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Inequivalent representations of ternary matroids

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Abstract

This paper considers representations of ternary matroids over fields other than GF(3). It is shown that a 3-connected ternary matroid representable over a finite field F has at most $|F|-2$ inequivalent representations over F. This resolves a special case of a conjecture of Kahn in the affirmative.

1. Introduction

Let F be a finite field. Kahn [2] has conjectured that there exists a positive integer k such that a 3-connected matroid M representable over F has no more than k inequivalent representations. The conjecture is certainly true if M is binary, since Brylawski and Lucas [1] have shown that a binary matroid representable over a field is uniquely representable over that field. In this paper we prove that Kahn's conjecture is true when M is ternary. Since it is known [1] that ternary matroids are uniquely representable over GF(3), this result may not surprise the reader. The point is that, in contrast to the binary case, if F is not GF(3), then a ternary matroid M representable over F may have a number of inequivalent representations over F.

It is also the case that the result no longer holds if the condition that M be 3-connected is dropped. For example let M_n denote a matroid obtained from n copies of $U_{2,4}$ by repeatedly taking 2-sums. That is,

$$M_n = \underbrace{U_{2,4}\oplus_2 \cdots \oplus_2 U_{2,4}}_{n\text{-terms}}.$$

It is easily seen that M_n is representable over any field other than GF(2). Moreover, it is not hard to show that if q is a prime power greater than 3, then M_n has at least 2^{n-1} inequivalent representations over GF(q).

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The problem which led to the research for this paper is that of characterising the class of matroids representable over both GF(3) and the rationals. There are a number of natural conjectures concerning this class, one being that the class consists of all matroids with a matrix representation over the rationals where all subdeterminants are in $\{0, \pm 2^i;\ i \text{ an integer}\}$. A major difficulty with these conjectures is the existence of inequivalent representations of ternary matroids over the rationals. Very loosely speaking, the technique used in this paper is to show that the 'space' of inequivalent representations of a 3-connected ternary matroid representable over a field F has at most one degree of freedom. In other words, representations are unique up to a single parameter. The hope is that, when F is the rationals, it could be shown that a choice of this parameter exists for which the corresponding representation is of the desired type.

Since first writing this paper progress has been made on the problems mentioned above. It is now known [4] that Kahn's conjecture is false for all fields other than GF(2), GF(3), GF(4), and GF(5). The proof in [4] that Kahn's conjecture holds for GF(5) makes use of the results of this paper. It is also known [8] that the matroids representable over GF(3) and the rationals are indeed the ones having a matrix representation over the rationals where all subdeterminants are in $\{0, \pm 2^i : i \text{ an integer}\}$. Furthermore, in [9], matrix characterisations are given of the matroids representable over GF(3) and F where F is any given field. The results of this paper are used in the proofs of [8,9].

2. Main results

We assume some familiarity with the theory of matroid representations. For a good discussion of the theory, see [3].

Let M be a rank r matroid with n elements, and assume that M is representable over a field F. Fix an ordering of the ground set of M such that the first r elements form a basis. Consider matrix representations of M over F which respect this ordering. We shall say that an $r \times n$ matrix representing M over F is in *normal form*, if it is of the form $[I_r|A]$, where I_r is the $r \times r$ identity matrix and the first non-zero entry of each row and column of A is a 1. It is well known that every representation of M is equivalent to one in normal form, and, since the first r elements of M form a basis, it is clear that any representation of M which respects the ordering of the ground set is equivalent to one in normal form which also respects the ordering of the ground set.

We are interested in inequivalent representations. Now, if $r(M) > 2$ and F has no non-trivial automorphisms, then each equivalence class of representations contains exactly one member in normal form which respects the given ordering on the elements of M, so distinct representations in normal form characterise distinct equivalence classes of representations. However, if F has a non-trivial automorphism, then a distinct, but equivalent, representation of M in normal form can be obtained by replacing each entry of $[I|A]$ by its image under this automorphism. In this paper we focus on the representations in normal form, and it follows from the above that, in general, this is *not* the same as examining the distinct equivalence classes of representations. Of course, if the automorphism group of F is known, then a characterisation of the inequivalent representations of a matroid M representable over F follows quickly from a knowledge of the distinct representations in normal form.

Let S be the set of distinct representations in normal form of the rank r matroid M over the field F, and assume that $|E(M)| = n$. Then M has *at most one degree of freedom over* F if there exists a pair (i,j) with $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$ such that for all $\alpha \in F$, there is at most one matrix $A \in S$ with $a_{ij} = \alpha$. The definition given here is relative to a given ordering of the ground set of M, but it is easily seen that M has at most one degree of freedom over F relative to one ordering if and only if it has at most one degree of freedom relative to any other ordering. The following theorem is our main result.

Theorem 2.1. *If M is a 3-connected ternary matroid representable over the field F, then M has at most one degree of freedom over F.*

It follows immediately from Theorem 2.1 that if F is a finite field with q elements, then M has at most q inequivalent representations over F. In fact, M has at most $q-2$ inequivalent representations as we show in Corollary 2.8. We first establish some lemmas.

Let $n \geqslant 2$ be a positive integer. Following Tutte [7, p. 78], we define the *wheel* $\mathcal{W}_n$ to be the graph that is formed from an n-edge cycle C_n by adding a single new vertex and then joining this new vertex to each vertex of C_n by a single new edge. These new edges are the *spokes* of $\mathcal{W}_n$, and the edge set of C_n is the *rim* of $\mathcal{W}_n$. The rim C_n is a circuit-hyperplane of $M(\mathcal{W}_n)$, and the *whirl* $\mathcal{W}^n$ is obtained from $M(\mathcal{W}_n)$ by relaxing C_n, that is, by declaring C_n to be a basis and leaving the remaining bases the same. Note that $\mathcal{W}^2$ is the matroid $U_{2,4}$. The terms *rim* and *spoke* will be used in the obvious way in $\mathcal{W}^n$.

The following result is a straightforward consequence of Seymour's Splitter Theorem [6]. For a discussion of this theorem and its consequences see [3, Ch. 11].

Lemma 2.2. *Let $M(E)$ be a non-binary, 3-connected matroid. If M is not a whirl, there exists $x \in E$ such that either $M \backslash x$ or M/x is non-binary and 3-connected.* $\square$

Recall that $\overline{M}$ denotes the simple matroid canonically associated with M.

Lemma 2.3. *Let M be a 3-connected non-binary matroid with $r(M) \geqslant 3$. Then there is an element $x \in E(M)$ with the property that $\overline{M/x}$ is 3-connected and non-binary.*

Proof. Assume that M is a whirl, say $M = \mathcal{W}^r$. Then it is well known (and easily seen) that if x is a rim element of M, then $\overline{M/x}$ is isomorphic to $\mathcal{W}^{r-1}$, and this is a 3-connected non-binary matroid. Assume that M is not a whirl, and assume that the result holds for all 3-connected, non-binary rank r matroids with fewer elements than M. By Lemma 2.2 there is an element a in $E(M)$ such that either $M\backslash a$ or M/a is 3-connected and non-binary. If M/a is 3-connected and non-binary, we are done. Assume not, so that $M\backslash a$ is 3-connected and non-binary. Consider $M\backslash a$. It follows from the induction assumption that there is an element b in $M\backslash a$ such that $\overline{M\backslash a/b}$ is 3-connected and non-binary. A routine argument then shows that $\overline{M/b}$ is 3-connected and non-binary. □

The following lemma is closely related to results of Kahn [2, Section 3]. For a proof see [5, Lemma 3.2].

Lemma 2.4. *Let M be a 3-connected, non-binary, spanning submatroid of* PG$(r-1,3)$. *Then for any pair $\{p,q\}$ of distinct points of* PG$(r-1,3)$, *there is a hyperplane H of M such that the closure of H in* PG$(r-1,3)$ *contains p but not q.* □

The following corollary is an immediate consequence of Lemma 2.4.

Corollary 2.5. *Let M be a rank r, connected, ternary matroid with an element $x \in E(M)$ such that $M\backslash x$ is 3-connected and non-binary. If a is an element of M with the property that $\{a,x\}$ is independent, then there exists a hyperplane H of $M\backslash x$ such that $cl_M(H)$ contains x but not a.* □

Lemma 2.6. *Let M be a rank r connected ternary matroid representable over a field F with an element $x \in E(M)$ such that $M\backslash x$ is 3-connected and non-binary. Let A be a matrix representation of $M\backslash x$ over F in normal form which extends to a representation of M. Then there exists a unique vector $\boldsymbol{x} \in F^r$ such that $[A|\boldsymbol{x}]$ represents M in normal form.*

Proof. We proceed by induction on r, the rank of M. It is routinely shown that the lemma holds if $r \in \{1,2\}$. Assume that $r > 2$, and that the result holds for all matroids satisfying the conditions of the lemma whose rank is less than r.

Label the columns of A by their corresponding elements in $E(M\backslash x)$. Certainly there exists a vector $\boldsymbol{x}$ such that $[A|\boldsymbol{x}]$ represents M in normal form. Assume, for a contradiction, that there exists a distinct vector $\boldsymbol{x}'$ such that $[A|\boldsymbol{x}']$ also represents M in normal form. Since both representations are in normal form, it follows that $\boldsymbol{x}$ and $\boldsymbol{x}'$ are independent. Let N be the matroid on $E(M)\cup\{x'\}$ represented by $[A|\boldsymbol{x},\boldsymbol{x}']$, where $\boldsymbol{x}$ and $\boldsymbol{x}'$ represent x and x', respectively.

Now $M\backslash x$ is non-binary and 3-connected, so by Lemma 2.3 there exists $a \in E(M\backslash x)$ such that $\overline{M\backslash x/a}$ is non-binary and 3-connected. We now show that $\{a,x,x'\}$ is independent in N. Assume not. Then a, x, and x' are collinear. Since M is ternary, and $M\backslash x$ is 3-connected and non-binary, it follows, by Corollary 2.5 that there is a hyperplane H of $M\backslash x$ whose closure in M contains x but not a. Now H spans a hyperplane of N. This hyperplane contains x but not a. It follows that this hyperplane does not contain x' (for otherwise it would contain $\mathrm{cl}_N(\{x,x'\})$, and $\mathrm{cl}_N(\{x,x'\})$ contains a). This clearly contradicts the assumption that both $[A|\boldsymbol{x}]$ and $[A|\boldsymbol{x}']$ represent M. Therefore $\{a,x,x'\}$ is independent in N.

It follows from the above that $\{x,x'\}$ is independent in N/a. But $\overline{M\backslash x/a}$ is non-binary and 3-connected. This means that there exists a set S such that $M\backslash x/a\backslash S$ is a 3-connected rank $r-1$ matroid. Let A' be a normal-form representation of $N/a\backslash(S\cup\{x,x'\})$ which extends to a normal-form representation $[A'|\boldsymbol{y},\boldsymbol{y}']$ of $N/a\backslash S$ where $\boldsymbol{y}$ and $\boldsymbol{y}'$ represent x and x', respectively. By the induction hypothesis, $[A'|\boldsymbol{y}]$ and $[A'|\boldsymbol{y}']$ are normal-form represention of distinct matroids. This contradicts the fact that both of these matrices represent $M/a\backslash S$. Therefore $\boldsymbol{x} = \boldsymbol{x}'$ and the lemma is proved. □

Let F be a field and $n > 1$ be an integer. It is well known that if $F \neq \mathrm{GF}(2)$, then $\mathcal{W}^n$ is representable over F.

Lemma 2.7. *If $F \neq \mathrm{GF}(2)$, then $\mathcal{W}^n$ has at most one degree of freedom over F. Moreover, if F is finite, then there are at most $|F| - 2$ inequivalent representations of $\mathcal{W}^n$ over F.*

Proof. For $n \geqslant 2$, let A_n^α be the matrix over F defined by

$$A_n^\alpha = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & \alpha \end{bmatrix}.$$

It is easily seen that if $\alpha \notin \{0,(-1)^{n-1}\}$, then $[I_n|A_n^\alpha]$ represents $\mathcal{W}^n$ over F, where that columns of I_n and A_n correspond to a certain ordering of the spoke and rim elements of $\mathcal{W}^n$, respectively. This shows that there is a unique representation of $\mathcal{W}^n$ in normal form which respects the given ordering for each choice of α other than 0 or $(-1)^{n-1}$. Therefore $\mathcal{W}^n$ has at most one degree of freedom over F. If F is finite, then there are $|F| - 2$ representations in normal form. It follows that there are at most $|F| - 2$ inequivalent representations of $\mathcal{W}^n$ over F. □

The reader may be forgiven for believing that if M is uniquely representable over F, then M has at most one degree of freedom over F. It is easily shown that this is indeed the case if $r(M) > 2$, but it is not the case if $r(M) = 2$. For example, $U_{2,5}$ is uniquely representable over the rationals. It is known [1, Theorem 3.7] that binary matroids representable over F are uniquely representable over F. It is not hard to see that in this case we can say in addition that binary matroids have at most one degree of freedom over F.

Proof of Theorem 2.1. It follows from the above discussion that the result holds if M is binary, so assume that M is non-binary. If M is a whirl the result follows from Lemma 2.7, so assume that M is not a whirl. Assume that the result holds for all matroids satisfying the conditions of the theorem whose ground set has cardinality less than that of M. By Lemma 2.2, there is an element $x \in E(M)$ such that either $M\backslash x$ or M/x is 3-connected and non-binary. It is evident that M has at most one degree of freedom over F if and only if M^* does. This means that we may assume without loss of generality that $M\backslash x$ is non-binary and 3-connected. By the induction assumption, $M\backslash x$ has at most one degree of freedom over F. By Lemma 2.6 each representation of $M\backslash x$ which extends to a representation of M extends uniquely to such a representation. It now follows routinely that M has at most one degree of freedom over F. □

Corollary 2.8. *If M is a 3-connected ternary matroid representable over the finite field* GF(q), *then M has at most $q-2$ inequivalent representations over* GF(q).

Proof. The arguments of the proof of Theorem 2.1 are easily adapted to show that the number of inequivalent representations of M over F is no greater than the number of inequivalent representations of a whirl over F. But, by Corollary 2.5, $\mathscr{W}^n$ has at most $q-2$ inequivalent representations over F. □

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