JOURNAL OF COMBINATORIAL THEORY (A) 18, 234–235 (1975)

Note

A Remark about Permutations

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Communicated by N. S. Mendelsohn

Received April 1, 1974

In [2] Ree proved a theorem about permutation groups by making use of a formula for the genus of Riemann surfaces. The purpose of this note is to provide a direct proof of Ree's theorem.

Let Ω be a finite set of n letters. For a permutation π on Ω define $v(\pi) = \sum_{i=1}^{k} (l_i - 1)$ where π is a product of disjoint cycles of lengths $l_1, \ldots, l_k$.

LEMMA 1. *Let T be a minimal set of transpositions generating a transitive group on Ω. Then*

(a) $|T| = n - 1$,

(b) *the product in any order of all elements of T is an n-cycle.*

Proof. Define a graph G_T on Ω by joining α to β whenever $(\alpha\beta) \in T$. The fact that T generates a transitive group means that G_T is connected. Moreover G_T is a tree because of the minimality of T. Now (*a*) is immediate.

Let $\tau_1, \ldots, \tau_{n-1}$ be the elements of T. We will show by induction on $|\Omega|$ that $\tau_1 \cdots \tau_{n-1}$ is an n-cycle. Of course we may rearrange the order of $\tau_1, \ldots, \tau_{n-1}$ by any cyclic permutation, so we may assume $\tau_1 = (\alpha\beta)$ where α is joined only to β in G_T. Thus $G_{T-\{\tau_1\}}$ is a tree on $\Omega - \{\alpha\}$. By induction $\tau_2 \cdots \tau_{n-1}$ is an $(n-1)$-cycle $(\beta\gamma \cdots)$. So $\tau_1\tau_2 \cdots \tau_{n-1} = (\beta\alpha\gamma\cdots)$ is an n-cycle as required.

* This work was partially supported by the N.S.F., and represents independent work of the second author, and the first and third author in collaboration.

REE'S THEOREM. *Let* $\pi_1, \ldots, \pi_m$ *be permutations on* Ω *such that* $\pi_1 \cdots \pi_m = 1$. *Then*

$$v(\pi_1) + \cdots + v(\pi_m) \geqslant 2(n - s),$$

where s *is the number of orbits of the group generated by* $\pi_1, \ldots, \pi_m$.

Proof. As Ree observes it clearly suffices to settle the case $s = 1$. We may also assume each π_i is a cycle, for if $\pi = \rho_1 \cdots \rho_k$ is a product of disjoint cycles, we have $v(\pi) = v(\rho_1) + \cdots + v(\rho_k)$. Finally if $\rho = (\alpha_1 \cdots \alpha_k)$ is a k-cycle, then $\rho = \tau_1 \cdots \tau_{k-1}$ where $\tau_i = (\alpha_1 \alpha_{i+1})$, and $v(\rho) = k - 1 = v(\tau_1) + \cdots + v(\tau_{k-1})$. So we may even assume that each π_i is a transposition.

Select from $\pi_1, \ldots, \pi_m$ a minimal subset $\pi_{i_1}, \ldots, \pi_{i_r}$ which generates a group transitive on Ω. By Lemma 1 $r = n - 1$ and $\pi_{i_1} \cdots \pi_{i_r}$ is an n-cycle. From the equation $\pi_1 \cdots \pi_m = 1$ we obtain, by moving $\pi_{i_1}, \ldots, \pi_{i_{n-1}}$ to the left, an equation $\pi_{i_1} \cdots \pi_{i_{n-1}} \pi_n' \cdots \pi_m' = 1$ where the π_j''s are conjugates of those π_i's not among $\pi_{i_1}, \ldots, \pi_{i_{n-1}}$.

Since $\pi_{i_1} \cdots \pi_{i_{n-1}}$ is an n-cycle, its inverse $\pi_n' \cdots \pi_m'$ is also an n-cycle. Thus $\pi_n', \ldots, \pi_m'$ generate a transitive group and so $m - n + 1 \geqslant n - 1$ by part (a) of Lemma 1. Hence $v(\pi_1) + \cdots + v(\pi_m) = m \geqslant 2(n - 1)$ and the proof is complete.

The function $v(\pi)$ has arisen previously in connection with Riemann surfaces. See for instance [1, Definition 3].

The following lemma provides alternative descriptions of $v(\pi)$.

Let $l(\pi)$ be the smallest integer l such that π is a product of l transpositions.

Let $t(\pi)$ be the multiplicity of 1 as a characteristic value of π considered as a permutation matrix.

LEMMA 2. $l(\pi) = v(\pi) = n - t(\pi)$.

Proof. Clearly $v(\pi) + t(\pi) = n$. Since a k-cycle is a product of $k - 1$ transpositions we see that $l(\pi) \leqslant v(\pi)$. Each transposition has the characteristic value 1 with multiplicity $n - 1$. Thus $t(\pi) \geqslant n - l(\pi)$. Therefore,

$$l(\pi) \leqslant v(\pi) = n - t(\pi) \leqslant l(\pi),$$

as required.

REFERENCES

1. M. FRIED, On a Conjecture of Schur, *Mich. Math. J.* **17** (1970), 41–55.
2. R. REE, A Theorem on Permutations, *J. Combinatorial Theory* **10** (1971), 174–175.