^\beta$ over $\beta X$ into the one point compactification $R^* = R \cup \{\infty\}$ of the real field $R$. Then $\nu X = \{ p \in \beta X : f^\beta(p) \neq \infty \text{ for every } f \in C(X) \}$. Recall that $X$ is called realcompact if $X = \nu X$ and pseudocompact if $\nu X = \beta X$.

We recall a space $X$ nearly pseudocompact if $\nu X - X$ is dense in $\beta X - X$. Clearly every pseudocompact space is nearly pseudocompact and every realcompact nearly pseudocompact space is compact.

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Perhaps our main result is that a space $X$ is nearly pseudocompact if and only if it is the union of two regular closed subsets $X_1$ and $X_2$ such that $X_1$ is pseudocompact and has a locally compact interior, no point of $X_2$ has a closed realcompact neighborhood and $\text{int}(X_1 \cap X_2) = \emptyset$. Spaces no point of which has a closed realcompact neighborhood are called anti-locally realcompact and are nearly pseudocompact. There is a countably compact space whose product with itself fails to be nearly pseudocompact, but our main theorem enables us to show that the product of two nearly pseudocompact spaces one of which is locally compact is nearly pseudocompact.

We obtain also a number of characterizations of nearly pseudocompact spaces, we show that a space $X$ is nearly pseudocompact if its projective cover (or absolute) is nearly pseudocompact (but not conversely), and that an almost realcompact nearly pseudocompact space is compact.

The principle tool used is the notion of a hard set introduced and used by M. Rayburn in [11] and [12]. More generally, if $\nu X \subseteq T \subseteq \beta X$, we call a subset of $X$ a $T$-hard set if it is closed in $X \cup \text{cl}_{\beta X}(T - X)$, and we call $X$ a $T$-pseudocompact space if $T - X$ is dense in $\beta X - X$. In case $T = \nu X$, a $T$-hard set is called hard and a $T$-pseudocompact space is nearly pseudocompact. In Section 2, we generalize known properties of hard sets to $T$-hard sets, which we use in Section 3 to show that $X$ is $T$-pseudocompact if and only if every regular $T$-hard set is compact.

The notation used is by and large that of [3]. For general background, see also [14] and [15].

2. $T$-Hard sets

If $\nu X \subseteq T \subseteq \beta X$, let $K^T X = K^T = \text{cl}_{\beta X}(T - X)$, and let $\delta^T X = \beta X - (K^T - X)$. If $T = \nu X$, then $K^T X$ is abbreviated by $KX$ and $\delta^T X$ by $\delta X$. Note that for spaces $T$ between $\nu X$ and $\beta X$, we have $X = \delta^T X$ if and only if $X$ is $T$-pseudocompact.

A subset $S$ of $X$ is called $T$-hard if it is closed in $X \cup K^T$, and is called hard if $T = \nu X$. Note that a closed subset $H$ of $X$ is $T$-hard if and only if $\text{cl}_{\beta X}(H) \cap K^T \subseteq X$. Let $\mathcal{H}^T(X)$ denote the family of $T$-hard subsets of $X$. The following results generalize known properties of hard sets.

2.1 Proposition. Suppose $\nu X \subseteq T \subseteq \beta X$.

(a) If $H \in \mathcal{H}^T(X)$, then $H$ is realcompact.

(b) Every compact subset of $X$ is in $\mathcal{H}^T(X)$, every finite union of elements of $\mathcal{H}^T(X)$ is in $\mathcal{H}^T(X)$, and each closed subset of any element of $\mathcal{H}^T(X)$ is in $\mathcal{H}^T(X)$.

(c) $\delta^T X = \bigcup \{\text{cl}_{\beta X} H : H \in \mathcal{H}^T(X)\}$.

(d) $H \in \mathcal{H}^T(X)$ if and only if $H = F \cap X$ for some compact $F \subseteq \delta^T X$.

Proof. Clearly $X \cup K^T = \nu X \cup K^T$ is the union of a realcompact and a compact subset of $\beta X$ and hence is realcompact by [3, 8.16]. Since $H$ is a closed subset of $X \cup K^T$, it is realcompact by [3, 8.10]. So (a) holds.
Part (b) follows immediately from the definition of a T-Hard set.

For part (c), suppose \( p \in \delta^T X - X \). Then \( p \notin K^T \), so there is a compact neighborhood \( F \) of \( p \) in \( \beta X \) such that \( F \cap K^T = \emptyset \). Now \( F \cap X \) is closed in \( X \cup K^T \) and is therefore \( T \)-hard. Also, \( p \in \text{cl}_{\beta X}(F \cap X) \). It follows that \( p \in \bigcup \{ \text{cl}_{\beta X} H: H \in \mathcal{H}^T(X) \} \).

Moreover, if \( H \subseteq X \) is \( T \)-hard, then \( \text{cl}_{\beta X}(H) \cap K^T \subseteq X \), so \( \text{cl}_{\beta X}(H) \subseteq \delta^T X \).

If \( H \subseteq X \) is \( T \)-hard, then by (c), \( \text{cl}_{\beta X}(H) \subseteq \delta^T X \). But \( H = X \cap \text{cl}_{\beta X}(H) \). Conversely, if \( F \subseteq \delta^T X \) is compact, then \( F \cap (K^T - X) = \emptyset \) and \( F \cap K^T \subseteq X \).

Thus \( F \cap X \in \mathcal{H}^T(X) \) and (d) holds.

The proof of the next lemma is an exercise.

2.2. Lemma. If \( X \) is dense in \( Y \) and \( B \) is a subset of \( X \) that is closed in \( Y \), then \( X - B \) is dense in \( Y - B \).

2.3. Theorem. Let \( \nu X \subseteq T \subseteq \beta X \) and let \( N_T(X) \) be the set of points of \( X \) which fail to have a \( T \)-hard neighborhood in \( X \). Then \( N_T(X) = X \cap K^T \).

Proof. Suppose \( p \in X - K^T \). Since \( K^T \) is closed in the regular space \( X \cup K^T \), there is a set \( V \) open in \( X \cup K^T \) containing \( p \) whose closure \( \bar{V} \) in \( X \cup K^T \) misses \( K^T \). By definition of \( T \)-hard sets, \( \bar{V} \) is a \( T \)-hard neighborhood of \( p \) contained in \( X \). Thus \( p \notin N_T(X) \).

Conversely, suppose \( p \notin N_T(X) \). Then there is a set \( V \) open in \( X \) such that \( \text{cl}_{X}(V) \) is \( T \)-hard and \( p \in V \). Since \( X \) is dense in \( X \cup K^T \) and \( \text{cl}_{X}(V) \) is closed in \( X \cup K^T \), by Lemma 2.2 \( X - \text{cl}_{X}(V) \) is dense in \( S = (X \cup K^T) - \text{cl}_{X}(V) \). Since \( X - \text{cl}_{X}(V) \subseteq X - V \), we have \( X - V \) dense in \( S \). Hence \( S = \text{cl}_{S}(X - V) = S \cap \text{cl}_{\beta X}(X - V) \),

so \( \text{cl}_{X}(V) \cup \text{cl}_{\beta X}(X - V) \supseteq \text{cl}_{X}(V) \cup \text{cl}_{S}(X - V) = \text{cl}_{X}(V) \cup S \)

\[ = \text{cl}_{X}(V) \cup [X \cup K^T - \text{cl}_{X}(V)] = X \cup K^T \supseteq T. \]

Thus \( T = [\text{cl}_{X}(V) \cup \text{cl}_{\beta X}(X - V)] \cap T \)

\[ = \text{cl}_{X}(V) \cup [T \cap \text{cl}_{\beta X}(X - V)] = \text{cl}_{X}(V) \cup \text{cl}_{T}(X - V). \]

Therefore

\[ T - \text{cl}_{T}(X - V) = [\text{cl}_{X}(V) \cup \text{cl}_{T}(X - V)] - \text{cl}_{T}(X - V) \subseteq \text{cl}_{X}(V). \]

So \( T - \text{cl}_{T}(X - V) \) is a \( T \)-open set contained in \( \text{cl}_{X}(V) \), whence \( T - \text{cl}_{T}(X - V) \subseteq V \).

On the other hand, since \( X - V \) is closed in \( X \), \( V \subseteq T - \text{cl}_{T}(X - V) \). Thus \( V = T - \text{cl}_{T}(X - V) \) is open in \( T \). Therefore \( p \notin \text{cl}_{T}(T - X) = T \cap \text{cl}_{\beta X}(T - X) \). But \( p \in X \subseteq T \), so \( p \notin \text{cl}_{\beta X}(T - X) = K^T \).
2.4. Theorem. Let \( vX \subseteq T \subseteq \beta X \) and let \( N_c(\delta^T X) \) be the set of points of \( \delta^T X \) which fail to have a compact neighborhood. Then \( N_c(\delta^T X) = X \cap K^T \).

Proof. Since \( X \subseteq \delta^T X \subseteq \beta X \), we have
\[
\beta(\delta^T X) - \delta^T X = \beta X - [\beta X - (K^T - X)] = K^T - X.
\]
It was observed by M. Henriksen and J. Isbell [6, p. 871] that for any space \( Y \),
\[ N_c(Y) = Y \cap \text{cl}_{\beta Y}(\beta Y - Y). \]
Thus
\[
N_c(\delta^T X) = \delta^T X \cap \text{cl}_{\beta X}(\beta X - \delta^T X)
= \delta^T X \cap \text{cl}_{\beta X}(K^T - X) = \delta^T X \cap K^T = X \cap K^T.
\]

2.5. Corollary. Let \( vX \subseteq T \subseteq \beta X \). Then \( N_c(\delta^T X) = N_T(X) \).

It is shown in [9, Lemma 2.10] that \( X \cap KX = N_{rc}(X) \), the set of points in \( X \) which fail to have a closed realcompact neighborhood in \( X \). So we have:

2.6. Corollary. For any space \( X \), the set of points of \( X \) that fail to have a realcompact neighborhood is precisely the set of points of \( \delta X \) that fail to have a compact neighborhood.

This last result was obtained more directly by Rayburn in [12]. Since a \( T \)-hard neighborhood is a realcompact neighborhood (Proposition 2.1(a)), it is apparent that for \( vX \subseteq T \subseteq \beta X \), \( N_{rc}(\delta^T X) \subseteq N_c(\delta^T X) \). If \( T \neq vX \), this inequality is not in general reversible, as we see in the following example.

2.7. Example. A realcompact space \( X \) with an extension \( T \subseteq \beta X \) such that \( N_{rc}(X) \neq N_c(\delta^T X) \).

Let \( X = Q \) be the space of rational numbers with the usual topology. For any \( f \in C(Q) \), let \( T(f) = \{ p \in \beta Q : f^p(p) \in R \} \). Note that \( Q \subseteq T(f) \subseteq \beta Q \), so \( T(f) \) is dense as well as open in \( \beta Q \). Let \( G \) be open in \( \beta Q - Q \) and \( G = \hat{G} \cap (\beta Q - Q) \) for some open set \( \hat{G} \) of \( \beta Q \). Then \( \hat{G} \cap T(f) \) is open in \( \beta Q \). Since \( \beta Q - Q \) is dense in \( \beta Q \),
\[
\emptyset \neq \hat{G} \cap T(f) \cap (\beta Q - Q) = G \cap [T(f) - Q].
\]
Thus \( T(f) - Q \) is dense in \( \beta Q - Q \), so \( \text{cl}_{\beta Q}(T(f) - Q) = \beta Q \). For any such \( f \), \( \delta^{T(f)} Q = Q \). But \( N_c(Q) = Q \), while \( N_{rc}(Q) \) is empty.

3. Nearly pseudocompact spaces

In this section, our main results on nearly pseudocompact spaces are presented. We give them in the more general context of \( T \)-pseudocompact spaces whenever possible.
Recall that a subset of a topological space $X$ that is the closure of its interior is called a *regular closed* set. A regular closed set that is also $T$-hard for some $T$ between $\nu X$ and $\beta X$ is called a *regular $T$-hard* set, and a *regular hard* set is defined similarly. We will make use of the following result which is proved in [3, 9.13].

**3.1. Lemma.** A space $X$ is pseudocompact if and only if every descending sequence of nonempty regular closed subsets of $X$ has nonempty intersection.

Using this result, we obtain:

**3.2. Theorem.** If $\nu X \subseteq T \subseteq \beta X$, the following are equivalent.

(a) $X$ is $T$-pseudocompact.

(b) Every $T$-hard set is compact.

(c) Every regular $T$-hard set is compact.

(d) Each descending sequence of nonempty regular $T$-hard sets has nonempty intersection.

**Proof.** If $X$ is $T$-pseudocompact then $X = \delta^T X$, so (a) implies (b) by Proposition 2.1(d).

Obviously (b) implies (c). Since any descending chain of nonempty compact sets has nonempty intersection, (c) implies (d).

We will conclude the proof by showing that (d) implies (c) and (c) implies (a).

Suppose (d) holds, Let $H$ be a regular $T$-hard set in $X$ and let $(G_n)$ be a sequence of open subsets of $X$ such that $\{G_n \cap H\}$ is a decreasing sequence of nonempty open subsets of $H$. For each $n$, $V_n = G_n \cap \text{int}_X H \neq \emptyset$ since $H$ is regular. Hence $\{V_n\}$ is a decreasing sequence of nonempty open subsets of $H$. By Lemma 2.1(b), $\text{cl}_X(V_n)$ is $T$-hard for every positive integer $n$. Now $\bigcap_{n=1}^{\infty} \text{cl}_X V_n \subseteq \bigcap_{n=1}^{\infty} \text{cl}_X (G_n \cap H)$, and the former is nonempty by assumption. By Lemma 3.1, $H$ is pseudocompact, and by Lemma 2.1(a), $H$ is realcompact. Hence $H$ is compact and (c) holds.

Suppose (c) holds and there is a point $p \in \delta^T X - X$. By Corollary 2.5, there is an open subset $V$ of $\delta^T X$ which contains $p$ and whose closure $\tilde{V}$ in $\delta^T X$ is compact. Since $X$ is dense in $\delta^T X$, $U = \tilde{V} \cap X$ is a nonempty open subset of $X$ and $\text{cl}_X U = \tilde{V} \cap X$ is a regular $T$-hard subset of $X$ by Proposition 2.1(d). By assumption, $\text{cl}_X U$ is compact and $V - \text{cl}_X U$ is an open subset of $\delta^T X$ which misses the dense subspace $X$, contrary to assumption. Thus $X = \delta^T X$ is $T$-pseudocompact, and the proof of Theorem 3.2 is complete.

Before proceeding further, we need to consider some questions about hardness as a relative property. To do this, we will make use of the following internal characterization of hardness of a closed subset due to M. Rayburn [12, 1.2 and 1.4].

**3.3. Lemma.** A closed subset $H$ of a space $X$ is hard if and only if there is a compact subset $K$ of $X$ such that for every open neighborhood $U$ of $K$, the closed set $H \setminus U$ is
completely separated from the complement of some realcompact set. In particular, \( H \) is hard if it is completely separated from the complement of a realcompact subset of \( X \).

3.4. Theorem (a). If \( Y \) is closed subspace of a space \( X \), and \( H \) is a subset of \( Y \) that is hard in \( X \), then \( H \) is a hard subset of \( Y \).

(b) If \( X \) is the union of a finite collection of closed nearly pseudocompact subspaces, then \( X \) is nearly pseudocompact.

Proof (a). Since \( H \) is hard in \( X \), there is a compact subset \( K \) of \( X \) satisfying the conditions of Lemma 3.3. If \( V \) is any open neighborhood in \( Y \) of \( K \cap Y \), then \( U = V \cup (X \setminus Y) \) is an open neighborhood in \( X \) of \( K \). So, by Lemma 3.3, there is a realcompact subset \( S \) of \( X \) and on \( f \in C(X) \) such that \( f[H \setminus U] = 0 \) and \( f[X \setminus S] = 1 \). Now \( H \setminus V = H \setminus U \) since \( H \subseteq Y \), and \( S \cap Y \) is realcompact since \( Y \) is closed. So \( g = f|_Y \) provides a complete separation of \( H \setminus V \) and \( (Y \setminus Y \setminus S) \), whence \( H \) is a hard subset of \( Y \) by Lemma 3.3.

Part (b) follows immediately from (a), Theorem 2.1(b) and Theorem 3.2(b) and is left as an exercise.

The roles of \( Y \) and \( X \) cannot be reversed in the statement of Theorem 3.4(a). For, as is noted in [12, p.22], the right hand edge \( N \) of the Tichonov plane \( T \) is a hard subset of itself, but is not hard in \( T \). Indeed the closed subspace \( N \) of the pseudocompact space \( T \) fails to be nearly pseudocompact. But, as we will show below, a regular closed subspace of a nearly pseudocompact space is nearly pseudocompact.

A space \( X \) will be called anti-locally realcompact if no point of \( X \) has a closed realcompact neighborhood; that is, if \( X = N_{rc}(X) \). Since every hard set is realcompact, then \( X = N_{rc}(X) \) implies that every regular hard subset of \( X \) is empty. So the following result is immediate from Theorem 3.2.

3.5. Corollary. Every anti-locally realcompact space is nearly pseudocompact.

The next proposition supplies us with a substantial number of anti-locally realcompact spaces that fail to be pseudocompact. Part (b) is based on a suggestion of R.G. Woods. As usual, we denote by \( W(\omega_1) \) the space of ordinals less than the first uncountable ordinal \( \omega_1 \).

3.6. Proposition. Each of the following spaces is anti-locally realcompact but not pseudocompact.

(a) A linearly ordered space \( L \) that is not countably compact, but is such that each interval of \( L \) contains a copy of \( W(\omega_1) \).

(b) A product \( \prod \{X_\alpha : \alpha \in I\} \) of spaces infinitely many of which fail to be realcompact and at least one of which is not pseudocompact.

(c) A product of an anti-locally realcompact space and a space that is not pseudocompact.
Proof. To see (a), it suffices to observe that every linearly ordered space is normal, and every normal pseudocompact space is countably compact.

Next, suppose $x = \{x_\alpha\}_{\alpha \in I}$ is a point in the product space of (b) and that $x$ has a closed realcompact neighborhood $V$. By definition of the product topology, there is a finite subset $J$ of $I$ and closed neighborhoods $W_\alpha$ of $x_\alpha$ for each $\alpha \in J$ such that

$$x \in W = \prod \{W_\alpha : \alpha \in J\} \times \prod \{X_\alpha : \alpha \in I - J\} \subseteq V.$$

Thus, for each $\alpha \in I - J$, the space $X_\alpha$ is realcompact since it is a closed subspace of the realcompact space $V$. This contradicts the assumption that infinitely many of the $X_\alpha$ fail to be realcompact. Hence $X$ is anti-locally realcompact. If it were pseudocompact, so would each $X_\alpha$ be pseudocompact since a continuous image of a pseudocompact space is pseudocompact.

Since every closed subspace of a realcompact space is realcompact, the proof of (c) is similar of (b).

An example of a linear ordered space satisfying the hypotheses of Proposition 3.5(a) is given in [2, Example 10.9] and many more such can be constructed using the techniques given in [2, Sections 6–10].

Next, $T$-pseudocompactness of a space $X$ is characterized in terms of the ring $C(X)$. To do so, use will be made of the following result of A. Hager and D. Johnson; see [15, 11.24].

3.7. Lemma. If $U$ is an open subset of a space $X$ and $\text{cl}_X U$ is compact, then $\text{cl}_X U$ is pseudocompact.

3.8. Theorem. If $\nu X \subseteq T \subseteq \beta X$, then the following are equivalent.

(a) Every $f \in C(X)$ is bounded on every $T$-hard subset of $X$.
(b) Every $f \in C(X)$ is bounded on every regular $T$-hard subset of $X$.
(c) $X$ is $T$-pseudocompact.

Proof. Obviously, (a) implies (b).

Suppose $H \in \mathcal{K}(X)$ is regular and (b) holds. Then $\text{cl}_X H$ is compact as is shown in [15, 11.25]. Hence, by Lemma 3.7, $H$ is pseudocompact. By Proposition 2.1(a), $H$ is also realcompact. So $H$ is compact, whence $X$ is $T$-pseudocompact by Theorem 3.2 and (b) implies (c).

If $X$ is $T$-pseudocompact and $H \in \mathcal{K}(X)$, then $H$ is compact, so every $f \in C(X)$ is bounded on $H$ and (c) implies (a).

As in [*], we call a space $X$ almost locally compact if the set of points of $X$ which have a compact neighborhood is dense in $X$.

3.9. Theorem. A regular closed almost locally compact subspace $Y$ of a nearly pseudocompact space $X$ is pseudocompact.
Suppose $Y$ were not pseudocompact. By Lemma 3.7, there is a sequence $(y_n)$ in $\text{int} Y$ and an $f \in C(X)$ such that $(f(y_n))$ increases strictly to infinity. For $n = 1, 2, \ldots$, let $U_n = \{y \in \text{int} Y : f(y_{2n}) < f(y) < f(y_{2n+2})\}$. Since $Y$ is almost locally compact, for each $n$, there is an $x_n$ with a compact neighborhood $K_n$ contained in $U_n$. Let $H = \{x_n\}_{n=1}^\infty$ and $K = \bigcup_{n=1}^\infty K_n$. By the definition of $f$, the union of any subfamily of $(K_n)_{n=1}^\infty$ is closed, and $K$ is a locally compact and $\sigma$-compact subspace of $X$. Thus $K$ is realcompact and by [3, 3.1], $H$ and $X \setminus K$ are completely separated. Hence $H$ is a hard set by Lemma 3.3. Since $f$ is unbounded on $H$, the space $X$ fails to be nearly pseudocompact by Theorem 3.8(a).

With the aid of Theorem 3.9, we obtain the following decomposition theorem for nearly pseudocompact spaces.

3.10. Theorem. A space $X$ is nearly pseudocompact if and only if $X = X_1 \cup X_2$, where $X_1$ is a regular closed almost locally compact pseudocompact subset, $X_2$ is a regular closed anti-locally realcompact subset and $\text{int}(X_1 \cap X_2)$ is empty.

Proof. Suppose $X$ is nearly pseudocompact and let $X_1 = \overline{\text{cl}}_X[\text{int} X - N_c(X)]$ be the closure of the set of points which have a compact neighborhood. Then $X_1$ is a regular closed almost locally compact set, which by Theorem 3.9 is pseudocompact. Let $X_2 = \overline{\text{cl}}_X(X - X_1)$. By Corollary 2.6, $X_2$ is anti-locally realcompact. Clearly $X_2$ is regular closed and $\text{int}(X_1 \cap X_2)$ is empty.

Conversely, by Corollary 3.4(b) if $X$ is the union of two regular closed nearly pseudocompact subspaces, then $X$ is nearly pseudocompact (an anti-locally realcompact space is nearly pseudocompact by 3.5). This completes the proof.

3.11. Corollary. A regular closed subspace $Y$ of a nearly pseudocompact space $X$ is nearly pseudocompact.

Proof. Since $U = \overline{\text{cl}}_X(\text{int} Y)$, we may write $Y = Y_1 \cup Y_2$ where $Y_i = \overline{\text{cl}}_X(\text{int} Y \cap X_i)$ for $i = 1, 2$. Then $Y_1$ is pseudocompact by Lemma 3.7, and $Y_2$ is anti-locally reacompact since it is a regular closed subspace of an anti-locally realcompact space. So $Y$ is nearly pseudocompact by Theorem 3.10.

Next, we consider the extent to which near pseudocompactness is preserved by some kinds of mappings and pullbacks.

A continuous surjection $\tau : X \to Y$ is called a perfect map if it is closed and $\tau^{-1}(y)$ is compact for every $y \in Y$. In [6], a perfect map is called a fitting map and it is shown that a continuous surjection $\tau : X \to Y$ is a perfect map if and only if its unique continuous extension $\tau_F : \beta X \to \beta Y$ maps $\beta X - X$ onto $\beta Y - Y$, and it is shown that if $F$ is compact in $Y$, then $\tau^{-1}[F]$ is compact.

A continuous surjection $\tau : X \to Y$ is called a tight map if it maps no proper closed subset of $X$ onto $Y$. It is known that if $\tau : X \to Y$ is closed and tight, then for every open subset $U$ of $X$, there is an open subset $V$ of $Y$ such that $\tau^{-1}[V] \subseteq U$ [13].

Use will be made below of the following well-known fact.
3.12. Lemma. If \( f : X \to Y \) is a closed, tight map, then \( f \) maps regular closed subsets of \( X \) onto regular closed subsets of \( Y \).

For a proof, see [17, Lemma 2.3] where it is shown that the map \( B \to f[B] \) is a Boolean algebra isomorphism.

Recall that a space \( X \) is called a weak-cb space if every locally bounded lower semicontinuous real-valued function on \( X \) is bounded above by a continuous function. Equivalently, \( X \) is a weak-cb space if and only if whenever \( \{F_n\} \) is a decreasing sequence of non-empty regular closed subsets of \( X \) with empty intersection, then there is a sequence \( \{Z_n\} \) of zero sets of \( X \) with empty intersection such that for each \( n \), \( F_n \subseteq Z_n \). Clearly then, every regular closed subset of a weak cb-space is a weak cb-space. See [8].

3.13. Theorem. Suppose \( \tau : X \to Y \) is a perfect map.

(a) If \( X \) is nearly pseudocompact, then so is \( Y \).

(b) If \( \tau \) is tight, and \( Y \) is weak cb and nearly pseudocompact, then \( X \) is nearly pseudocompact.

Proof. (a). In [12], M. Rayburn shows that the pullback of a hard set under a perfect map is a hard set. Thus if \( H \) is a hard subset of \( Y \), then \( \tau^{-1}[H] \) is a hard subset of the nearly pseudocompact space \( X \), and therefore is compact. Hence \( H = \tau[\tau^{-1}(H)] \) is compact, and \( Y \) is nearly pseudocompact by Theorem 3.2.

(b). Suppose \( H \) is a regular hard subset of \( X \). It is easy to check that the restriction of \( \tau \) to \( H \) is perfect and tight. By Lemma 3.12, \( \tau(H) \) is a regular closed subset of the nearly pseudocompact space \( Y \) and hence is nearly pseudocompact by Corollary 3.11. Also, \( H \) is realcompact by Proposition 2.1(a). In [1, Corollary 1.4], N. Dykes showed that perfect maps onto weak cb-spaces preserve realcompactness. So \( \tau(H) \) is also realcompact. Thus \( \tau[H] \) is compact, as is \( \tau^{-1}[\tau(H)] \) since \( \tau \) is perfect. We conclude that the closed subset \( H \) of \( \tau^{-1}[\tau(H)] \) is compact. So \( X \) is nearly pseudocompact by Theorem 3.2.

Our last theorem brings some questions to mind immediately.

(A) Are there nearly pseudocompact spaces that fail to be weak cb?

(B) Can either the hypothesis in 3.13(b) that \( \tau \) is a tight map or that \( Y \) is a weak cb-space be omitted?

It is easy to see that (A) has an affirmative answer. For the product of an anti-locally realcompact space and a space that fails to be weak cb is a nearly pseudocompact space that fails to be weak cb (by definition of weak cb-space).

Before answering B, we digress to introduce some needed concepts.

Recall that a space \( X \) is called extremely disconnected if \( \text{cl}_X(U) \) is open for every open subset \( U \) of \( X \). For any space \( X \), there is an extremely disconnected space \( E(X) \) and a tight perfect map \( \pi : E(X) \to X \). The space \( E(X) \) is called a projective cover or absolute of \( X \) and is a projective object in the category of completely regular spaces and perfect maps. That is, if \( \tau : Y \to X \) is a perfect map, then there is a perfect
\( \rho : E(X) \to Y \) such that \( \pi = \tau \circ \rho \). Any two projective covers of \( X \) are homeomorphic, and \( F \) is a regular closed subset of \( X \) if and only if \( \tau^{-1}[F] \) is both open and closed in \( E(X) \). For background, see [10, 13, 16, or 17].

We call a space \( X \) almost realcompact if every ultrafilter of regular closed sets with the countable intersection property converges; see [5, Theorem 1.2]. In [1, Theorem 1.7], N. Dykes shows that \( E(X) \) is realcompact if and only if \( X \) is almost realcompact, after showing in [1, Theorem 1.2] that an almost realcompact weak cb-space is realcompact.

The next example supplies a negative answer to (part of) question B since the projection of \( E(X) \) onto \( X \) is both perfect and tight.

3.14. Example. A nearly pseudocompact space \( X \) such that \( E(X) \) fails to be nearly pseudocompact.

Let \( Y \) denote any almost realcompact space that is not realcompact. (See, for example [15, 16.12] or [17, 4.7].) Then the product \( X \) of \( \omega \) copies of \( Y \) is almost realcompact and is nearly pseudocompact by Proposition 3.5 and 3.6(b). By the result of N. Dykes cited above, \( E(X) \) is realcompact. If \( E(X) \) were also nearly pseudocompact, it would be compact. But then \( X \) would be compact, contrary to assumption.

Since every extremally disconnected space is weak cb [8], the last example shows also that the hypothesis in 3.13(b) that the range space be weak-cb cannot be replaced by the assumption that the domain is weak-cb. Note also that \( \nu E(X) \) contains \( E(X) \) properly as a subspace of \( E(\beta X) = \beta E(X) \) (See [17, Section 4] and [16].)

Recall from [8] that every pseudocompact space is weak cb. It is not difficult to see that the spaces described in Proposition 3.6(a) are both anti-locally realcompact and (weak) cb (since they are both normal and countably paracompact).

In [16, Proposition 2.5], R. G. Woods shows that \( X \) is pseudocompact if and only if \( E(X) \) is pseudocompact. Since any continuous image of a pseudocompact space is pseudocompact, the following may be regarded as an improvement of Woods’ theorem.

3.15. Theorem. If \( \tau : X \to Y \) is a tight closed map and \( Y \) is pseudocompact, so is \( X \).

Proof. By Lemma 3.1, it suffices to show that if \( \{ R_n \} \) is a decreasing sequence of nonempty regular closed subsets of \( X \), then their intersection is nonempty. Let \( S_1 = \text{cl} \bigcup \{ \tau^{-1}[V] : V \text{ is open in } Y \text{ and } \tau^{-1}[V] \subseteq \text{int}(R_1) \} \). Since \( \tau \) is a tight map, we know that \( S_1 \) is not empty. Since \( \tau \) is closed and tight, it follows from Lemma 3.12 that \( \tau[S_1] \) is a regular closed subset of \( Y \), and \( \text{int}(S_1 \cap R_2) \neq \emptyset \). Let \( S_2 = \text{cl} \bigcup \{ \tau^{-1}[V] : V \text{ is open in } Y \text{ and } \tau^{-1}[V] \subseteq \text{int}(S_1 \cap R_2) \} \). Continuing inductively in this way, we obtain a sequence \( \{ S_n \} \) of nonempty regular closed subsets of \( X \) such that

\( \text{int}(S_{n+1} \cap R_{n+1}) \neq \emptyset \) for all \( n \).
for all \( n, S_{n+1} \subseteq S_{n} \subseteq R_{n} \), \( \tau^{-}[S_{n}] \) is regular closed, and \( \tau^{-}[\tau(S_{n})] \subseteq R_{n} \). By Lemma 3.1, \( \bigcap_{n=1}^{\infty} \tau[S_{n}] \) contains a point \( y \) of \( Y \), and so \( \tau^{-}(y) \subseteq \bigcap_{n=1}^{\infty} R_{n} \). Hence \( X \) is pseudocompact.

That \( \tau \) must be tight in the previous theorem is shown by the following example where it appears that the pullback of a pseudocompact (and hence weak cb-) space under a perfect map need not be nearly pseudocompact. This completes the negative answer to question B.

3.16. Example. Let \( Y = W(\omega_{1} + 1) \times W(\omega_{0} + 1) - \{(\omega_{1}, \omega_{0})\} \) denote the Tichonov plank and let \( X \) denote the free union of \( Y \) and a copy of the set \( N \) of positive integers with the discrete topology. Let \( \tau : X \to Y \) be the identity map on \( Y \) and map each point \( n \) of \( N \) to the point \( (\omega_{1}, n) \) of \( Y \). As is observed in [6, Example 2.3], \( Y \) is pseudocompact and \( \tau \) is perfect. But \( N \) is a noncompact regular hard subset of \( X \), so \( X \) fails to be nearly pseudocompact by Theorem 3.2.

In [3, 9.15], an example is given of a countably compact space \( G \) such that \( G \times G \) contains an open and closed copy of \( N \). Hence \( G \times G \) fails to be nearly pseudocompact by Corollary 3.11. We can, however, show the following about products of nearly pseudocompact spaces.

If \( X \) is any space, we call \( \text{cl}_{X}[X - N_{e}(X)] \) the almost locally compact part of \( X \). By Theorem 3.9, if \( X \) is nearly pseudocompact, then its almost locally compact part is pseudocompact.

3.17. Theorem. Suppose \( X \) and \( Y \) are nearly pseudocompact spaces. Then \( X \times Y \) is nearly pseudocompact if and only if the product of their almost locally compact parts is pseudocompact. In particular, if the almost locally compact part of either \( X \) or \( Y \) is locally compact, then \( X \times Y \) is nearly pseudocompact.

Proof. As in the proof of Theorem 3.10, we write \( X = X_{1} \cup X_{2} \), where \( X_{1} \) is the almost locally compact part of \( X \) and \( X_{2} = \text{cl}_{X}(X - X_{1}) \) is anti-locally realcompact, and similarly for \( Y \). Then

\[
X \times Y = (X_{1} \times Y_{1}) \cup (X_{1} \times Y_{2}) \cup (X_{2} \times Y_{1}) \cup (X_{2} \times Y_{2}).
\]

The last three terms each have at least one anti-locally realcompact factor, hence they are anti-locally realcompact.

Since each \( X_{i} \times Y_{j} \) is regular for \( i, j = 1, 2 \), it follows from Theorems 3.4 and 3.11 that \( X \times Y \) is nearly pseudocompact if and only if \( X_{1} \times Y_{1} \) is nearly pseudocompact. But, by Theorem 3.9, the almost locally compact space \( X_{1} \times Y_{1} \) is nearly pseudocompact (if and) only if it is pseudocompact.

Finally, if either \( X_{1} \) or \( Y_{1} \) is locally compact, then by a result of Tamano given by R. Walker in [13, p. 203], its product with any other pseudocompact space is pseudocompact. It now follows from the preceding paragraph that \( X \times Y \) is nearly pseudocompact.
We close with another characterization of nearly pseudocompact spaces.

In [4], I. Glicksberg has shown that a space $X$ is pseudocompact if and only if every countable family of pairwise disjoint regular closed sets has a limit point in $X$.

3.18. Theorem. A space $X$ is nearly pseudocompact if and only if every countable family $\{H_n\}$ of (non-empty) pairwise disjoint regular hard subsets of $X$ has a limit point in $X$.

Proof. Since every regular hard subset of an anti-locally realcompact space is empty, it follows from Theorem 3.10 that each $H_n$ is in the pseudocompact almost locally compact part of $X$. So Theorem 3.18 follows from Glicksberg's theorem cited above.

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References