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Modules with finite Cousin cohomologies have uniform local cohomological annihilators

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Abstract

Let A be a Noetherian ring. It is shown that any finite A-module M of finite Krull dimension with finite Cousin complex cohomologies has a uniform local cohomological annihilator. The converse is also true for a finite module M satisfying (S_2) which is over a local ring with Cohen–Macaulay formal fibers.

Keywords: Cousin complexes; Local cohomology

1. Introduction

Throughout let A denote a commutative Noetherian ring and M a finite (i.e. finitely generated) A-module. Recall that an A-module M is called *equidimensional* (or unmixed) if $\mathrm{Min}_A(M) = \mathrm{Assh}_A(M)$ (i.e. for each minimal prime $\mathfrak{p}$ of $\mathrm{Supp}_A(M)$, $\dim_A(M) = \dim(A/\mathfrak{p})$). For an ideal $\mathfrak{a}$ of A, write $\mathrm{H}^i_{\mathfrak{a}}(M)$ for the ith local cohomology module of M with support in $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(A)\colon \mathfrak{p} \supseteq \mathfrak{a}\}$. An element $x \in A$ is called a *uniform local cohomological annihilator* of M if $x \in A \setminus \bigcup_{\mathfrak{p} \in \mathrm{Min}_A(M)} \mathfrak{p}$ and for each maximal ideal $\mathfrak{m}$ of A, $x\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for all $i < \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$. The existence of a local cohomological annihilator is studied by Hochster and Huneke [6] and

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proved its importance for the existence of big Cohen–Macaulay algebras and a uniform Artin–Rees theorem [7].

In [12], Zhou studied rings with a uniform local cohomological annihilator. Hochster and Huneke, in [5], proved that if A is locally equidimensional (i.e. $A_{\mathfrak{m}}$ is equidimensional for every maximal ideal $\mathfrak{m}$ of A) and is a homomorphic image of a Gorenstein ring of finite dimension, then A has a strong uniform local cohomological annihilator (i.e. A has an element which is a uniform local cohomological annihilator of $A_{\mathfrak{p}}$ for each $\mathfrak{p} \in \operatorname{Spec}(A)$). In [12], Zhou showed that if a locally equidimensional ring A of positive dimension is a homomorphic image of a Cohen–Macaulay ring of finite dimension (or an excellent local ring), then A has a uniform local cohomological annihilator.

Cousin complexes were introduced by Hartshorne in [4] and have a commutative algebra analogue given by Sharp in [10]. Recently, Cousin complexes have been studied by several authors. In [2,3] and [8], Dibaei, Tousi, and Kawasaki studied finite Cousin complexes (i.e. the Cousin complexes with finitely generated cohomologies). In [9, Proposition 9.3.5], Lipman, Nayak, and Sastry generalized these results to complexes on formal schemes.

In Section 2, it is proved that any finite A-module of finite Krull dimension with finite Cousin complex cohomologies has a uniform local cohomological annihilator (Theorem 2.7). As a result it follows that if $(A, \mathfrak{m})$ is local, satisfies Serre's condition (S_2), and such that all of its fibers of $A \to \widehat{A}$ are Cohen–Macaulay, then A has a uniform local cohomological annihilator (Corollary 2.10). For a finite module M over a local ring $(A, \mathfrak{m})$ satisfying (S_2) and with Cohen–Macaulay formal fibers, it is proved that the following conditions are equivalent: (i) $\widehat{M}$, the completion of M with respect to $\mathfrak{m}$-adic topology, is equidimensional; (ii) $\mathcal{C}_A(M)$, the Cousin complex of M is finite; (iii) M has a uniform local cohomological annihilator (Theorem 2.13).

In Section 3, for certain modules M, the relationship between the cohomology modules of the Cousin complex of M and the local cohomology modules of M with respect to an arbitrary ideal of A is studied. It is shown that the M-height of $\mathfrak{a}$ is equal to the infimum of numbers r for which $0 :_A \mathrm{H}^r_{\mathfrak{a}}(M)$ does not contain the product of all the annihilators of the Cousin cohomologies of M (Theorem 3.2).

2. Cousin complexes

Let M be an A-module and let $\mathcal{H} = \{H_i\colon i \geqslant 0\}$ be the family of subsets of $\operatorname{Supp}_A(M)$ with $H_i = \{\mathfrak{p} \in \operatorname{Supp}_A(M)\colon \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geqslant i\}$. The family $\mathcal{H}$ is called the M-height filtration of $\operatorname{Supp}_A(M)$. Define the Cousin complex of M as the complex

$$\mathcal{C}_A(M)\colon \quad 0 \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \longrightarrow \cdots, \qquad (*)$$

where $M^{-1} = M$, $M^i = \bigoplus_{\mathfrak{p} \in H_i \setminus H_{i+1}} (\operatorname{Coker} d^{i-2})_{\mathfrak{p}}$ for $i > -1$. The homomorphism $d^i : M^i \to M^{i+1}$ has the following property: for $m \in M^i$ and $\mathfrak{p} \in H_i \setminus H_{i+1}$, the component of $d^i(m)$ in $(\operatorname{Coker} d^{i-1})_{\mathfrak{p}}$ is $\overline{m}/1$, where $\bar{\ }\colon M^i \to \operatorname{Coker} d^{i-1}$ is the natural map (see [10] for details).

Throughout, for the Cousin complex $(*)$, we use the following notations:

$$K^i := \operatorname{Ker} d^i, \qquad D^i := \operatorname{Im} d^{i-1}, \qquad H^i := K^i/D^i, \quad i = -1, 0, \ldots.$$

We call the Cousin complex $\mathcal{C}_A(M)$ *finite* if, for each i, the cohomology module H^i is finite. Recall that for an ideal $\mathfrak{a}$ of A and an A-module M, the M-height of $\mathfrak{a}$ is defined by $\operatorname{ht}_M(\mathfrak{a}) :=$

$\inf\{\dim M_{\mathfrak{p}}\colon \mathfrak{p} \in \operatorname{Supp}_A(M) \cap \mathrm{V}(\mathfrak{a})\}$. Note that $\operatorname{ht}_M(\mathfrak{a}) \geqslant 0$ whenever $M \neq \mathfrak{a}M$. If M is finitely generated then $\operatorname{ht}_M(\mathfrak{a}) = \operatorname{ht}(\frac{\mathfrak{a}+I}{I})$, where $I = \operatorname{Ann}_A(M)$.

We begin by the following lemma which for the first part we adopt the argument in [11, Theorem].

Lemma 2.1. *Let M be an A-module. For any integer k with $0 \leqslant k < \operatorname{ht}_M(\mathfrak{a})$, the following statements are true.*

(a) $H^s_{\mathfrak{a}}(M^k) = 0$ *for all integers* $s \geqslant 0$.
(b) $\operatorname{Ext}^s_A(A/\mathfrak{a}, M^k) = 0$ *for all integers* $s \geqslant 0$.

Proof. (a) Set $C_{k-1} := \operatorname{Coker} d^{k-2} = M^{k-1}/D^{k-1}$ so that $M^k = \bigoplus_{\substack{\mathfrak{p}\in \operatorname{Supp}_A(M)\\ \operatorname{ht}_M(\mathfrak{p})=k}} (C_{k-1})_{\mathfrak{p}}$. For each $k < \operatorname{ht}_M(\mathfrak{a})$ and each $\mathfrak{p} \in \operatorname{Supp}_A(M)$ with $\operatorname{ht}_M(\mathfrak{p}) = k$, there exists an element $x \in \mathfrak{a} \setminus \mathfrak{p}$. Thus the multiplication map $(C_{k-1})_{\mathfrak{p}} \xrightarrow{x} (C_{k-1})_{\mathfrak{p}}$ is an automorphism and so the multiplication map $\mathrm{H}^s_{\mathfrak{a}}((C_{k-1})_{\mathfrak{p}}) \xrightarrow{x} \mathrm{H}^s_{\mathfrak{a}}((C_{k-1})_{\mathfrak{p}})$ is also an automorphism for all integers s. One may then conclude that $\mathrm{H}^s_{\mathfrak{a}}((C_{k-1})_{\mathfrak{p}}) = 0$. Now, from additivity of local cohomology functors, it follows that $\mathrm{H}^s_{\mathfrak{a}}(M^k) = 0$.

(b) Assume in general that N is an A-module such that $\mathrm{H}^s_{\mathfrak{a}}(N) = 0$ for all $s \geqslant 0$. We show, by induction on i, $i \geqslant 0$, that $\operatorname{Ext}^i_A(A/\mathfrak{a}, N) = 0$. For $i = 0$, one has $\operatorname{Hom}_A(A/\mathfrak{a}, N) = \operatorname{Hom}_A(A/\mathfrak{a}, \mathrm{H}^0_{\mathfrak{a}}(N))$ which is zero. Assume that $i > 0$ and the claim is true for any such module N and all $j \leqslant i - 1$. Choose E to be an injective hull of N and consider the exact sequence $0 \to N \to E \to N' \to 0$, where $N' = E/N$. As $\mathrm{H}^0_{\mathfrak{a}}(E) = 0$, it follows that $\mathrm{H}^s_{\mathfrak{a}}(N') = 0$ for all $s \geqslant 0$. Thus $\operatorname{Ext}^{i-1}_A(A/\mathfrak{a}, N') = 0$, by our induction hypothesis. As, by the above exact sequence $\operatorname{Ext}^{i-1}_A(A/\mathfrak{a}, N') \cong \operatorname{Ext}^i_A(A/\mathfrak{a}, N)$, the result follows. □

The following technical result is important for the rest of the paper.

Proposition 2.2. *Let M be an A-module and let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a}M \neq M$. Then, for each non-negative integer r with $r < \operatorname{ht}_M(\mathfrak{a})$,*

$$\prod_{i=0}^{r}\bigl(0 :_A \operatorname{Ext}^{r-i}_A\bigl(A/\mathfrak{a}, H^{i-1}\bigr)\bigr) \subseteq 0 :_A \operatorname{Ext}^r_A(A/\mathfrak{a}, M).$$

Here $\prod$ is used for product of ideals.

Proof. For each $j \geqslant -1$, there are the natural exact sequences

$$0 \longrightarrow M^{j-1}/K^{j-1} \longrightarrow M^j \longrightarrow M^j/D^j \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow H^{j-1} \longrightarrow M^{j-1}/D^{j-1} \longrightarrow M^{j-1}/K^{j-1} \longrightarrow 0. \tag{2}$$

Let $0 \leqslant r < \operatorname{ht}_M(\mathfrak{a})$.

We prove by induction on j, $0 \leqslant j \leqslant r$, that

$$\prod_{i=0}^{j}\bigl(0 :_A \operatorname{Ext}^{r-i}_A\bigl(A/\mathfrak{a}, H^{i-1}\bigr)\bigr) \cdot \bigl(0 :_A \operatorname{Ext}^{r-j}_A\bigl(A/\mathfrak{a}, M^{j-1}/K^{j-1}\bigr)\bigr) \subseteq 0 :_A \operatorname{Ext}^r_A(A/\mathfrak{a}, M). \tag{3}$$

In case $j=0$, the exact sequence (2) implies the exact sequence

$$\operatorname{Ext}_A^r\big(A/\mathfrak{a}, H^{-1}\big) \longrightarrow \operatorname{Ext}_A^r(A/\mathfrak{a}, M) \longrightarrow \operatorname{Ext}_A^r\big(A/\mathfrak{a}, M^{-1}/K^{-1}\big)$$

so that

$$\big(0:_A \operatorname{Ext}_A^r\big(A/\mathfrak{a}, H^{-1}\big)\big) \cdot \big(0:_A \operatorname{Ext}_A^r\big(A/\mathfrak{a}, M^{-1}/K^{-1}\big)\big) \subseteq 0:_A \operatorname{Ext}_A^r(A/\mathfrak{a}, M)$$

and thus the case $j=0$ is justified.

Assume that $0 \leqslant j < r$ and formula (3) is settled for j. Therefore, by Lemma 2.1(b), formula (1) implies that

$$\operatorname{Ext}_A^{r-j}\big(A/\mathfrak{a}, M^{j-1}/K^{j-1}\big) \cong \operatorname{Ext}_A^{r-j-1}\big(A/\mathfrak{a}, M^j/D^j\big). \tag{4}$$

On the other hand the exact sequence (2) implies the exact sequence

$$\operatorname{Ext}_A^{r-j-1}\big(A/\mathfrak{a}, H^j\big) \longrightarrow \operatorname{Ext}_A^{r-j-1}\big(A/\mathfrak{a}, M^j/D^j\big) \longrightarrow \operatorname{Ext}_A^{r-j-1}\big(A/\mathfrak{a}, M^j/K^j\big),$$

from which it follows that

$$\big(0:_A \operatorname{Ext}_A^{r-j-1}\big(A/\mathfrak{a}, H^j\big)\big) \cdot \left(0:_A \operatorname{Ext}_A^{r-j-1}\left(A/\mathfrak{a}, \frac{M^j}{K^j}\right)\right) \subseteq 0:_A \operatorname{Ext}_A^{r-j-1}\left(A/\mathfrak{a}, \frac{M^j}{D^j}\right). \tag{5}$$

Now (4) and (5) imply that

$$\big(0:_A \operatorname{Ext}_A^{r-j-1}\big(A/\mathfrak{a}, H^j\big)\big) \cdot \left(0:_A \operatorname{Ext}_A^{r-j-1}\left(A/\mathfrak{a}, \frac{M^j}{K^j}\right)\right) \subseteq 0:_A \operatorname{Ext}_A^{r-j}\left(A/\mathfrak{a}, \frac{M^{j-1}}{K^{j-1}}\right). \tag{6}$$

From (6), it follows that

$$\begin{aligned}
&\prod_{i=0}^{j+1}\left(0:_A \operatorname{Ext}_A^{r-i}\left(\frac{A}{\mathfrak{a}}, H^{i-1}\right)\right) \cdot \left(0:_A \operatorname{Ext}_A^{r-j-1}\left(\frac{A}{\mathfrak{a}}, \frac{M^j}{K^j}\right)\right)\\
&\quad= \textstyle\prod_{i=0}^{j}\left(0:_A \operatorname{Ext}_A^{r-i}\left(\frac{A}{\mathfrak{a}}, H^{i-1}\right)\right) \cdot \left(0:_A \operatorname{Ext}_A^{r-j-1}\left(\frac{A}{\mathfrak{a}}, H^j\right)\right)\\
&\qquad \cdot \left(0:_A \operatorname{Ext}_A^{r-j-1}\left(\frac{A}{\mathfrak{a}}, \frac{M^j}{K^j}\right)\right)\\
&\quad\subseteq \prod_{i=0}^{j}\left(0:_A \operatorname{Ext}_A^{r-i}\left(\frac{A}{\mathfrak{a}}, H^{i-1}\right)\right) \cdot \left(0:_A \operatorname{Ext}_A^{r-j}\left(\frac{A}{\mathfrak{a}}, M^{j-1}/K^{j-1}\right)\right),
\end{aligned}$$

and, by the induction hypothesis (3), it follows that

$$\prod_{i=0}^{j+1}\big(0:_A \operatorname{Ext}_A^{r-i}\big(A/\mathfrak{a}, H^{i-1}\big)\big) \cdot \big(0:_A \operatorname{Ext}_A^{r-j-1}\big(A/\mathfrak{a}, M^j/K^j\big)\big) \subseteq 0:_A \operatorname{Ext}_A^r(A/\mathfrak{a}, M).$$

This is the end of the induction argument. Putting $j = r$ in (3) gives the result, because $\operatorname{Ext}^0_A(A/\mathfrak{a}, M^r) = 0$ by Lemma 2.1(b) and, as by (1) for $j = r$ there is an embedding $\operatorname{Ext}^0_A(A/\mathfrak{a}, M^{r-1}/K^{r-1}) \hookrightarrow \operatorname{Ext}^0_A(A/\mathfrak{a}, M^r)$, it follows that $\operatorname{Ext}^0_A(A/\mathfrak{a}, M^{r-1}/K^{r-1}) = 0$. □

An immediate corollary to the above result is the following.

Corollary 2.3. *Assume that M is a finite A-module and that $\mathfrak{a}$ is an ideal of A such that $\mathfrak{a}M \neq M$. Then, for each integer r with $0 \leqslant r < \operatorname{ht}_M(\mathfrak{a})$,*

$$\prod_{i=-1}^{r-1} \left(0 :_A H^i\right) \subseteq \bigcap_{i=0}^{r} \left(0 :_A \operatorname{Ext}^i_A(A/\mathfrak{a}, M)\right).$$

Proof. It follows by Proposition 2.2 and the fact that the extension functors are linear. □

Corollary 2.4. *Let M be a finite A-module of dimension n and let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a}M \neq M$. Assume that x is an element of A such that $xH^i = 0$ for all i. Then x^n annihilates all the modules $\operatorname{Ext}^r_A(A/\mathfrak{a}, M)$, $r = 0, 1, \ldots, \operatorname{ht}_M(\mathfrak{a}) - 1$ for all ideals $\mathfrak{a}$ of A.*

Proof. It follows clearly from Corollary 2.3 □

The following lemma states an easy but essential property of annihilators of Cousin cohomologies.

Lemma 2.5. *Assume that M is a finite A-module of finite $\dim_A(M) = n$ and that $\mathcal{C}_A(M)$ is finite, then $\bigcap_{i \geqslant -1}(0 :_A H^i) \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Min}_A(M)} \mathfrak{p}$.*

Proof. By [10, (2.7), vii], $\operatorname{V}(0 :_A H^i) = \operatorname{Supp}_A(H^i) \subseteq \{\mathfrak{p} \in \operatorname{Supp}_A(M)\colon \dim_{A_\mathfrak{p}}(M_\mathfrak{p}) \geqslant i + 2\}$ for all $i \geqslant -1$. Hence $(0 :_A H^i) \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Min}_A(M)} \mathfrak{p}$. Now Prime Avoidance Theorem implies that $\bigcap_{i \geqslant -1}(0 :_A H^i) \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Min}_A(M)} \mathfrak{p}$. □

We are now in a position to prove that the modules with finite Cousin complexes have uniform local cohomological annihilators. But one can state more.

Proposition 2.6. *Assume that M is a finite A-module of finite $\dim_A(M) = n$ and that $\mathcal{C}_A(M)$ is finite. Then there exists an element $x \in A \setminus \bigcup_{\mathfrak{p} \in \operatorname{Min}_A(M)} \mathfrak{p}$ such that $x \operatorname{Ext}^i_A(A/\mathfrak{m}^j, M) = 0$ for all $i < \operatorname{ht}_M(\mathfrak{m})$, all $j \geqslant 0$ and all maximal ideals $\mathfrak{m}$ in $\operatorname{Supp}_A(M)$.*

Proof. It follows by Lemma 2.5 and Corollary 2.4. □

Theorem 2.7. *Assume that M is a finite A-module of finite $\dim_A(M) = n$ and that $\mathcal{C}_A(M)$ is finite, then M has a uniform local cohomological annihilator.*

Proof. By Proposition 2.6, there is an element $x \in A \setminus \bigcup_{\mathfrak{p} \in \operatorname{Min}_A(M)} \mathfrak{p}$ such that $x \operatorname{Ext}^i_A(A/\mathfrak{m}^j, M) = 0$ for all $i < \operatorname{ht}_M(\mathfrak{m})$, all $j \geqslant 0$, and all maximal ideals $\mathfrak{m}$ in $\operatorname{Supp}_A(M)$. Choose a maximal ideal $\mathfrak{m}$ in $\operatorname{Supp}_A(M)$ and $i < \operatorname{ht}_M(\mathfrak{m})$. As $x \in \operatorname{Ann}_A(\operatorname{Ext}^i_A(A/\mathfrak{m}^j, M))$ for all j, we have $x \in \operatorname{Ann}_A(\varinjlim_j(\operatorname{Ext}^i_A(A/\mathfrak{m}^j, M)))$, i.e. $x\operatorname{H}^i_\mathfrak{m}(M) = 0$ for all $i < \operatorname{ht}_M(\mathfrak{m})$. □

Corollary 2.8. *Assume that A has finite dimension and that $\mathcal{C}_A(A)$ is finite. Then A has a uniform local cohomological annihilator, and so A is locally equidimensional and is universally catenary.*

Proof. It is clear from Theorem 2.6 and [12, Theorem 2.1]. □

In [12, Corollary 3.3], Zhou proved that any locally equidimensional Noetherian ring has a uniform local cohomological annihilator provided it is a homomorphic image of a Cohen–Macaulay ring of finite dimension. Here we have the following result:

Corollary 2.9. *Assume that $(A, \mathfrak{m})$ is local with Cohen–Macaulay formal fibers. Let M be a finite A-module such that it satisfies (S_2) and that $\operatorname{Min}_{\widehat{A}}(\widehat{M}) = \operatorname{Assh}_{\widehat{A}}(\widehat{M})$. Then M has a uniform local cohomological annihilator.*

Proof. By [2, Theorem 2.1], $\mathcal{C}_A(M)$ is finite. Now Theorem 2.7 implies the result. □

Corollary 2.10. *(Compare with [12, Corollary 3.3(i)].) Assume that $(A, \mathfrak{m})$ is local and that it satisfies (S_2) and all of its formal fibers are Cohen–Macaulay. Then A has a uniform local cohomological annihilator.*

Proof. See [2, Corollary 2.2]. □

Proposition 2.11. *Let M be a finite A-module such that it has a uniform local cohomological annihilator. Then M is locally equidimensional.*

Proof. Let $\mathfrak{m} \in \operatorname{Max}\operatorname{Supp}_A(M)$. We will show that $\dim_{A_\mathfrak{m}}(M_\mathfrak{m}) = \dim A_\mathfrak{m}/\mathfrak{p}A_\mathfrak{m}$ for all $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathfrak{p} \in \operatorname{Min}_A(M)$ and $\mathfrak{p} \subseteq \mathfrak{m}$. By assumption, there exists an element $x \in A \setminus \bigcup_{\mathfrak{p}\in\operatorname{Min}_A(M)} \mathfrak{p}$ such that $x\mathrm{H}^i_\mathfrak{m}(M) = 0$ for all $i < \dim_{A_\mathfrak{m}}(M_\mathfrak{m})$. As $x \in A_\mathfrak{m} \setminus \bigcup_{\mathfrak{p}A_\mathfrak{m}\in\operatorname{Min}_{A_\mathfrak{m}}(M_\mathfrak{m})} \mathfrak{p}A_\mathfrak{m}$, and $\mathrm{H}^i_\mathfrak{m}(M) \cong \mathrm{H}^i_{\mathfrak{m}A_\mathfrak{m}}(M_\mathfrak{m})$ by using the definition of local cohomology, we may assume that $(A, \mathfrak{m})$ is local with the maximal ideal $\mathfrak{m}$ and write $d := \dim_A(M)$.

Assume, to the contrary, that there exists $\mathfrak{p} \in \operatorname{Min}_A(M)$ with $c := \dim A/\mathfrak{p} < d$. Set $S = \{\mathfrak{q} \in \operatorname{Min}_A(M)\colon \dim A/\mathfrak{q} \leqslant c\}$ and $T = \operatorname{Ass}_A(M) \setminus S$. There exists a submodule N of M such that $\operatorname{Ass}_A(N) = T$ and $\operatorname{Ass}_A(M/N) = S$. Note that $\dim_A(M/N) = c$ and that $\dim_A(N) = d$. As $\sqrt{0 :_A N} = \bigcap_{\mathfrak{q}\in T} \mathfrak{q}$, it follows that there exists an element $y \in 0 :_A N \setminus \bigcup_{\mathfrak{q}\in S} \mathfrak{q}$. Thus, trivially, $y\mathrm{H}^i_\mathfrak{m}(N) = 0$ for all $i \geqslant 0$. The exact sequence $0 \to N \to M \to M/N \to 0$ implies the exact sequence $\mathrm{H}^i_\mathfrak{m}(M) \to \mathrm{H}^i_\mathfrak{m}(M/N) \to \mathrm{H}^{i+1}_\mathfrak{m}(N)$. As $x\mathrm{H}^i_\mathfrak{m}(M) = 0$ for all $i < d$, it follows that $xy\mathrm{H}^i_\mathfrak{m}(M/N) = 0$ for all $i < d$. In particular, $xy\mathrm{H}^c_\mathfrak{m}(M/N) = 0$. Thus $xy \in \bigcap_{\mathfrak{q}\in\operatorname{Assh}_A(M/N)} \mathfrak{q}$ (cf. [1, Proposition 7.2.11 and Theorem 7.3.2]). Therefore $xy \in \mathfrak{p}$ by the choice of $\mathfrak{p}$. As $\mathfrak{p} \in S \cap \operatorname{Min}_A(M)$, this is a contradiction. □

Now we can state the following result which partially extends Corollary 2.8.

Corollary 2.12. *Let M be a finite A-module such that its Cousin complex $\mathcal{C}_A(M)$ is finite. Then M is locally equidimensional.*

Proof. The proof is clear from Theorem 2.7 and Proposition 2.11. □

Now it is easy to provide an example of a module whose Cousin complex has at least one non-finite cohomology.

Example. Consider a Noetherian local ring A of dimension $d > 2$. Choose any pair of prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of A with conditions $\dim A/\mathfrak{p} = 2$, $\dim A/\mathfrak{q} = 1$, and $\mathfrak{p} \nsubseteq \mathfrak{q}$. Then $\mathrm{Min}_A(A/\mathfrak{pq}) = \{\mathfrak{p}, \mathfrak{q}\}$ and so $A/\mathfrak{pq}$ is not an equidimensional A-module and thus its Cousin complex is not finite.

We are now ready to present the following result which, for a finite module M, shows connections of finiteness of its Cousin complex, existence of a uniform local cohomological annihilator for M, and equidimensionality of $\widehat{M}$.

Theorem 2.13. *Let A be a local ring with Cohen–Macaulay formal fibers. Assume that M is a finite A-module which satisfies the condition (S_2) of Serre. Then the following statements are equivalent.*

(i) $\mathrm{Min}_{\widehat{A}}(\widehat{M}) = \mathrm{Assh}_{\widehat{A}}(\widehat{M})$.
(ii) *The Cousin complex of M is finite.*
(iii) *M has a uniform local cohomological annihilator.*

Proof. (i) $\Rightarrow$ (ii) by [2, Theorem 2.1].

(ii) $\Rightarrow$ (iii). This is Theorem 2.7.

(iii) $\Rightarrow$ (i). There exists an element $x \in A \setminus \bigcup_{\mathfrak{p} \in \mathrm{Min}_A(M)} \mathfrak{p}$ such that $x\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for all $i < \dim_A(M)$, and, by artinianness of local cohomology modules, $x\mathrm{H}^i_{\widehat{\mathfrak{m}}}(\widehat{M}) = 0$ for all $i < \dim_{\widehat{A}}(\widehat{M})$. Assume that $\mathcal{Q}$ is an element of $\mathrm{Min}_{\widehat{A}}(\widehat{M})$. Note that $0 :_A M \subseteq \mathcal{Q} \cap A$ and, by Going Down Theorem, $\mathcal{Q} \cap A \in \mathrm{Min}_A(M)$. Hence $x \notin \mathcal{Q}$. Therefore $\widehat{M}$ has a uniform local cohomological annihilator. Now, Proposition 2.11 implies that $\mathrm{Min}_{\widehat{A}}(\widehat{M}) = \mathrm{Assh}_{\widehat{A}}(\widehat{M})$. □

We end this section by showing that any finite A-module M which has a uniform local cohomological annihilator is universally catenary, that is the ring $A/(0 :_A M)$ is universally catenary.

Theorem 2.14. *Let M be a finite A-module that has a uniform local cohomological annihilator. Then $A/(0 :_A M)$ has a uniform local cohomological annihilator and so $A/(0 :_A M)$ is universally catenary.*

Proof. By Proposition 2.11, $A/(0 :_A M)$ is locally equidimensional. By [12, Theorem 3.2], it is enough to show that $\frac{A}{0:_A M}/\frac{\mathfrak{p}}{0:_A M} \cong A/\mathfrak{p}$ has a uniform local cohomological annihilator for each minimal prime ideal $\mathfrak{p}$ of M. We prove it by using the ideas given in the proof of [12, Theorem 3.2].

Assume that $\mathfrak{p} \in \mathrm{Min}_A(M)$ and that $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{p}$. As $M_{\mathfrak{p}}$ is an $A_{\mathfrak{p}}$-module of finite length we set $t := l_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Then there exists a chain of submodules $0 \subset N_1 \subset N_2 \subset \cdots \subset N_t \subseteq M$ such that the following sequences are exact.

$$
\begin{aligned}
&0 \longrightarrow A/\mathfrak{p} \longrightarrow M \longrightarrow M/N_0 \longrightarrow 0,\\
&0 \longrightarrow A/\mathfrak{p} \longrightarrow M/N_0 \longrightarrow M/N_1 \longrightarrow 0,\\
&\vdots\\
&0 \longrightarrow A/\mathfrak{p} \longrightarrow M/N_{t-2} \longrightarrow M/N_{t-1} \longrightarrow 0,\\
&0 \longrightarrow A/\mathfrak{p} \longrightarrow M/N_{t-1} \longrightarrow M/N_t \longrightarrow 0.
\end{aligned}
$$

Since $M_\mathfrak{m}$ is equidimensional, $\mathrm{ht}_M(\mathfrak{m}/\mathfrak{p}) = \mathrm{ht}_M(\mathfrak{m})$. As, by definition of t, $\mathfrak{p} \notin \mathrm{Ass}_A(M/N_t)$, it follows that $0 :_A (M/N_t) \nsubseteq \mathfrak{p}$. Localizing the above exact sequences at $\mathfrak{m}$ implies the following exact sequences.

$$\begin{array}{l} 0 \longrightarrow (A/\mathfrak{p})_\mathfrak{m} \longrightarrow M_\mathfrak{m} \longrightarrow (M/N_0)_\mathfrak{m} \longrightarrow 0, \\ 0 \longrightarrow (A/\mathfrak{p})_\mathfrak{m} \longrightarrow (M/N_0)_\mathfrak{m} \longrightarrow (M/N_1)_\mathfrak{m} \longrightarrow 0, \\ \vdots \\ 0 \longrightarrow (A/\mathfrak{p})_\mathfrak{m} \longrightarrow (M/N_{t-2})_\mathfrak{m} \longrightarrow (M/N_{t-1})_\mathfrak{m} \longrightarrow 0, \\ 0 \longrightarrow (A/\mathfrak{p})_\mathfrak{m} \longrightarrow (M/N_{t-1})_\mathfrak{m} \longrightarrow (M/N_t)_\mathfrak{m} \longrightarrow 0. \end{array}$$

Choose an element $y \in 0 :_A (M/N_t) \setminus \mathfrak{p}$. By assumption, there is an element $x \in A \setminus \bigcup_{\mathfrak{q} \in \mathrm{Min}_A(M)} \mathfrak{q}$ such that $x\mathrm{H}^i_{\mathfrak{m}A_\mathfrak{m}}(M_\mathfrak{m}) = 0$ for all $i < \mathrm{ht}_M(\mathfrak{m})$. Now, with a similar technique as in the proof of [12, Lemma 3.1(i)] one can deduce that $(xy)^l \mathrm{H}^i_\mathfrak{m}(A/\mathfrak{p})_\mathfrak{m} = 0$ for all $i < \mathrm{ht}_M(\mathfrak{m})$ and for some integer $l > 0$. □

Corollary 2.15. *Let M be a finite A-module of finite dimension such its Cousin complex $\mathcal{C}_A(M)$ is finite. Then the ring $A/0 :_A M$ is universally catenary.*

Proof. By Theorem 2.7, M has a uniform local cohomological annihilator. Now, the result follows by Theorem 2.14. □

3. Height of an ideal

As mentioned in Corollary 2.3 and in the proof of Theorem 2.7, we may write the following corollary.

Corollary 3.1. *For any finite A-module M and any ideal $\mathfrak{a}$ of A with $\mathfrak{a}M \neq M$,*

$$\prod_{-1 \leqslant i} \left(0 :_A H^i\right) \subseteq 0 :_A H_\mathfrak{a}^{\mathrm{ht}_M(\mathfrak{a})-1}(M).$$

We now raise the question that whether it is possible to improve the upper bound restriction.

Question. Does the inequality

$$\prod_{-1 \leqslant i} \left(0 :_A H^i\right) \subseteq 0 :_A \mathrm{H}_\mathfrak{a}^{\mathrm{ht}_M(\mathfrak{a})}(M)$$

hold?

It will be proved that the answer is negative for the class of finite A-modules M with finite Cousin cohomologies. More precisely,

Theorem 3.2. *Assume that M is a finite A-module of finite dimension and that its Cousin complex $\mathcal{C}_A(M)$ is finite. Then*

$$\operatorname{ht}_M(\mathfrak{a}) = \inf\left\{r\colon \prod_{-1\leqslant i}\left(0:_A H^i\right) \nsubseteq 0:_A H^r_{\mathfrak{a}}(M)\right\},$$

for all ideals $\mathfrak{a}$ with $\mathfrak{a}M \neq M$.

Proof. By Corollary 2.3, $\prod_{i\geqslant -1}(0:_A H^i) \subseteq 0:_A \operatorname{Ext}^r_A(A/\mathfrak{a}^n, M)$ for all r, $0 \leqslant r < \operatorname{ht}_M(\mathfrak{a})$ and all $n \geqslant 0$. Passing to the direct limit, as in the proof of Theorem 2.7, one has $\prod_{i\geqslant -1}(0:_A H^i) \subseteq 0:_A \mathrm{H}^r_{\mathfrak{a}}(M)$ for all $r < \operatorname{ht}_M(\mathfrak{a})$. Hence we have

$$\operatorname{ht}_M(\mathfrak{a}) \leqslant \inf\left\{r\colon \prod_{-1\leqslant i}\left(0:_A H^i\right) \nsubseteq 0:_A \mathrm{H}^r_{\mathfrak{a}}(M)\right\}.$$

Thus it is sufficient to show that $\prod_{-1\leqslant i}(0:_A H^i) \nsubseteq 0:_A \mathrm{H}^{\operatorname{ht}_M(\mathfrak{a})}_{\mathfrak{a}}(M)$. By Independence Theorem of local cohomology (cf. [1, Theorem 4.2.1]), $\mathrm{H}^{\operatorname{ht}_M(\mathfrak{a})}_{\mathfrak{a}}(M) = \mathrm{H}^{\operatorname{ht}_M(\mathfrak{b})}_{\mathfrak{b}}(M)$ as $\overline{A} = A/(0:_A M)$-module, where $\mathfrak{b} = \mathfrak{a} + 0:_A M/0:_A M$. Note that $\operatorname{ht}_M(\mathfrak{a}) = \operatorname{ht}_M(\mathfrak{b})$ and that $\mathcal{C}_A(M) \cong \mathcal{C}_{\overline{A}}(M)$ (see [2, Lemma 1.2]).

Hence we may assume that $0:_A M = 0$. Set $h := \operatorname{ht}_M(\mathfrak{a})$. Let $x \in 0:_A \mathrm{H}^h_{\mathfrak{a}}(M)$. As $\mathfrak{a}M \neq M$, there exists a minimal prime $\mathfrak{q}$ over $\mathfrak{a}$ in $\operatorname{Supp}_A(M)$ such that $\dim(A_{\mathfrak{q}}) = \operatorname{ht}_M(\mathfrak{a})$. Hence $x/1 \in 0:_{A_{\mathfrak{q}}} \mathrm{H}^h_{\mathfrak{q}A_{\mathfrak{q}}}(M_{\mathfrak{q}})$. Thus, by any choice of $\mathfrak{p}A_{\mathfrak{q}} \in \operatorname{Assh}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}})$ we have $x/1 \in \mathfrak{p}A_{\mathfrak{q}}$ (see [1, Proposition 7.2.11(ii) and Theorem 7.3.2]) and so $x \in \mathfrak{p}$. Therefore, one has $0:_A \mathrm{H}^h_{\mathfrak{a}}(M) \subseteq \bigcup_{\mathfrak{p}\in\operatorname{Min}_A(M)} \mathfrak{p}$. On the other hand, by Lemma 2.5, $\prod_{i\geqslant -1}(0:_A H^i) \nsubseteq \bigcup_{\mathfrak{p}\in\operatorname{Min}_A(M)} \mathfrak{p}$, from which it follows that

$$\prod_{i\geqslant -1}\left(0:_A H^i\right) \nsubseteq 0:_A \mathrm{H}^h_{\mathfrak{a}}(M). \qquad \square$$

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