Effective elastic modulus of heterogeneous peristatic bar of random structure

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Abstract

The basic feature of the peridynamic model considered is a continuum description of a material behavior as the integrated nonlocal force interactions between infinitesimal particles. In contrast to these classical theories, the peridynamic equation of motion introduced by Silling (2000) is free of any spatial derivatives of displacement. A heterogeneous bar of statistically homogeneous random structure of constituents with the peristatic mechanical properties is analyzed by the standard averaging tool of micromechanics. The applicability of local elasticity theory is demonstrated for description of effective elastic behavior of this bar. The approach proposed is based on numerical solution (for both the displacements and peristatic stresses) for one heterogeneity inside infinite homogeneous bar loaded by a pair of self-equilibrated concentrated remote forces. This solution is substituted into the general scheme of micromechanics of locally elastic media adapted for the considered case of 1D peristatic structures. A convergence of effective modulus estimations is demonstrated for both the peristatic composite bar and locally elastic bar.

1. Introduction

In local elasticity the stress at a given point uniquely depends on the deformation and temperature at that point only. However, such a consideration implies that the material can be treated as a continuum at an arbitrarily small scale. Real materials, however, have complicated internal structures with length scales ranging over many orders of magnitude. Simultaneous modeling over the entire range of length scales is prohibitively expensive for practical applications since it requires coupling of continuum mechanics approaches with both molecular mechanics and quantum mechanics models. Affordable material models should select some resolution level below which the microstructural details are not explicitly “visible” to the model and need to be taken into account approximately and implicitly, by an appropriate definition of “effective” material properties (see, e.g., Bažant and Jirásek, 2002). A gap between the continuum mechanics and the discrete point-mass models of molecular dynamics (or interatomic potentials based on quantum mechanics) is bridged by a special mechanism using various forms of generalized (so-called nonlocal) continuum formulations, dealing with materials that are nonsimple or polar, or both, where the nonlocal variables are simply volume averages of the corresponding local variables over some nonlocal representative volume element (RVE) around the material point being considered. The standard micromechanical arguments (see, e.g., Chapter 12 in Buryachenko (2007)) show that the size of this nonlocal RVE and their constitutive law should depend on the local variable distribution itself or, in other words, on the solution at the fine (molecular) resolution level. One usually emerges from this difficulty by the use of representations of the corresponding nonlocal operators through either the differential or integral forms. These were refined by Kröner et al. (see for references Kröner, 1967; Kunin, 1983; Rogula, 1982; Eringen, 2002; Bažant and Jirásek, 2002), frequently motivated by homogenization of the atomic theory of Bravais lattices. Most concepts of these nonlocal models use the nonlocal forms (either the integral or differential ones) of constitutive equations and the differential equation of motion.

In contrast to these classical local and nonlocal theories, the peridynamic equation of motion introduced in Silling (2000) (see also Silling and Lehoucq, 2008, 2010; Weckner et al., 2009) is free of any spatial derivatives of displacement. The basic feature of the peridynamic model is a continuum description of a material behavior as the integrated nonlocal force interactions between infinitesimal particles. This might be an attractive feature especially for the problems involving discontinuities in the deformation process. The effectiveness of peridynamic models has already been demonstrated in several sophisticated applications, including...
damage accumulation, the fracture and failure of composites of deterministic (e.g., periodic) structure, crack instability, the fracture of polycrystals, and nanofiber networks (see, e.g., Askari et al., 2006, 2008; Alali and Lipton, 2012). While the similar problems were solved in the framework of local elasticity, their peristatic formulation requires considerably higher computational costs. However, the mentioned drawback of peridynamics is with urgency compensated by the opportunities opening by peridynamics in solution of problems with discontinuities.

Analysis of random structure composites (see for references Buryachenko, 2007) even for locally elastic constitutive phases in general requires essentially high computation costs in comparison with the deterministic structures (e.g., periodic one). In so doing, random structure composites with the peristatic properties of constituents were only considered in the framework of assumption of replacement of the random structure by a deterministic one (e.g., mixture theory and laminate theory, see Askari et al., 2006; Kilic, 2008; Hu et al., 2012). Moreover, historically micromechanics usually work with the differential equation of motion rather than with the integral one (as in peridynamics). Because of this even the background concepts of micromechanics (see for details, e.g., Buryachenko, 2007) such as the effective moduli, effective field, and especially the general integral equations (connecting the displacement fields in the inclusion being considered and the surrounding inclusions, see Buryachenko, 2010a,b, 2011b, 2014) are not yet defined in the theory of peristatic composites. The current paper is dedicated to closing this gap between micromechanical approaches and peridynamic models. It will be considered by an example of an infinite 1D peristatic bar of random structure while a locally elastic counterpart of the corresponding problem is exactly solved. Analysis of the simplified 1D structure makes it possible to dispense with the some assumptions (widely used in 2D and 3D cases) and to focus our attention to direct use of a large body of both the analytical and numerical results obtained for 1D homogeneous and inhomogeneous peridynamic bar of deterministic structure (see, e.g., Silling et al., 2003; Silling and Askari, 2005; Weckner and Abeyaratne, 2005; Weckner and Emmrich, 2005; Emmrich and Weckner, 2007; Bobaru et al., 2009; Mikata, 2012). In this paper the term “peristatics” is used analogously to Mikata (2012) to differentiate the static problems considered in the current paper from the dynamic problems.

The paper is originated as follows. In Section 2 we give a short introduction into the 1D peristatic theory of solids. In Section 3 previous research for one heterogeneity inside infinite homogeneous bar is presented in the form adapted for subsequent exploitation in Section 4 in the model for the heterogeneous bar of statistically homogeneous random structure. At the beginning of Section 4 one summarizes the known exact solution for the heterogeneous bar of statistically homogeneous random structure in the framework of local elasticity. The mentioned form of the solution is most suitable for generalization to the peristatic model also considered in Section 4. The numerical results are presented in Section 5 where one also demonstrates a convergence of effective modulus estimations obtained for peristatic composite bar to the corresponding exact effective moduli evaluated for the local elastic theory.

2. Preliminaries

2.1. Basic equations of peristatics

In this section, we first summarize the linear peristatic 1D theory (see the references in Introduction) for an infinitely long bar of a constant cross section A = 1, assume that the bar is parallel to the X1 = x axis. We reproduce (see for details Silling et al., 2003) the constitutive law for a peristatic bar directly in the one-dimensional setting, omitting the calculations requiring the cross section:

$$\int_{-\infty}^{\infty} C(x, \xi)|u(\xi) - u(x)| \, d\xi + b(x) = 0,$$

(2.1)

where u is the displacement field, b is a prescribed external force density field, and C is a stiffness distribution density or micromodulus function. The body force density function b(x) is assumed to be self-equilibrated

$$\int_{-\infty}^{\infty} b(x) \, dx = 0,$$

(2.2)

and vanished outside some loading region: b(x) = 0 for |x| > a'. For consistency with Newton's third law, the micromodulus function C for the homogeneous materials must be even (\(\xi = x - \tilde{x}\)):

$$C(\xi) = C(-\xi) \quad \text{for} \quad -\infty < \xi < \infty.$$  

(2.3)

It is assumed that C(\(\xi\)) has a compact support, i.e., the material has a “horizon”, when there is no interaction between particles that are more than some finite distance \(l_0\) away, then C(\(\xi\)) = 0 for all \(|\xi| > l_0\). Thus, the integration domain \(R = (-\infty, \infty) = \mathbb{R}\) in Eq. (2.1) can be limited by a neighborhood \(\mathcal{N}_l(x) = \{\xi : |\xi - x| < l\}\) of the point x. For example, for the micromodulus functions with the step-function and triangular profiles

$$C(\xi) = \begin{cases} C, & \text{for } |\xi| < l, \\ 0, & \text{for } |\xi| > l. \end{cases},$$  

$$C(\xi) = \begin{cases} C(1 - |\xi|/l), & \text{for } |\xi| < l, \\ 0, & \text{for } |\xi| > l, \end{cases}$$

(2.4)

respectively.

For two phase bar, the domain R contains a homogeneous matrix \(\nu^0\) and a statistically homogeneous field X = (\(\xi_i\)) of identical inclusions \(v_i \subset \nu^1\) with indicator functions \(\nu_i (\nu^0 \cup \nu^1 = R, \nu^0 \cap \nu^1 = \emptyset)\). We consider a dilute approximation when interaction of inclusions \(v_i \subset \nu^1\) are absent, and the peridynamic horizon \(l_0\) is chosen to be smaller than the spacing separating the inclusions. For any two points x and \(\tilde{x}\) in R, \(C(\xi) = C(x, \tilde{x}) (\xi = x - \tilde{x})\) is given by the formula \((v_i \subset \nu^1, i = 1, 2, \ldots)\)

$$C(x, \tilde{x}) = \begin{cases} C^{(1)}(x, \tilde{x}), & \text{for } x, \tilde{x} \in v_i, \\ C^{(0)}(x, \tilde{x}), & \text{for } x, \tilde{x} \in \nu^0, \\ 0, & \text{for } |x - \tilde{x}| > l_0, \end{cases}$$

(2.5)

which can also be presented in the form

$$C(x, \tilde{x}) = C^{(1)}(x, \tilde{x}) V^{(1)}(x) V^{(1)}(\tilde{x}) + C^{(0)}(x, \tilde{x}) V^{(0)}(x) V^{(0)}(\tilde{x})$$

$$+ C(x, \tilde{x}) V^{(0)}(x) V^{(0)}(\tilde{x}) + V^{(0)}(x) V^{(1)}(\tilde{x})],$$

(2.6)

where \(V^{(k)}(x)\) is an indicator function of \(\nu^k\) equals 1 at \(x \in \nu^k\) and 0 otherwise (\(k = 0, 1\)). The material parameters \(C^{(1)}\) and \(C^{(0)}\) are intrinsic to each phase and can be determined through the experiments. Bonds connecting particles in the different materials are characterized by micromodulus C, which can be chosen such that

$$C^{(1)}(x, \tilde{x}) \geq C(x, \tilde{x}) \geq C^{(0)}(x, \tilde{x}),$$

which can also be presented in the form

$$C(x, \tilde{x}) = (C^{(0)}(x, \tilde{x}) + C^{(1)}(x, \tilde{x}))/2,$$

(2.7)

$$C(x, \tilde{x}) = \min\{C^{(0)}(x, \tilde{x}), C^{(1)}(x, \tilde{x})\},$$

(2.8)

see Silling and Askari (2005) and Alali and Lipton (2012). The peridynamic theory is traditionally based on the using of the displacement field u(x) rather than either the stress \(\sigma(x)\) or strain \(\varepsilon(x)\) fields which are not conceptually necessary. However, introduction of the notion of the stress is helpful, as one can use it to formulate the stress–strain relations for exploiting of well-developed tool of classical elasticity theory in a subsequent application of the present theory for heterogeneous materials. So, by
adapting the Cauchy’s notion of stress in a crystal, one can define the “peristatic stress” \( \sigma(z) \) at the point \( z \) to be the total force that all material particles \( \hat{x} \) to the right of \( z \) exert on all material particles to its left (see e.g., Silling et al., 2003; Weckner and Abeyaratne, 2005)

\[
\sigma(z) = \int_{-\infty}^{z} \left\{ \int_{-\infty}^{x} C(x, \hat{x})[u(\hat{x}) - u(x)]dx \right\}dx \\
= \int_{0}^{\infty} \left\{ \int_{0}^{z} C(z + r, z - s)[u(z + r) - u(z - s)]ds \right\}dr, \tag{2.9}
\]

where (2.9a) is obtained from (2.9) by the change of the variables \( x = z + r, \hat{x} = z - s \). However, due to the vanishing of \( C(x, \hat{x}) \) outside the domains \( \mathcal{H}(\hat{x}) \) and \( \mathcal{H}(x) \), e.g., Eq. (2.9) can be presented in the form

\[
\sigma(z) = \int_{-\infty}^{\hat{z}} \left\{ \int_{\hat{z} - l}^{\hat{z} + l} C(x, \hat{x})[u(\hat{x}) - u(x)]dx \right\}dx, \tag{2.10}
\]

where \( \mathcal{H}_{\hat{z} - l} = \{ x : \hat{x} - l < x < \hat{x} \} \) and \( \mathcal{H}_{\hat{z} + l} = \{ x : \hat{x} < x < \hat{x} + l \} \). In the particular case of a homogeneous deformation, the strain in any bond does not depend on location

\[
u(x) = \hat{x} \quad \text{for some } \hat{x} = \text{const} \tag{2.11}
\]

and Eq. (2.9a) is simplified to the “stress–strain relation” (see Silling et al., 2003)

\[
\sigma = E^0 l^3, \quad E^0 = \int_{0}^{\infty} r^2 C(r)dr, \tag{2.12}
\]

establishing a representation of the Young’s modulus \( E^0 \) in the classical theory of elasticity through the micromodulus \( C(r) \) in the peridynamic theory. If the stress–strain relation (2.12) is found, constitutive model can be calibrated by choosing of the micromodulus function \( C(\hat{z}) \) which corresponds to measured material values (2.12b). So, substituting of Eqs. (2.4a) and (2.4b) into Eq. (2.12) leads to (see Silling et al., 2003)

\[
C = 3E^0 l^3, \quad \text{and} \quad C = 12E^0 / l^3, \tag{2.13}
\]

respectively. Schematic representations of the micromodi \( C(\hat{z}) \) corresponding to Eqs. (2.4a) (2.13a) and (2.4b) (2.13b) respectively, are presented in Fig. 1.

Boundary conditions corresponding to Eq. (2.11) are called homogeneous. Due to nonlocality of the peristatic motion Eq. (2.1), the boundary displacement and forces are prescribed on a boundary layer with non-zero volumetric measure (in opposite to the local elasticity case where the boundary conditions are imposed precisely at the bounding surface) [see for details Silling (2000), Kilic (2008), and Section 3.1 of the current paper]. In practice (see Macke and Silling, 2007), a thickness of this layer equals to \( l \).

2.2. Statistical description of the composite microstructures

It is assumed that all material properties \( g = g(C, E) \) are decomposed as \( g = g^0 + g_1(x) = g^0 + g^1(x) \). The upper index \( ^0 \) indicates the components and the lower index \( ^1 \) indicates the individual identical inclusions \( \eta_i \) with the centers \( x_i \) and the size \( 2a_i \). Thus \( g^0 = w \cup v^0 \) and \( v^1 = \cup \eta_i \) denote the matrix and inclusions, respectively, of the bar \( w = v^0 \cup v^1 \), \( V(x) = V^0(x) = \sum \eta_i \), and \( V(x) \) is an indicator function of \( \eta_i \) equals 1 at \( x \) in \( \eta_i \) and 0 otherwise, \( i = 1, 2, \ldots \). The bar \( w \) contains a statistically large number of realizations \( x \) (providing validity of the standard probability technique) of inclusions \( \eta_i \subset v^0 \) \( i = 1, 2, \ldots \). A random parameter \( x \) belongs to a sample space \( \mathcal{A} \), over which a probability density \( p(x) \) is defined (see, e.g., Willis, 1981). For any given \( x \), any random function \( g(x, \eta) \) (e.g., \( g = V, V^1, \sigma \)) is defined explicitly as one particular member, with label \( x \), of an ensemble realization. Then, the mean, or ensemble average is defined by the angle brackets enclosing the quantity \( \langle g \rangle(x) = \int_{A} g(x, \eta)p(x)dx \). \tag{2.14}

At first all the random quantities under discussion in Section 4.1 are described by statistically inhomogeneous random fields while in Sections 5 and 5 only statistically homogeneous random fields of heterogeneities subjected to homogeneous boundary conditions are analyzed. \( \phi(\eta_i, x) \) is a number density, \( n^1 = n^1(x) \) of component \( \eta^1 \) at the point \( x \) and \( c = c(\eta) \) is the concentration, i.e., volume fraction, of the component \( \eta^1 \) at the point \( x \) (see for details Silling (2000), Kilic (2008), and Section 3.1 of the current paper). In practice (see Macke and Silling, 2007), a thickness of this layer equals to \( l \).

\[
\langle g \rangle(x) = \int_{A} g(x, \eta)p(x)dx \tag{2.14}
\]

is only fulfilled for statistically homogeneous media subjected to the homogeneous boundary conditions. However, although in a general case

\[
\phi(\eta_i, x) = \lim_{x \to \text{fixed point}} \int_{A} g(x, \eta)p(x, x)dx \tag{2.15}
\]

is possible to establish a straightforward relation between these averages for the identical inclusions \( \eta_i \). Indeed, at first we build some auxiliary set \( \eta^0_i(x) = \{ x : |x - x_i| < 2a_i \} \) around the fixed point \( x \). Then we can get a relation between the mentioned averages:

\[
\langle g \rangle(x) = \int_{A} n^1(y)g(x, y)p(y)dy \tag{2.16}
\]
It should be mentioned that for statistically homogeneous fields \((\rho, v)^{(1)}(x) = \rho^{(1)} = \text{const.}; \ 0 < \nu \leq 1\), at the micro-coordinate \(x \in \nu\). Because of this, the unknown will arise below in notations of the average \((\rho, v)^{(1)}(x)\) and conditional average \((\rho, v)(x)\) at the fixed inclusion \(\nu\) which are the functions of macro-coordinate (with “resolution” equal to \(\Lambda \gg a\)) and micro-coordinate \(x \in \nu\) (used in the case of fixed inclusion \(\nu\), respectively. Formula (2.17) is valid for any material inhomogeneity of inclusions of any concentration in the bar. Obviously, the general Eq. (2.17) is reduced to Eq. (2.15) for both the statistically homogeneous media subjected to homogeneous boundary conditions and statistically homogeneous fields \(g\) (e.g., \(g = \text{const.}\)). However, in a general case \(g(\nu, y)(x) = f(x, y)g(\nu, y)\) is a statistically homogeneous field and \(f(x, y)\) is a continuous function of \(x, y\). Eq. (2.17) is not reduced to Eq. (2.15).

It is interesting that the relations (2.12a) and (2.12b) are physically meaningful only in the context of homogeneous deformations of a homogeneous bar. However, for a statistically homogeneous heterogeneous bar subjected to a homogeneous external loading, we will demonstrate in the next section that an effective Young’s modulus \(E\), connecting the statistical averages stresses \((\sigma)\) and strains \((e)\)

\[
(\sigma) = E(\varepsilon)
\]  
(2.18)
can also be defined.

3. Infinite bar with a single heterogeneity

3.1. Displacement field for a single heterogeneity in an infinite bar

Let us consider the governing equation for 1D peristatic bar of the infinite length and cross-section area \(A = 1\)

\[
\mathcal{L} \ast u(x) + b(x) = 0, \quad (\infty < x < \infty),
\]  
(3.1)

where the nonlocal operator \(\mathcal{L}\) is defined as

\[
\mathcal{L} = \int C(x, y)u(y) - u(x)dy,
\]  
(3.2)

loaded by a pair \((-P, P)\) of self-equilibrated concentrated remote forces

\[
b(x) = \frac{P}{A} \delta(x - a) - \delta(x + a), \quad (a \rightarrow \infty),
\]  
(3.3)

with \(\delta\) being the Dirac-delta distribution.

At first we consider a single fixed inclusion \(\nu(0)\) with the properties \(C^{(1)}(x)\) in the infinite homogeneous bar with the properties \(C^{(0)}(x)\). Then the solution of (3.1) can be split into two ones as

\[
u(x) = \nu_0(x) + u_1(x),
\]  
(3.4)

where

\[
\mathcal{L}^{(0)} \ast \nu_0(x) + b(x) = 0,
\]  
(3.5)

\[
\mathcal{L} \ast u_1(x) + \mathcal{L}_1 \ast \nu_0(x) = 0
\]  
(3.6)

and \(\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}_1\). The peristatic solution of Eq. (3.5) is in detail investigated by both numerical and analytical methods (see Silling et al., 2003; Weckner and Abeyaratne, 2005; Bobaru et al., 2009; Mikata, 2012). When the length scale \(\ell_1/a \rightarrow 0\), this solution converges to the classical elasticity solution almost everywhere, e.g., for \(|x| < a^2/2\)

\[
u_0 = \nu_0(x), \quad \nu_0 = \frac{P}{AE^{(0)}},
\]  
(3.7)

where \(E^{(0)}\) for the homogeneous peristatic bar is defined by Eq. (2.12). In so doing, Eq. (3.6) is more preferable to solve than Eq. (3.1) (see, e.g., Weckner and Abeyaratne, 2005).

4. Effective modulus of heterogeneous bar

4.1. Local elasticity

For the local elasticity theory, the constitutive law has the form

\[
\sigma(x) = E(x)\varepsilon(x), \quad \varepsilon(x) = E(x)\sigma(x),
\]  
(4.1)

where \(E, E^{(0)}\) are macroscale Young’s moduli and \(\varepsilon, \varepsilon^{(0)}\), while \(\sigma, \sigma^{(0)}\) are corresponding stress fields. Averages \((\varepsilon)\) and \((\sigma)\) are taken over volumes \(\mathcal{V}\) in the entire domain or its subsets, respectively.
where \( \hat{E}(x) := E^{-1}(x) \). For statistically homogeneous medium subjected to the homogeneous loading, the statistical averages of the stresses and strains are invariant to the translation \( \langle \sigma \rangle(x) = \langle \sigma \rangle \equiv \text{const.}, \quad \langle \varepsilon \rangle(x) = \langle \varepsilon \rangle \equiv \text{const.} \), and, due to the problem’s linearity, connected by the linear relation
\[
\langle \sigma \rangle = \hat{E}(x) \langle \varepsilon \rangle, \quad \langle \varepsilon \rangle = \hat{E}^{-1}(\langle \sigma \rangle),
\]
where \( \hat{E} = (\hat{E}(x))^{-1} \).

We will reproduce the well known exact solutions of \( \hat{E} \) and \( \hat{E} \) in the form which is the most convenient for the subsequent generalization to the peristatic heterogeneous bar. For both 2D and 3D cases of the local elasticity the problem of estimation of effective moduli is in general cannot be exactly solved. In 1D case considered, the stress field is constant for any heterogeneous (even statistically inhomogeneous) bar
\[
\sigma(x) = \langle \sigma \rangle \equiv \text{const.},
\]
while the other two parameters \( \varepsilon(x) \) and \( E(x) \) [or \( \hat{E}(x) \)] vary. In so doing, a deformation in any point is homogeneous \( (x \in \nu^0), \quad k = 0, 1 \)
\[
\varepsilon(x) \equiv \varepsilon^0 = \langle \varepsilon \rangle \equiv \text{const.}, \quad u(x) = \langle u \rangle(x) \equiv \varepsilon^0 x,
\]
in a coordinate system connected with the centers of either the inclusions \( (x \in \nu^1) \) or the segments of the matrix \( (x \in \nu^0) \). Because of this, a comparison of a simple estimation of a statistical average of Eq. (4.12) leads to the exact effective elastic modulus representation
\[
\langle \varepsilon \rangle = \langle \hat{E} \rangle \langle \sigma \rangle, \quad \hat{E} = \langle \hat{E} \rangle,
\]
while extraction of \( \langle \varepsilon \rangle \) from \( (\hat{E}(x)\langle \varepsilon \rangle(x)) \) in the averaged Eq. (4.11) requires more detailed consideration of the strain concentration factors inside the inclusions. Indeed, the averaged Eq. (4.11) at \( E(x) = E^0 \sqrt{\nu(x)} \) \( (x \in \nu^0), \quad k = 0, 1 \) can be presented in the form
\[
\langle \sigma \rangle = E^0 \langle \varepsilon \rangle + \langle \varepsilon \rangle \tau,
\]
where \( \langle \varepsilon \rangle \tau \) is the strain polarization parameter averaged over \( R = (-\infty, \infty) \)
\[
\langle \varepsilon \rangle \tau(x) = (E(x) - E^0) \langle \varepsilon \rangle(x).
\]

Eq. (4.6) can be exactly solved in an accompany with the exact equations
\[
\langle \varepsilon \rangle = E^0 \varepsilon^0 + C_1 E^1, \quad E^1 \equiv \langle \varepsilon \rangle \tau = (E^1 - E^0) \langle \varepsilon \rangle = \hat{E} \langle \varepsilon \rangle,
\]
that leads to the exact representation for the effective compliance \( \hat{E} \) [4.52] presented for a two phase bar in the form
\[
\hat{E}^{-1} = C^1 \hat{E}^{-1} + C^0 \hat{E}^{-1}.
\]

It is interesting to consider the exact results obtained (4.5)-(4.9) in the light of some basic assumptions and concepts widely used in 2D and 3D problems of micromechanics of composites (see for references and details Buryachenko, 2007). So, the exact equality (4.3) coincides with the Reuss, 1929 assumption necessarily leading to the representation (4.52). From other side, homogeneity of strain fields inside the inclusions conjures up the idea to consider the effective field \( \bar{\varepsilon}_i \) concept (see for details Buryachenko, 2007) when a behavior of a representative inclusion \( v_i \) in a composite material is equivalent to the behavior of this inclusion \( v_i \) inside the infinite homogeneous matrix with the modulus \( E^0 \) subjected to the homogeneous field \( \bar{\varepsilon}_i \). One can concludes from Eqs. (3.13) that this effective field is given by
\[
\bar{\varepsilon}_i = E^0 \langle \sigma \rangle E \frac{\bar{E}}{E^0} \langle \bar{\varepsilon} \rangle,
\]
where the equality (4.10) is in fact the Mori and Tanaka, 1973 approximation (MTA) according to which the effective field \( \bar{\varepsilon}_i \) coincides with the volume average of the strain inside the matrix. In so doing Eq. (4.10) is equivalent to the dilute approximation where each inclusion behaves as an isolated one inside the infinite homogeneous matrix. Furthermore, Eqs. (4.41) imply that the quasi-crystalline approximation by Lax (1952) is exactly fulfilled. Acceptance of the quasi-crystalline approximation for statistically homogeneous fields of identical inclusions, in turns, leads to an equivalence between the multiparticle effective field method, the method of effective field, and the Mori and Tanaka (1973) method (see for details Buryachenko, 2007). This seemingly unusual coincidence of results obtained by the different methods is explained by the exact conditions (4.3) and (4.4).

Obtaining of the exact representation (4.9) essentially uses a statistic homogeneity of the bar and the homogeneity of inclusions. If either one or both mentioned conditions are violated then two step operation (2.17) is preferable that enables one to perform averaging of Eq. (4.11) in a spirit of the dilute approximation (4.10) as
\[
\langle \sigma \rangle(z) = E^0 \langle \varepsilon \rangle(z) + \int_{z-a}^{z+a} n(r) \frac{1}{2a} \int_{r-a}^{r+a} E^0 \langle \sigma \rangle(z) - E^0 \langle \sigma \rangle \langle \varepsilon \rangle ds dr,
\]
which is reduced to the result (4.52) for statistically homogeneous bar \( n(r) \equiv \text{const.} \).

4.2. Peristatic heterogeneous bar

For a randomly heterogeneous bar at the fixed \( (x, \tilde{x}), C(x, \tilde{x}) \) is also random, and Eq. (2.8) can be recast in the form
\[
\langle \sigma \rangle(z) = \int_{y = -a}^{y = a} \left\{ \int_{y = -a}^{y = a} C^0(x, \tilde{x}) |u(x) - u(\tilde{x})| dy \right\} dx
\]
\[
+ \int_{y = -a}^{y = a} \left\{ \int_{x = -a}^{x = a} C_1(x, \tilde{x}) |u(x) - u(\tilde{x})| dx \right\} dy.
\]

A statistically homogeneous \( n(x) = n \equiv \text{const.} \) bar subjected to the remote homogeneous loading behaves as a macroscopically homogeneous bar subjected to the homogeneous loading: \( \langle \sigma \rangle(z) = \langle \sigma \rangle \equiv \text{const.} \) and \( \langle u \rangle(x) = \langle \sigma \rangle x \langle \varepsilon \rangle \equiv \text{const.} \). Therefore statistical average of Eq. (4.12) leads to
\[
\langle \sigma \rangle = \int_{y = -a}^{y = a} \left\{ \int_{y = -a}^{y = a} C^0(x, \tilde{x}) |u(x) - u(\tilde{x})| dy \right\} dx
\]
\[
+ \int_{y = -a}^{y = a} \left\{ \int_{x = -a}^{x = a} C_1(x, \tilde{x}) |u(x) - u(\tilde{x})| dx \right\} dy.
\]

In the first integrals in Eq. (4.13) \( u(x) \) and \( u(\tilde{x}) \) are only random while all the other functions (such as \( C^0(x, \tilde{x}) \) and the integration limits) are deterministic at the fixed \( x \) and \( \tilde{x} \). Because of this, analogously to the averaged Eq. (4.12), a deterministic function \( C^0(x, \tilde{x}) \) can be carried out from the averaging operation \( C^0(x, \tilde{x}) |u(x) - u(\tilde{x})| \). Then the first integral in Eq. (4.13) can be transformed similarly to reduction of Eq. (4.11) to Eq. (4.6)
\[
\langle \sigma \rangle = E^0 \langle \varepsilon \rangle + \int_{y = -a}^{y = a} \left\{ \int_{y = -a}^{y = a} [C_1(x, \tilde{x}) u(x)] - [C_1(x, \tilde{x}) u(\tilde{x})] dx \right\} dy.
\]

where \( E^0 \) is defined by Eq. (2.12) at \( C(\xi) \equiv C^0(x, \tilde{x}) \langle \xi - \tilde{x} \rangle \). It is interesting, that the reduction of Eq. (2.10) to Eq. (2.12) was obtained for a homogeneously loaded homogeneous bar while Eqs. (4.13) and (4.14) describe a behavior of the heterogeneous bar of statistically homogeneous random structure. Nevertheless
we have proved that appearing of the term $E^0(\varepsilon)$ in Eq. (4.14) is statistically exact. Buryachenko (2014) has obtained a generalization of Eq. (4.14) to 2D and 3D thermostatic composites. Unfortunately, a corresponding simplification of the term $\langle C_1(x, \lambda)u(\lambda) \rangle$ by taking $C_1(x, \lambda)$ out from the averaging operation is impossible, and it is necessary to estimate a strain distribution inside each representative moving inclusion $v_i$ with the center $r \in [a - z, a + z]$ (similarly to Eq. (2.17)) that can be done by two step averaging procedure in a like manner of Eq. (4.11)

$$
\langle \sigma \rangle = E^0(\varepsilon) + \frac{n}{2(a + l')} \int_{a - z}^{a + z} \int_{a - l}^{a + l} \int_{\mathcal{H}_{v_i}} \left\{ \left[ C_1(x, \lambda)u(\lambda) \right] \right\} dx ds dr,
$$

where the use of the dilute approximation (4.10a) presenting in fact a solution (3.10) for a homogeneous peristatic bar is justified. Comparison of Eq. (4.19) with Eq. (2.18) defining the effective modulus $E$ yields the final representation

$$
E' = E^0 \left[ 1 - \frac{nE^0}{2(a + l')} \int_{a - z}^{a + z} \int_{a - l}^{a + l} \int_{\mathcal{H}_{v_i}} \left\{ \frac{1}{2} \left[ C_1(x, \lambda)u_0(\lambda) - L_u(x) \right] \right\} dx ds dr \right]^{-1}.
$$

It should be mentioned that Eq. (4.20) is only exact in the limit of the horizon going to zero $l'/a \rightarrow 0$ because the dilute approximation (4.10a) is not exactly fulfilled (contrary to the local elasticity case) due to impossibility of overlapping of domains $v_i' \cap v_j' = \emptyset$ (4.18). In such a case, the peristatic horizon $l'$ is chosen to be smaller than the spacing separating the inclusions (this assumption was also used for the peridynamic composites of periodic structures, see Ali and Lipton, 2012). However, estimation of errors generated by the popular assumptions of micromechanics (see, e.g., (2.7), (2.8), (4.10), and (4.18)) is beyond the scope of the current publication (although the numerical errors generated by the integral estimations are analyzed at the consideration of Fig. 6). Accuracy improvement of the approach proposed is based on the abandonment of the used assumptions that is possible for specific mechanical properties of constituents. So in the framework of the new background of micromechanics proposed, Buryachenko and Brum (2012) analyzed elastically homogeneous media with statistically homogeneous field of identical aligned inclusions with stress free strain $f_0(x) \neq 0$. The residual stresses were estimated for 2D thermoelastic composite materials (CMs) with any prescribed numerical accuracy without any hypotheses; in so doing, the classical effective field hypothesis (similar to Eq. (4.10)) leads to the error 40% at the estimation of the inhomogeneous effective field. However, generalization of the mentioned results by Buryachenko and Brum (2012) to the thermoelastic CMs is beyond the scope of the current study.

Another dissimilarity between the results obtained by both the local elasticity and peristatics is that Eq. (4.9) (in contradiction to Eq. (4.20)) is invariant to the replacements $u^{(0)} \rightarrow u^{(1)}$, $E^{(0)} \rightarrow E^{(1)}$. Moreover, Eq. (4.19) looks similar to its local exact counterpart (4.11) which is also exact for statistically inhomogeneous bar. However, generalization of Eq. (4.19) to the case of statistical inhomogeneous structures is questionable because a reduction of the first integral in Eq. (4.13) to $E^{(0)}(\varepsilon)$ is only possible for a composite bar of statistically homogeneous random structure.

5. Numerical results

1D example of an infinite heterogeneous peristatic bar is analyzed for the next geometrical and mechanical properties of constituents. An infinitely long bar of a constant cross section $A = 1$ contains the randomly distributed identical inclusions of the size $2a$. At first, the micromodulus functions with the step-function profile (2.4i) with the same horizon $l/a = 0.02, 0.1, 0.25$ are considered for both the inclusions and matrix with the ratios of the corresponding elastic counterparts (2.13) $u^{(1)}/E^{(0)} = 3$ and $E^{(1)}/E^{(0)} = 0.3$. For a homogeneous bar, the parameters of loading $P/A = 1$ correspond to the homogeneous stress distribution of the local elastic counterpart (4.3) $\sigma(x) = 1$ with the displacement distribution $u(x)$ (3.12) $(-\infty < x < \infty)$. The midpoint quadrature scheme (3.8) is realized for the uniform grid with a constant step size $\Delta x = a/1000$ (corresponding to $l'/\Delta x = 20, 100, 250$, respectively); an accuracy of the numerical solution for a single inclusion in the infinite homogeneous bar is analyzed by comparison of results obtained for $\Delta x = l'/250$, $l'/100$ and $l'/20$.

At first a single inclusion $v_0(0)$ in the infinite homogeneous bar is considered for $\Delta x = a/1000$. The odd functions of the displacement perturbation $u_0(x)$ introduced by the inclusion are presented in Figs. 2 and 3 for $E^{(1)}/E^{(0)} = 3$ and $E^{(1)}/E^{(0)} = 0.3$, respectively, and $l'/a = 0.02, 0.1, 0.25$. The curves $u_0(x)$ vs $x/a$ behave according to the general features of peristatic solutions for a homogeneous peristatic bar mentioned by Silling et al. (2003), and Bobaru et al. (2009),
Indeed, the fictitious body force \( b^f(x) = C + u^f(x) \) (3.6) has a discontinuity at \( x = \pm a \) and, therefore, \( u_l(x) \) also has a discontinuity at \( x = \pm a \) (that is visualized in Figs. 2 and 3 in the forms of the sharp variations of the functions \( u_l(x) \) in the vicinity of \( x = a \)) because in general the displacement field \( u(x) \) has the same smoothness as the body force field (see Silling et al., 2003). In addition, the discontinuity in the micromodulus \( C(\xi) \) at \( |\xi| = l_c \) has a further effect on the smoothness of \( u_l(x) \) with a corresponding discontinuity of the derivatives \( u_n^{(k)}(x) \) of the order \( k = 1, 2, \ldots \) at \( x = \pm (a+k \cdot l_c) \) [it is well observed in Figs. 2 and 3 in the form of breaks of \( u_l(x) \) \((k = 0) \) at \( x = a \) and \( k \cdot l_c \). As it was mentioned in Section 4, \( u_l(x) \rightarrow \text{const.} \) at \( x \rightarrow \infty \), and \( u_l(x) \rightarrow u_l(3.12) \) in the limit of the horizon going to zero \( l_c/a \rightarrow 0 \).

For the same problem, we consider now the peristatic stresses \( \sigma(x) \) (3.14) presented in Figs. 4 and 5 for \( E^{(1)}/E^{(0)} = 3 \) and \( E^{(1)}/E^{(0)} = 0.3 \), respectively. An inhomogeneity of the peristatic stresses \( \sigma(x) \) in comparison with the local elastic limit \( ^c \sigma(x) \equiv \text{const.} \) reflects a nonlocal nature of peristatic phenomena. As can be seen, the inhomogeneities of \( \sigma(x) \) at \( l_c/a \rightarrow 0 \) are localized in the vicinity of discontinuities \( x = a \) of displacements \( u_l(x) \), and a width of the “hills” \( \sigma(x) \sim x \) becomes thinner at \( l_c/a \rightarrow 0 \) while a high of these “hills” at \( x = a \) is not diminished.

It is interesting to analyze a dependence of a long-range action of

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**Fig. 2.** Displacement perturbations \( u_l(x) \) vs \( x/a \) estimated for \( E^{(1)}/E^{(0)} = 3 \) and \( l_c/a = 0 \) (1), \( l_c/a = 0.02 \) (2), \( l_c/a = 0.1 \) (3), and \( l_c/a = 0.25 \) (4).

**Fig. 3.** Displacement perturbations \( u_l(x) \) vs \( x/a \) estimated for \( E^{(1)}/E^{(0)} = 0.3 \) and \( l_c/a = 0 \) (1), \( l_c/a = 0.02 \) (2), \( l_c/a = 0.1 \) (3), and \( l_c/a = 0.25 \) (4).

**Fig. 4.** Peristatic stresses \( \sigma(x) \) vs \( x/a \) estimated for \( E^{(1)}/E^{(0)} = 3 \) and \( l_c/a = 0 \) (1), \( l_c/a = 0.02 \) (2), \( l_c/a = 0.1 \) (3), and \( l_c/a = 0.25 \) (4).

**Fig. 5.** Peristatic stresses \( \sigma(x) \) vs \( x/a \) estimated for \( E^{(1)}/E^{(0)} = 0.3 \) and \( l_c/a = 0 \) (1), \( l_c/a = 0.02 \) (2), \( l_c/a = 0.1 \) (3), and \( l_c/a = 0.25 \) (4).

**Fig. 6.** Peristatic stresses \( \sigma(x) \) vs \( x/a \) estimated for: elastic case (1); \( (2.131) \), \( \Delta x = l_c/250 \); \( (2.132) \), \( \Delta x = l_c/250 \); \( (2.133) \), \( \Delta x = l_c/5 \).
nonlocal stress constituents $\sigma(x) - \sigma(x)$ on the material stiffness of the phases $\nu^{(0)}$ and $\nu^{(1)}$ for the same their horizons $l_i$. It is visible that for $l_i/a = 0.02, 0.1$ the curves $\sigma(x) \sim x$ are not symmetric with respect to $x = a$ and the distributions $\sigma(x) - \sigma(x)$ are displaced into a side of stiffer materials (to the left hand side and to the right hand side for $E^{(1)} > E^{(0)}$ and $E^{(1)} < E^{(0)}$, respectively), i.e. a stiffer material exhibits most long-range action of nonlocal stress constituents in comparison with a softer material.

As an example of parametric analysis, we compare the peristatic stresses $\sigma(x)$ estimated for the micromoduli (2.41) and (2.42) with the same horizons $l_i$, the parameters $C$ corresponding (2.131) and (2.132), respectively, and the ratio $E^{(1)}/E^{(0)} = 3$. As can be seen in Fig. 6, both the peristatic stresses $\sigma(x)$ evaluated for Eqs. (2.131) and (2.132), respectively, differ from the local elastic stresses $\sigma(x) \equiv \text{const}$, just on a few percents. However, this similitude becomes much more meaningful when the comparison is only performed for the relative (nonlocal) constituent of stresses $\bar{\sigma}(x) = \sigma(x) \sigma(x)$, in such a case $\bar{\sigma}(x)$ estimated for the micromoduli (2.131) and (2.132) differ on 13% (see Fig. 6). In addition, the triangle profile (2.132) provides in some average sense the thinner and higher distribution than the corresponding step one (2.131) with the same $E$ and $l_i$ (see Fig. 1). As a result, the “hill” of the peristatic stresses $\sigma(x) \sim x$ estimated for the triangle $C(z)$ (2.132) is thinner and higher than the “hill” corresponding to the step-function $C(z)$ (2.131). We also perform a comparative analysis of $\bar{\sigma}(x)$ estimated for the same horizons $l_i/a = 0.25$ with the different step sizes $\Delta x = a/1000$ and $\Delta x = a/20$ corresponding to the values $l_e = 250a$ and $l_e = 5a$. An error of the coarse mesh $\Delta x = l_i/5$ at the estimation of $\bar{\sigma}(x)$ equals 47%; this error diminished to 8% and 2% for more fine meshes $\Delta x = l_i/20$ and $\Delta x = l_i/50$, respectively. The errors at the estimation of other values (such as $u_i(x), \tau(x)$, and $E'$) essentially less than the corresponding errors at the evaluation of the relative parameter $\bar{\sigma}(x)$.

The classical problem of micromechanics is estimation of effective elastic properties which are directly defined by the stress polarization parameters (4.17) rather than the stress distribution (3.14). Figs. 7 and 8 demonstrate convergence of the peridynamic solutions to that of the classical, local elasticity solution (4.7) in the limit of vanishing length scale $l_i/a \rightarrow 0$. The peristatic stress polarizations $\bar{\tau}(x)$ (4.17) presented in Figs. 6 and 7 for $E^{(1)}/E^{(0)} = 3$ and $E^{(1)}/E^{(0)} = 0.3$, respectively, have the compact support $x \in \nu_i$ while the local elastic counterparts $\tau^{(0)}(x)$ vanish outside $v_i \subset \nu_i$. Moreover, $\bar{\tau}(x)$ varies at $x \in \nu_i$ (4.17) while $\tau^{(0)}(x) \equiv \text{const}$ at $x \in \nu_i$.
by Eq. (4.20), see Figs. 9 and 10. The limit of vanishing length scale $l_0/a \to 0$, reduces the peristatic solution $E^* \sim c (4.20)$ to the classical exact result $\frac{1}{l_0} E^* \sim c (4.9)$. It is interesting, that taking the non-local peristatic properties of constituents into account leads to the softening and stiffening of $E^*$ in comparison with $\frac{1}{l_0} E^*$ for the composite bar with the stiff (Fig. 9) and soft (Fig. 10) inclusions, respectively.

It should be mentioned that Buryachenko (2011a,b) analyzed the 2D composites consisting of constituents with another non-local properties introduced by Kröner (1967) and Eringen (2002). It was demonstrated that for statistically homogeneous composites subjected to the homogeneous loading, the effective properties are described by the classical local elasticity theory and, moreover, the effective properties in both the nonlocal and local theories do not significantly differ from one another while the local fields estimated by both the nonlocal and local theories can differ by a sign. In a similar manner, the effective moduli for the composite bars with the nonlocal and local properties do not significantly differ from one another (see Figs. 9 and 10) while the local fields (see Figs. 7 and 8) can be essentially different. However, the current estimations (4.20) are obtained in the framework of the dilute approximation when the direct long-range actions between inclusions vanish that is appropriate for 1D case (see Section 4.1). In subsequent publications, the author expects obtaining of fundamentally new results based on consideration of multiparticle interactions of inclusions in 2D and 3D cases.

Acknowledgments

This work was partially funded by IllinoisRocstar LLC. Both the helpful comments of the reviewers and their encouraging recommendations are gratefully acknowledged.

References