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Abstract Forced Symmetry Breaking and Forced Frequency Locking of Modulated Waves

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We consider abstract forced symmetry breaking problems of the type $F(x, \lambda) = y$. It is supposed that for all λ the maps $F(\cdot, \lambda)$ are equivariant with respect to the action of a compact Lie group, that $F(x_0, \lambda_0) = 0$ and, hence, that $F(x, \lambda_0) = 0$ for

parameters λ and y), their dynamic stability and their asymptotic behavior for y tending to zero. Further, generalizations are given to problems of the type $F(x, \lambda) = y(x, \lambda)$. Finally, our results are applied to a forced frequency locking problem of the type $\dot{x}(t) = f(x(t), \lambda) - y(t)$. Here it is supposed that the vector fields $f(\cdot, \lambda)$ are $\mathbf{S}^1$-equivariant, that the unperturbed equation $\dot{x} = f(x, \lambda_0)$ has an orbitally stable modulated wave solution and that the forcing $y(t)$ is a modulated wave.

1. INTRODUCTION

In this paper we consider abstract forced symmetry breaking problems of the type

$$F(x, \lambda) = y. \tag{1.1}$$

In (1.1), F is a smooth mapping such that for all λ the maps $F(\cdot, \lambda)$ are equivariant with respect to representations of a given compact Lie group Γ, that $F(x_0, \lambda_0) = 0$ and, hence, that $F(x, \lambda_0) = 0$ for all elements x of the group orbit $\mathcal{O}(x_0)$ of x_0. We look for solutions to (1.1) which bifurcate from the solution family $\mathcal{O}(x_0)$ as λ and y move away from λ_0 and zero, respectively. Thus, x is the "state parameter", λ is the "internal, symmetry preserving control parameter", and y is the "external, symmetry breaking control parameter".

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The aim of this work is to present an analytic and geometric strategy for predicting, or engineering, solutions to (1.1) in the case of $\dim \mathcal{O}(x_0) = \dim \Gamma > 0$. The strategy is founded on a Lyapunov–Schmidt reduction, certain scaling techniques (Hadamard's lemma) and the Implicit Function Theorem. Our results make it possible to determine the number of different solutions x near $\mathcal{O}(x_0)$ to (1.1) (for fixed control parameters λ near λ_0 and y near zero), their dynamic stability, their asymptotic behavior for y tending to zero and the structural stability of all these results.

In fact, this work is partially an application and partially a generalization of results of Hale and Táboas [12, 19, 20, 34], Vanderbauwhede [35, 37], Dancer [14] and Chillingworth [8, 9] in order to solve certain forced frequency locking problems for modulated wave solutions of $\mathbf{S}^1$-equivariant evolution equations arising from laser modelling (cf. [27]).

The paper is organized as follows.

Using an approach of Vanderbauwhede [36] and Dancer [14], in Section 2 we carry out a Lyapunov–Schmidt reduction for (1.1) which leads to a smooth bifurcation equation (though we do not suppose the Lie group to act smoothly on the state space).

In Section 3 we describe the solution behavior of (1.1) in the case of vanishing symmetry breaking control parameter:

$$F(x, \lambda) = 0. \tag{1.2}$$

We show that, generically, there exists a smooth submanifold $\mathcal{M}$ in the λ-space with $\lambda_0 \in \mathcal{M}$ and tangential space $T_{\lambda_0}\mathcal{M} = \{\lambda : \partial_\lambda F(x_0, \lambda_0)\lambda \in \operatorname{im} \partial_x F(x_0, \lambda_0)\}$ such that (1.2) is solvable near $\mathcal{O}(x_0)$ iff $\lambda \in \mathcal{M}$. Here we use and generalize results of Dancer [13, 14, 15], who considered the case of $\operatorname{codim} \mathcal{M} = 0$, i.e., the case that (1.2) is solvable for all λ near λ_0. We are mainly interested (because of the applications in Section 6 and [27]) in the case that $\operatorname{codim} \mathcal{M} = \dim \mathcal{O}(x_0)$ (that is the largest generically possible codimension of $\mathcal{M}$).

In Section 4 we describe solution families of (1.1) that are obtained by a scaling technique. These families are smoothly parametrized by the control parameter (λ, y) belonging to certain open subsets (so-called locking cones, cf. Definition 4.1) of the (λ, y)-space. To be more precise, let Λ_2 be a topological complement of $T_{\lambda_0}\mathcal{M}$ in the λ-space, and let $\hat{\lambda}_2 : T_{\lambda_0}\mathcal{M} \to \Lambda_2$ be a parametrization of $\mathcal{M}$ near λ_0, i.e.,

$$\mathcal{M} = \{\lambda_0 + \lambda_1 + \hat{\lambda}_2(\lambda_1) : \lambda_1 \in T_{\lambda_0}\mathcal{M}, \lambda_1 \approx 0\}.$$

Then the scaling used in Section 4 is

$$\lambda = \lambda_0 + \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \qquad y = \varepsilon z, \tag{1.3}$$

where $\varepsilon \in \mathbb{R}$ and $\lambda_1 \in T_{\lambda_0}\mathcal{M}$ are small, and $\mu \in \Lambda_2$ and z are new scaled control parameters. Each isolated solution $\gamma = \gamma_0$ to the so-called reduced bifurcation equation

$$(I - \tilde{P})[\partial_\lambda F(x_0, \lambda_0)\, \mu_0 - \gamma^{-1} \cdot z_0] = 0 \tag{1.4}$$

generates a family of solutions to (1.1), and the corresponding locking cone is the set of all control parameters (λ, y) of the type (1.3), where ε and λ_1 vary near zero, μ near μ_0 and z near z_0. In (1.4), $\tilde{P}$ is a projector onto $\operatorname{im} \partial_x F(x_0, \lambda_0)$ that commutes with the action of the isotropy subgroup of x_0. The reduced bifurcation Eq. (1.4) is the vanishing-condition of the first order term of the ε-expansion of the equation, which is created by inserting (1.3) (with $\mu = \mu_0$ and $z = z_0$) and the ansatz

$$x = \gamma \exp(\varepsilon a_1 + \varepsilon^2 a_2 + \cdots) \cdot (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots),$$
$$x_j \in \operatorname{im} \partial_x F(x_0, \lambda_0) \qquad \text{for} \quad j > 0,$$

into (1.1).

There exists a remarkable difference between the solution behavior of problem (1.1) with $y \neq 0$ and that of problem (1.2): The parameter λ_2 is a "state parameter" for (1.2), because (1.2) determines λ_2 to be a function of the "control parameter" λ_1 (which may vary in an open subset of $T_{\lambda_0}\mathcal{M}$). But for equation (1.1) with $y \neq 0$, λ_2 is a "control parameter" like λ_1, because, for all $\lambda = (\lambda_1, \lambda_2)$ belonging to the locking cones, (1.1) is solvable near $\mathcal{O}(x_0)$. The reason for this behavior is that $y = 0$ is a singular value of the map $F(\cdot, \lambda)$, while generic values $y \neq 0$ are regular one's.

In Section 5 we present a simple criterion that implies linearized stability (resp. linearized instability) simultaneously for all solutions to (1.1) belonging to the solution family corresponding to a solution $\gamma = \gamma_0$ to (1.4). Essentially, the criterion consists in the question whether all eigenvalues of a matrix representation of the linearization of (1.4) with respect to γ in the solution $\gamma = \gamma_0$ have negative real parts or not.

We confine us to forced symmetry breaking problems of type (1.1) for reasons of simplicity only (and because the applications we have in mind are of this type). There exist straightforward generalizations of our results to forced symmetry breaking problems of the more general type $F(x, \lambda) = y(x, \lambda)$. Such generalizations are presented in the Remarks 2.3, 4.11 and 5.3.

In Section 6 we apply our results on abstract forced symmetry breaking to a forced frequency locking problem for ordinary differential equations of the type

$$\dot{\xi}(t) = f(\xi(t), \lambda) - \eta(t). \tag{1.5}$$

In (1.5), $f\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is a smooth parameter-depending vector field, and we suppose $S(e^{i\gamma})\, f(\xi, \lambda) = f(S(e^{i\gamma})\, \xi, \lambda)$ for all $\gamma \in \mathbb{R}$, $\xi \in \mathbb{R}^m$ and

$\lambda \in \mathbb{R}^n$, where S is an $\mathbf{S}^1$-representation on $\mathbb{R}^m$. Thus, λ is an "internal, symmetry preserving" control parameter, and $\eta(t)$ is an "external" control parameter (varying in a certain function space) which breaks the symmetry and the autonomy of Eq. (1.5). It is supposed that the unperturbed equation $\dot{\xi}(t) = f(\xi(t), \lambda_0)$ has an orbitally stable modulated wave solution $\xi_0(t) = S(e^{i\alpha_0 t})\, x_0(t)$ (with $x_0(t) = x_0(t + (2\pi/\beta_0))$ for all t). We describe the quasiperiodic frequency locking of this solution to a forcing of modulated wave type $\eta(t) = S(e^{i\alpha t})\, y(t)$ (with $y(t) = y(t + (2\pi/\beta)))$, where $y(t)$ is near zero, α near α_0 and β near β_0.

The motivation for our investigations comes from problems in laser dynamics. At present, self-pulsations (i.e. periodic intensity change in the output power with frequencies of tenth of gigahertz; for self-pulsations of multisection DFB semiconductor lasers see, e.g., [25, 33, 40]) and frequency locking of self-pulsations to optically injected modulations (cf. [3, 17, 23, 32, 27]) are topics of intensive experimental and theoretical research. The mathematical models are equivariant with respect to an $\mathbf{S}^1$-action on the state space. For a description of the physical nature of this equivariance see [26]. The frequencies α and α_0 (resp. β and β_0) are the so-called optical frequencies (resp. the power frequencies) of the external light signal and the self-pulsation, respectively, and the internal, symmetry preserving control parameter λ describes the internal laser parameters (laser currents, geometric and material parameters, facet reflectivities).

Let us introduce some notation.

If X and $\tilde{X}$ are normed vector spaces, then $\mathscr{L}(X, \tilde{X})$ is the vector space of all linear bounded operators from X into $\tilde{X}$. Further, we set $\mathscr{L}(X) := \mathscr{L}(X, X)$, and $X^* := \mathscr{L}(X, \mathbb{R})$ is the dual space to X. For $L \in \mathscr{L}(X, \tilde{X})$ we denote by $\ker L := \{x : Lx = 0\}$ and $\operatorname{im} L := \{Lx : x \in X\}$ the kernel and the image of L, respectively.

Partial derivatives will be denoted as usually. For example, if Λ is a further normed vector space and $F: X \times \Lambda \to \tilde{X}$ is a C^1-map, then $\partial_x F(x_0, \lambda_0) \in \mathscr{L}(X, \tilde{X})$ denotes the partial derivative of F with respect to $x \in X$ in the point (x_0, λ_0).

If Γ is a group working linearly on X, then, for $x \in X$, we denote the group orbit and the isotropy subgroup of x by $\mathcal{O}(x) := \{\gamma \cdot x \in X : \gamma \in \Gamma\}$ and $\Gamma x := \{\gamma \in \Gamma : \gamma \cdot x = x\}$, respectively.

2. SETUP AND LIAPUNOV–SCHMIDT REDUCTION

Let X and $\tilde{X}$ be Banach spaces, Λ a normed vector space, $k \geqslant 2$ a natural number, $F: X \times \Lambda \to \tilde{X}$ a C^k-map, and $x_0 \in X$ and $\lambda_0 \in \Lambda$ points such that

$$F(x_0, \lambda_0) = 0. \tag{2.1}$$

We denote $L := \partial_x F(x_0, \lambda_0)$ and suppose

$$L \text{ is a Fredholm operator from } X \text{ into } \tilde{X}. \tag{2.2}$$

Further, let Γ be a compact Lie group which works linearly on X and $\tilde{X}$, respectively. We suppose

$$F(\gamma \cdot x, \lambda) = \gamma \cdot F(x, \lambda) \text{ for all } x \in X,\ \lambda \in \Lambda, \text{ and } \gamma \in \Gamma, \tag{2.3}$$

$$\gamma \in \Gamma \mapsto (\gamma \cdot x, \gamma \cdot \tilde{x}) \in X \times \tilde{X} \text{ is continuous for all } x \in X \text{ and } \tilde{x} \in \tilde{X}. \tag{2.4}$$

The assumptions (2.1)–(2.4) imply that the map $\gamma \in \Gamma \mapsto \gamma \cdot x_0 \in X$ is C^k-smooth (cf. [15]). Hence, the group orbit $\mathcal{O}(x_0)$ is a C^k-submanifold in X, the map $\gamma \in \Gamma \mapsto \gamma \cdot x \in \mathcal{O}(x_0)$ is a submersion, and

$$\dim \mathcal{O}(x_0) = \dim \Gamma - \dim \Gamma x_0 \tag{2.5}$$

(cf. [36]). Moreover, the tangential space $T_{x_0}\mathcal{O}(x_0)$ at $\mathcal{O}(x_0)$ in x_0 is a subspace of ker L. We assume that this kernel is as small as possible under (2.3), i.e.,

$$\ker L = T_{x_0}\mathcal{O}(x_0). \tag{2.6}$$

The subspaces ker L and im L are invariant with respect to the subgroup Γx_0. Hence (cf. [36]), there exist projectors $P \in \mathcal{L}(X)$ and $\tilde{P} \in \mathcal{L}(\tilde{X})$ such that

$$\begin{gathered} \ker P = \ker L, \qquad \operatorname{im} \tilde{P} = \operatorname{im} L, \\ P \text{ and } \tilde{P} \text{ commute with the actions of } \Gamma x_0 \text{ on } X \text{ and } \tilde{X}, \text{ respectively.} \end{gathered} \tag{2.7}$$

In most of the applications it holds that

$$\begin{gathered} X \text{ is continuously embedded into } \tilde{X},\ \tilde{X} = \ker L \oplus \operatorname{im} L, \text{ and} \\ \text{the action of } \Gamma \text{ on } X \text{ equals the restriction of the action on } \tilde{X}. \end{gathered} \tag{2.8}$$

In that case the projectors P and $\tilde{P}$ will be chosen such that, in addition to (2.7), we have im $P = X \cap \operatorname{im} L$ and ker $\tilde{P} = \ker L$ and, hence, $Px = \tilde{P}x$ for all $x \in X$.

Finally, let Y be a normed vector space such that Y is continuously embedded into $\tilde{X}$, $\gamma \cdot y \in Y$ for all $y \in Y$, and

$$(\gamma, y) \in \Gamma \times Y \mapsto \gamma \cdot y \in \tilde{X} \text{ is } C^k\text{-smooth}. \tag{2.9}$$

Throughout Sections 2–5 of this paper we suppose (2.1)–(2.4), (2.6), (2.7), and (2.9) to be satisfied.

The following proposition is due to Vanderbauwhede (cf. [35, 36, 37]). It describes a parametrization of an invariant (with respect to Γ) tubular neighbourhood of $\mathcal{O}(x_0)$:

PROPOSITION 2.1. *There exist neighbourhoods $U \subseteq \operatorname{im} P$ of zero and $V \subseteq X$ of $\mathcal{O}(x_0)$ such that the map*

$$(\gamma, u) \in \Gamma \times U \mapsto \gamma \cdot (x_0 + u) \in V \tag{2.10}$$

is surjective. Moreover, for $(\gamma_j, u_j) \in \Gamma \times U$ $(j = 1, 2)$ we have $\gamma_1 \cdot (x_0 + u_1) = \gamma_2 \cdot (x_0 + u_2)$ if and only if $\gamma_1 \cdot x_0 = \gamma_2 \cdot x_0$ and $\gamma_1 \cdot u_1 = \gamma_2 \cdot u_2$.

Let us consider, for x near $\mathcal{O}(x_0)$, λ near λ_0 and y near zero, the abstract forced symmetry breaking problem

$$F(x, \lambda) = y. \tag{2.11}$$

This equation, written in the new coordinates (2.10), is equivalent to

$$F(x_0 + u, \lambda) = \gamma^{-1} \cdot y. \tag{2.12}$$

The following lemma proceeds with a Lyapunov-Schmidt reduction for Eq. (2.12). It is similar to [36, Lemma 8.2.10].

LEMMA 2.2. *There exist neighbourhoods $W \subseteq \Lambda \times Y$ of $(\lambda_0, 0)$ and $U \subseteq \operatorname{im} P$ of zero and a C^k-map $\hat{u}\colon \Gamma \times W \to \operatorname{im} P$ such that:*

(i) $\tilde{P}[F(x_0 + u, \lambda) - \gamma^{-1} \cdot y] = 0$, $u \in U$, $(\lambda, y) \in W$ *if and only if* $u = \hat{u}(\gamma, \lambda, y)$.

(ii) $\hat{u}(\gamma, \lambda_0, 0) = 0$ *for all* $\gamma \in \Gamma$.

(iii) $\hat{u}(\delta\gamma, \lambda, y) = \hat{u}(\gamma, \lambda, \delta^{-1} \cdot y)$ *for all* $\gamma, \delta \in \Gamma$ *and* $(\lambda, y) \in W$.

(iv) $\hat{u}(\gamma\delta, \lambda, y) = \delta^{-1} \cdot \hat{u}(\gamma, \lambda, y)$ *for all* $\gamma \in \Gamma$, $\delta \in \Gamma x_0$ *and* $(\lambda, y) \in W$.

Proof. The partial derivative of $\tilde{P}[F(x_0 + u, \lambda) - \gamma^{-1} \cdot y]$ with respect to u in $u = 0$, $\lambda = \lambda_0$, $y = 0$ (and in an arbitrary γ) is equal to the restriction of L on im P. But the assumption 2.2 yields that

$$L \text{ is an isomorphism from } \operatorname{im} P \text{ onto } \operatorname{im} \tilde{P}. \tag{2.13}$$

Therefore, the Implicit Function Theorem (together with the compactness of Γ) implies assertions (i) and (ii) of the lemma. Further, from (2.3) and (2.7) it follows that

$$\tilde{P}[F(x_0 + u, \lambda) - (\gamma\delta)^{-1} \cdot y] = \delta^{-1} \cdot \tilde{P}[F(x_0 + \delta \cdot u, \lambda) - \gamma^{-1} \cdot y]$$

for all $\gamma \in \Gamma$, $\delta \in \Gamma x_0$ and $(\lambda, y) \in W$. Therefore, assertion (iv) follows from the uniqueness assertion of the Implicit Function Theorem. A similar argument proves (iii). ▮

Let us define a map $G: \Gamma \times W \to \ker \tilde{P}$ by

$$G(\gamma, \lambda, y) := (I - \tilde{P})[F(x_0 + \hat{u}(\gamma, \lambda, y), \lambda) - \gamma^{-1} \cdot y]. \tag{2.14}$$

In (2.14), I is the identity in the space $\tilde{X}$. For all $\gamma \in \Gamma$ and $(\lambda, y) \in W$ we have

$$\begin{aligned} &G(\gamma, \lambda_0, 0) = 0, \\ &G(\delta\gamma, \lambda, y) = G(\gamma, \lambda, \delta \cdot y) \qquad \text{for all} \quad \delta \in \Gamma, \\ &G(\gamma\delta, \lambda, y) = \delta^{-1} \cdot G(\gamma, \lambda, y) \qquad \text{for all} \quad \delta \in \Gamma x_0. \end{aligned} \tag{2.15}$$

The correspondence between the solutions to (2.11) and those of the Lyapunov–Schmidt bifurcation equation

$$G(\gamma, \lambda, y) = 0 \tag{2.16}$$

may be described in the following way: Let (x, λ, y) be a solution to (2.11) such that x is close to $\mathcal{O}(x_0)$, λ is close to λ_0, and y is close to zero. Then there exists a $\gamma_* \in \Gamma$ such that $x = \gamma_* \cdot (x_0 + \hat{u}(\gamma_*, \lambda, y))$ and such that (γ, λ, y) is a solution to (2.16) iff $\gamma = \gamma_* \delta$ with $\delta \in \Gamma x_0$. And conversely, let (γ, λ, y) be a solution to (2.16). Then, for all $\delta \in \Gamma x_0$, $(\gamma\delta, \lambda, y)$ is a solution to (2.16), too, $x := (\gamma\delta) \cdot (x_0 + \hat{u}(\gamma\delta, \lambda, y))$ does not depend on δ, and (x, λ, y) is a solution to (2.11).

Remark 2.3. We do not assume the actions of Γ on X and $\tilde{X}$ to be smooth, because in most of the applications they are not smooth. Therefore, the parametrization (2.10) is not smooth, in general. But Eq. (2.12) is C^k-smooth already because of assumption (2.9).

If the symmetry breaking parameter does not appear as a right-hand side in (2.11), such an approach is not possible, in general. Nevertheless, there exist straightforward generalizations of Lemma 2.2 to equations of the type

$$\mathcal{F}(x, \lambda, y) = 0 \tag{2.17}$$

with $\mathcal{F}(x_0, \lambda_0, 0) = 0$ and

$$\mathcal{F}(\gamma \cdot x, \lambda, \gamma \cdot y) = \gamma \cdot \mathcal{F}(x, \lambda, y) \text{ for all } x \in X,\ y \in Y,\ \lambda \in \Lambda \text{ and } \gamma \in \Gamma \tag{2.18}$$

under the assumption that Γ works also on the space Y of the symmetry breaking parameters y and that the map $(x, \lambda, y, \gamma) \in X \times \Lambda \times Y \times \Gamma \mapsto \mathscr{F}(x, \lambda, \gamma \cdot y) \in \tilde{X}$ is C^k-smooth. In particular, forced symmetry breaking problems $F(x, \lambda) = y(x, \lambda)$ are of the type (2.17), where Y is a suitable subspace of the space of all C^k-maps $y: X \times \Lambda \to \tilde{X}$ such that the map

$$(x, \lambda, y, \gamma) \in X \times \Lambda \times Y \times \Gamma \mapsto \gamma \cdot y(\gamma^{-1} \cdot x, \lambda) \in \tilde{X} \tag{2.19}$$

is C^k-smooth. In this case (2.18) is satisfied if (2.3) holds and if the Γ-action on Y is defined by

$$(\gamma \cdot y)(x, \lambda) := \gamma \cdot y(\gamma^{-1} \cdot x, \lambda). \tag{2.20}$$

Let us indicate a typical example of the situation described above. Let $\tilde{X}$ be the space of all continuous 2π-periodic maps $\tilde{x}: \mathbb{R} \to \mathbb{R}^m$, and let X be the space of all C^1-smooth elements of $\tilde{X}$ (with the usual supremum norms). Let Γ be the rotation group $\mathbf{S}^1 := \{e^{i\varphi} \in \mathbb{C} : \varphi \in \mathbb{R}\}$, and let the action of Γ on X and $\tilde{X}$ be defined by $(e^{i\varphi} \cdot x)(t) := x(t + \varphi)$. Let $\Lambda := \mathbb{R}^n$ and $[F(x, \lambda)](t) := \dot{x}(t) + f(x(t), \lambda)$ with a C^k-smooth map $f: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$. Finally, let Y be the space of all superposition operators $y: X \to \tilde{X}$ of the type $[y(x)](t) := \tilde{y}(t, x(t))$ with a C^k-smooth generating map $\tilde{y}: \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ such that $\tilde{y}(\cdot, x)$ is 2π-periodic and $\tilde{y}$ and all its derivatives up to the kth one, $\tilde{y}^{(k)}$, are bounded. In Y we use the norm $\sup\{\|\tilde{y}^{(l)}(t, x)\| : t \in \mathbb{R},\ x \in \mathbb{R}^m,\ l = 0, 1, ..., k\}$, where $\|\cdot\|$ is a norm in $\mathbb{R}^m$. From (2.20) it follows that

$$[(e^{i\varphi} \cdot y)(x)](t) = \tilde{y}(t + \varphi, x(t)),$$

and, hence, (2.18) is satisfied. Moreover, the map (2.19) is C^k-smooth, because the so-called evaluation map $(t, x, y) \in \mathbb{R} \times \mathbb{R}^m \times Y \mapsto \tilde{y}(t, x) \in \mathbb{R}$ is C^k-smooth (cf., e.g., [2, Proposition 2.4.17]).

Analogously, forced symmetry breaking problems for symmetric elliptic boundary value problems on symmetric domains may be formulated in this way. Here one has to use known smoothness properties of superposition operators between Sobolev or Hölder spaces.

3. SOLUTIONS IN CASE OF VANISHING SYMMETRY BREAKING PARAMETER

In this section we describe the solution behavior of the bifurcation Eq. (2.16) and, hence, of the original Eq. (2.11) in case of vanishing parameter y:

$$F(x, \lambda) = 0. \tag{3.1}$$

Because of (2.15), $G(\gamma, \lambda, 0)$ is independent of γ. Hence, it is correct to define $G_0(\lambda) := G(\gamma, \lambda, 0)$. Here G_0 is a C^k-map which is defined for all $\lambda \in \Lambda$ near λ_0 and which takes values in $\ker \tilde{P}$, and (2.14) implies

$$G_0(\lambda) = (I - \tilde{P})\, F(x_0 + \hat{u}(\gamma, \lambda, 0), \lambda). \tag{3.2}$$

Let

$$X_0 := \{x \in X : \gamma \cdot x = x \text{ for all } \gamma \in \Gamma x_0\},$$
$$\tilde{X}_0 := \{\tilde{x} \in \tilde{X} : \gamma \cdot \tilde{x} = \tilde{x} \text{ for all } \gamma \in \Gamma x_0\}$$

be the isotropy subspaces corresponding to the isotropy subgroup Γx_0. Then, because of (2.3), $F(\cdot, \lambda)$ maps X_0 into $\tilde{X}_0$ for all λ. Hence,

$$LX_0 \subseteq \tilde{X}_0, \qquad \partial_\lambda F(x_0, \lambda_0)\, \Lambda \subseteq \tilde{X}_0. \tag{3.3}$$

Moreover, (2.13) yields

$$LX_0 = \tilde{X}_0 \cap LX. \tag{3.4}$$

Thus, LX_0 is a closed subspace of finite codimension in $\tilde{X}_0$, and we denote this codimension by $\operatorname{codim}_{\tilde{X}_0} LX_0$. From (3.4) it follows that

$$\operatorname{codim}_{\tilde{X}_0} LX_0 = \dim(\tilde{X}_0 \cap \ker \tilde{P}). \tag{3.5}$$

The following theorem describes the solution behavior of Eq. (3.1) under the assumption that the subspaces LX_0 and $\partial_\lambda F(x_0, \lambda_0)\, \Lambda$ are transversal in $\tilde{X}_0$:

Theorem 3.1. *Let Λ_2 be a subspace in Λ such that*

$$\dim \Lambda_2 = \operatorname{codim}_{\tilde{X}_0} LX_0, \qquad \tilde{X}_0 = LX_0 \oplus \partial_\lambda F(x_0, \lambda_0)\, \Lambda_2. \tag{3.6}$$

Further, let Λ_1 be a complement of Λ_2 in Λ, and let $\lambda_0 = \lambda_{01} + \lambda_{02}$ with $\lambda_{0j} \in \Lambda_j$ $(j = 1, 2)$.

Then there exist neighbourhoods $V \subseteq X$ of $\mathcal{O}(x_0)$ and $W_j \subseteq \Lambda_j$ of λ_{0j}, and C^k-maps $\hat{x}_0 \colon W_1 \to X_0$ and $\hat{\lambda}_2 \colon W_1 \to \Lambda_2$ with $\hat{x}_0(\lambda_{01}) = x_0$ and $\hat{\lambda}_2(\lambda_{01}) = \lambda_{02}$ such that the following is true: It holds that $F(x, \lambda_1 + \lambda_2) = 0$ with $x \in V$ and $\lambda_j \in W_j$ if and only if $\lambda_2 = \hat{\lambda}_2(\lambda_1)$ and $x = \gamma \cdot \hat{x}_0(\lambda_1)$ for some $\gamma \in \Gamma$.

Proof. Because of (2.15) we have $G_0(\lambda) \in \tilde{X}_0 \cap \ker \tilde{P}$ for all λ. We denote by $G_0'(\lambda_0) \in \mathcal{L}(\Lambda; \tilde{X}_0 \cap \ker \tilde{P})$ the derivative of G_0 in λ_0. Then (3.2) yields

$$G_0'(\lambda_0) = (I - \tilde{P})\, \partial_\lambda F(x_0, \lambda_0). \tag{3.7}$$

Let us show that the restriction of $G_0'(\lambda_0)$ to Λ_2 is injective. Thus, let $G_0'(\lambda_0)\lambda_2=0$ with $\lambda_2\in\Lambda_2$. Now (3.7) yields that $\partial_\lambda F(x_0,\lambda_0)\lambda_2\in \operatorname{im}\tilde{P}$. Hence, (3.3) and (3.4) imply that $\partial_\lambda F(x_0,\lambda_0)\lambda_2\in LX_0$, and from (3.6) it follows that $\partial_\lambda F(x_0,\lambda_0)\lambda_2=0$. However (3.6) further provides that $\partial_\lambda F(x_0,\lambda_0)$ is injective on Λ_2. Therefore $\lambda_2=0$.

On the other hand, from (3.3), (3.6) and (3.7) it follows that $G_0'(\lambda_0)\,\Lambda_2=\tilde{X}_0\cap\ker\tilde{P}$. Hence, the restriction of $G_0'(\lambda_0)$ to Λ_2 is an isomorphism from Λ_2 onto $\tilde{X}_0\cap\ker\tilde{P}$, and the Implicit Function Theorem solves equation $G_0(\lambda)=0$ for $\lambda\approx\lambda_0$ in the form of $\lambda_2=\hat{\lambda}_2(\lambda_1)$. Thus, the theorem is proved with

$$\hat{x}_0(\lambda_1):=x_0+\hat{u}(\gamma,\lambda_1+\hat{\lambda}_2(\lambda_1),0). \tag{3.8}$$

Remark that, because of Lemma 2.2, the right-hand side of (3.8) belongs to X_0 and does not depend on γ. ∎

Using a more geometrical language, the conclusion of Theorem 4.1 can be formulated as follows: There exists a C^k-submanifold $\mathcal{M}$ in Λ, namely $\mathcal{M}:=\{\lambda_1+\hat{\lambda}_2(\lambda_1):\lambda_1\approx\lambda_{01}\}$, with $\lambda_0\in\mathcal{M}$ and

$$\begin{aligned}T_{\lambda_0}\mathcal{M}&=\{\lambda\in\Lambda:\partial_\lambda F(x_0,\lambda_0)\lambda\in\operatorname{im}L\},\\ \operatorname{codim}\mathcal{M}&=\operatorname{codim}_{\tilde{X}_0}LX_0\end{aligned} \tag{3.9}$$

such that (3.1) is solvable near $\mathcal{O}(x_0)$ iff $\lambda\in\mathcal{M}$.

The following lemma states, under assumption (2.8), a sufficient condition for a subspace Λ_2 of Λ to satisfy (3.6):

LEMMA 3.2. *Suppose* (2.8), *and let* Λ_2 *be a closed subspace of* Λ *such that*

$$\dim\Lambda_2=\dim[X_0\cap T_{x_0}\mathcal{O}(x_0)],\ \partial_\lambda F(x_0,\lambda_0)\,\Lambda_2=X_0\cap T_{x_0}\mathcal{O}(x_0). \tag{3.10}$$

Then (3.6) *holds.*

Proof. Because of (2.8) we have $\tilde{X}_0=[X_0\cap\ker L]\oplus[\tilde{X}_0\cap\operatorname{im}L]$. Hence, (2.6) and (3.4) yield $\tilde{X}_0=[X_0\cap T_{x_0}\mathcal{O}(x_0)]\oplus LX_0$. This proves the lemma. ∎

Let us consider two particular situations described by Theorem 3.1.

In the first case the codimension of the submanifold $\mathcal{M}$ is as large as possible under assumption (3.6). Because of (2.5), (3.5), (3.6) and (3.9) this is the case if

$$\operatorname{codim}_{\tilde{X}_0}LX_0=\dim\Gamma-\dim\Gamma x_0. \tag{3.11}$$

For example, if Γx_0 consists of the unit element only (and, hence, $X_0 = X$ and $\tilde{X}_0 = \tilde{X}$) then (3.11) is satisfied. If, in addition to (3.6), condition (2.8) holds, then (3.11) is equivalent to $T_{x_0}\mathcal{O}(x_0) \subseteq X_0$. This condition is fulfilled, for example, if Γ is Abelian.

In the second case the codimension of $\mathcal{M}$ is as small as possible:

$$\operatorname{codim}_{\tilde{X}_0} LX_0 = 0. \tag{3.12}$$

In that case (3.6) is satisfied with $\Lambda_2 = \{0\}$, and Theorem 4.1 states that for all $\lambda \approx \lambda_0$ there exists exactly one orbit of solutions to (3.1). This is the so-called G-Invariant Implicit Function Theorem of Dancer [13, 14, 15]. Moreover, (3.12) is fulfilled if (2.8) and $X_0 \cap T_{x_0}\mathcal{O}(x_0) = \{0\}$, the so-called $\mathcal{P}$-property of Dancer, hold.

4. SYMMETRY BREAKING AND LOCKING CONES

Let Λ_1 and Λ_2 be subspaces of Λ such that $\Lambda = \Lambda_1 \oplus \Lambda_2$ and (3.6) is satisfied.

We introduce new control parameters $\varepsilon \in \mathbb{R}$, $\lambda_1 \in \Lambda_1$, $\mu \in \Lambda_2$ and $z \in Y$ to the original problem (2.11) and in the corresponding bifurcation Eq. (2.16) by scaling the old control parameters $\lambda \in \Lambda$ and $y \in Y$ in the following way:

$$\begin{gathered} \lambda = \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \qquad y = \varepsilon z, \\ (\mu, z) \in \mathcal{S} := \{(\lambda_2, y) \in \Lambda_2 \times Y : \|\lambda_2\| + \|y\| = 1\}. \end{gathered} \tag{4.1}$$

In (4.1), the symbol $\|\cdot\|$ is used for the norms in Λ and Y, respectively, and $\hat{\lambda}_2$ is the map given by Theorem 4.1.

Because of Theorem 3.1, $G(\gamma, \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \varepsilon z)$ vanishes for $\varepsilon = 0$. Consequently $G(\gamma, \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \varepsilon z) = \varepsilon H(\gamma, \varepsilon, \lambda_1, \mu, z)$ holds with

$$\begin{aligned} H(\gamma, \varepsilon, \lambda_1, \mu, z) := \int_0^1 & [\partial_\lambda G(\gamma, \lambda_1 + \hat{\lambda}_2(\lambda_1) + t\varepsilon\mu, t\varepsilon z)\, \mu \\ & + \partial_y G(\gamma, \lambda_1 + \hat{\lambda}_2(\lambda_1) + t\varepsilon\mu, t\varepsilon z)\, z]\, dt. \end{aligned} \tag{4.2}$$

In particular, for $\varepsilon = 0$ we have (cf. (2.14))

$$H_0(\gamma, \mu, z) := H(\gamma, 0, \lambda_{01}, \mu, z) = (I - \tilde{P})[\partial_\lambda F(x_0, \lambda_0)\, \mu - \gamma^{-1} \cdot z]. \tag{4.3}$$

The solutions with $\varepsilon \neq 0$ to the so-called scaled bifurcation equation

$$H(\gamma, \varepsilon, \lambda_1, \mu, z) = 0, \tag{4.4}$$

correspond, via (4.1), to solutions of (2.16) and, hence, to solutions of (2.11).

The aim of this section is to look for solutions to the so-called reduced bifurcation equation

$$H_0(\gamma_0, \mu_0, z_0) = 0, \ \gamma_0 \in \Gamma, \ (\mu_0, z_0) \in \mathcal{S} \tag{4.5}$$

such that in these solutions the Implicit Function Theorem works with respect to γ. Such solutions produce families of solutions to (4.4) with $\gamma \approx \gamma_0$, $\varepsilon \approx 0$, $\lambda_1 \approx \lambda_{01}$, $\mu \approx \mu_0$ and $z \approx z_0$ and, hence, families of solutions to (2.11) with control parameters $(\lambda, y) \in \Lambda \times Y$ defined by (4.1) with $\varepsilon \approx 0$, $\lambda_1 \approx \lambda_{01}$, $\mu \approx \mu_0$ and $z \approx z_0$. In order to describe the sets of such control parameters (λ, y), we introduce the following notation:

Definition 4.1. For $\varepsilon_0 > 0$, $(\mu_0, z_0) \in \mathcal{S}$ and for neighborhoods $W \subseteq \Lambda_1 \times \mathcal{S}$ of $(\lambda_{01}, \mu_0, z_0)$ we call the set

$$K(\varepsilon_0, \mu_0, z_0, W)$$
$$:= \{(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \varepsilon z) \in \Lambda \times Y : 0 < |\varepsilon| < \varepsilon_0, (\lambda_1, \mu, z) \in W\}$$

a locking cone (corresponding to ε_0, μ_0, z_0, and W).

Let (γ_0, μ_0, z_0) be a solution to (4.5). The Implicit Function Theorem works in this solution in order to solve (4.4) with respect to γ iff the operator

$$\partial_\gamma H_0(\gamma_0, \mu_0, z_0) = -(I - \tilde{P}) \frac{d}{d\gamma} [\gamma^{-1} \cdot z_0]_{\gamma = \gamma_0} \tag{4.6}$$

is an isomorphism from the tangential space $T_{\gamma_0}\Gamma$ onto $\ker \tilde{P}$. Obviously, for that condition to be fulfilled it is necessary that

$$\dim \Gamma x_0 = 0, \tag{4.7}$$

because $H_0(\gamma_0 \delta, \mu_0, z_0) = \delta^{-1} \cdot H_0(\gamma_0, \mu_0, z_0)$ holds for all $\delta \in \Gamma x_0$ (cf. (2.15)).

Let $\mathcal{A}$ be the Lie algebra of the Lie group Γ, $\exp: \mathcal{A} \to \Gamma$ the corresponding exponential map, $n := \dim \Gamma$ and $\{a_1, ..., a_n\}$ a basis in the vector space $\mathcal{A}$. Assumption (4.7) implies that the vectors

$$v_j := \frac{d}{dt} [\exp(t a_j) \cdot x_0]_{t=0} \tag{4.8}$$

form a basis in $T_{x_0}\mathcal{O}(x_0) = \ker L$. Further, let $L^* \in \mathcal{L}(\tilde{X}^*, X^*)$ be the adjoint operator to L. Then (2.5), (2.6) and (4.7) imply that $\dim \ker L^* = n$. If, moreover, (2.8) is satisfied, then there exists a basis $\{v_1^*, ..., v_n^*\}$ in $\ker L^*$ such that

$$\langle v_i, v_j^* \rangle = \delta_{ij} \text{ and } \tilde{P}\tilde{x} = \tilde{x} - \sum_{j=1}^{n} \langle \tilde{x}, v_j^* \rangle v_j \text{ for } \tilde{x} \in \tilde{X}. \tag{4.9}$$

Here $\langle \cdot, \cdot \rangle : \tilde{X} \times \tilde{X}^* \to \mathbb{R}$ is the dual pairing, and δ_{ij} is the Kronecker symbol.

The following lemma states two necessary and sufficient conditions for the operator (4.6) to be an isomorphism:

LEMMA 4.2. *Suppose* (4.7). *Then the following is true*:

(i) *The operator* (4.7) *is an isomorphism from* $T_{\gamma_0}\Gamma$ *onto* $\ker \tilde{P}$ *if and only if*

$$\tilde{X} = T_{\gamma_0^{-1} \cdot z_0} \mathcal{O}(z_0) \oplus \text{im } L. \tag{4.10}$$

(ii) *Suppose* (2.8). *Then the operator* (4.6) *is an isomorphism from* $T_{\gamma_0}\Gamma$ *onto* $\ker \tilde{P}$ *if and only if the matrix*

$$\left[\left\langle \frac{d}{dt} [\exp(ta_i)\, \gamma_0^{-1} \cdot z_0]_{t=0}, v_j^* \right\rangle \right]_{i,\, j=1}^{n} \tag{4.11}$$

has a non-vanishing determinant.

Proof. Because of (4.7) we have $\dim \Gamma = \dim \ker \tilde{P}$. Hence, (4.6) is an isomorphism from $T_{\gamma_0}\Gamma$ onto $\ker \tilde{P}$ iff it is injective.

(i) Suppose (4.6) to be injective. Then $\dim \Gamma z_0 = 0$ and, hence, $\dim \mathcal{O}(z_0) = \dim \Gamma = \operatorname{codim} \text{im } L$. Thus, for (4.10) it remains to show that

$$T_{\gamma_0^{-1} \cdot z_0} \mathcal{O}(z_0) \cap \text{im } L = \{0\}. \tag{4.12}$$

Let $\tilde{x}$ be an element of the left-hand side of (4.12). Then there exists a $\bar{\gamma} \in T_{\gamma_0}\Gamma$ such that $\tilde{x} = (d/d\gamma)[\gamma^{-1} \cdot z_0]_{\gamma=\gamma_0} \bar{\gamma}$, on the one hand, and (4.6) maps $\bar{\gamma}$ into zero, on the other hand. But (4.6) is injective, therefore $\bar{\gamma} = 0$.

Now, conversely, suppose (4.10). Then, as above, $\dim \mathcal{O}(z_0) = \operatorname{codim} \text{im } L = \dim \Gamma$ and, hence, $\dim \Gamma z_0 = 0$. Therefore, $(d/d\gamma)[\gamma^{-1} \cdot z_0]_{\gamma=\gamma_0}$ is injective, and (4.10) yields that (4.6) is so, too.

(ii) The map $\gamma \in \Gamma \mapsto -(I - \tilde{P})\, \gamma^{-1} \cdot z_0 \in \ker \tilde{P}$ is a local diffeomorphism in $\gamma = \gamma_0$ iff the map

$$a \in \mathscr{A} \mapsto -(I - \tilde{P}) \exp(-a)\, \gamma_0^{-1} \cdot z_0 \in \ker \tilde{P} \tag{4.13}$$

is a local diffeomorphism in $a=0$. However (4.11) is the matrix representation with respect to the bases $\{a_1, ..., a_n\}$ of $\mathscr{A}$ and $\{v_1, ..., v_n\}$ of $\ker \tilde{P} = T_{x_0}\mathcal{O}(x_0)$ of the derivative of (4.13) in $a=0$. ▮

The following theorem is the main result of this section. In its formulation we use the maps $\hat{x}_0$ and $\hat{\lambda}_2$, given by Theorem 3.1.

THEOREM 4.3. *Suppose* (4.7), *and let* (γ_0, μ_0, z_0) *be a solution to* (4.5) *with* (4.10).

Then there exist $\varepsilon_0 > 0$, *neighbourhoods* $V \subseteq X$ *of* $\gamma_0 \cdot x_0$ *and* $W \subseteq \Lambda_1 \times \mathscr{S}$ *of* $(\lambda_{01}, \mu_0, z_0)$, *a* C^{k-1}-*map* $\hat{\gamma}: W \to \Gamma$ *with* $\hat{\gamma}(\lambda_{01}, \mu_0, z_0) = \gamma_0$ *and a* C^k-*map* $\hat{x}: K(\varepsilon_0, \mu_0, z_0, W) \to X$ *such that the following is true*:

(i) *Let* $x \in V$ *and* $(\lambda, y) \in K(\varepsilon_0, \mu_0, z_0, W)$. *Then* $F(x, \lambda) = y$ *if and only if* $x = \hat{x}(\lambda, y)$.

(ii) *Let* $(\lambda_1, \mu, z) \in W$ *be fixed. Then* $\hat{x}(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \varepsilon z)$ *tends to* $\hat{\gamma}(\lambda_1, \mu, z) \cdot \hat{x}_0(\lambda_1)$ *for* $\varepsilon \to 0$.

Proof. The Implicit Function Theorem yields a relation $\gamma = \check{\gamma}(\varepsilon, \lambda_1, \mu, z)$ solving (4.4) near the solution $\gamma = \gamma_0$, $\varepsilon = 0$, $\lambda_1 = \lambda_{01}$, $\mu = \mu_0$ and $z = z_0$ (in particular, it holds that $\check{\gamma}(0, \lambda_{01}, \mu_0, z_0) = \gamma_0$). Hence, (i) follows with

$$\hat{x}(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \varepsilon z) := \check{\gamma}(\varepsilon, \lambda_1, \mu, z) \cdot (x_0 + \hat{u}(\check{\gamma}(\varepsilon, \lambda_1, \mu, z), \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \varepsilon z)). \quad (4.14)$$

In (4.14), $\hat{u}$ is the map given by Lemma 2.2, and Lemma 2.2(ii) and (3.8) imply assertion (ii) with $\hat{\gamma}(\lambda_1, \mu, z) := \check{\gamma}(0, \lambda_1, \mu, z)$.

Remark that the map H is only C^{k-1}-smooth in arguments with $\varepsilon = 0$, therefore the map $\hat{\gamma}$ is only C^{k-1}-smooth, in general. ▮

By means of Theorem 4.3, there exists a straightforward procedure to construct control parameters λ and y such that (2.11) is solvable near $\mathcal{O}(x_0)$: Just take $(\mu_0, z_0) \in \mathscr{S}$ such that the orbit $\mathcal{O}(z_0)$ intersects the affine subspace $\partial_\lambda F(x_0, \lambda_0)\, \mu_0 + LX$ in at least one point transversally. Then $\lambda = \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu$ and $y = \varepsilon z$ with arbitrary $\varepsilon \in \mathbb{R}$ near zero, $\lambda_1 \in \Lambda_1$ near λ_{01}, $\mu \in \Lambda_2$ near μ_0 and $z \in Y$ near z_0 are parameters of the type required.

Remark 4.4. Let us consider the uniqueness assertion of Theorem 4.3(i) in more detail. Let $(\mu_0, z_0) \in \mathscr{S}$ be fixed, and let $\gamma = \gamma_0$ be a solution to

$$H_0(\gamma, \mu_0, z_0) = 0 \quad (4.15)$$

with (4.10). Then Theorem 4.3 claims that, for $(\lambda, y) \in K(\varepsilon_0, \mu_0, z_0, W)$, there exists exactly one solution x near $\gamma_0 \cdot x_0$ to (2.11). Of course, there may exist other solutions x near $\mathcal{O}(x_0)$ to (2.11) (with the same control parameter (λ, y)), not close to $\gamma_0 \cdot x_0$. Now, suppose that all solutions γ to

(4.15) satisfy (4.10). Then the number of these solutions is finite (because Γ is compact), each such solution generates a family of solutions to (2.11), and we have the following "global" uniqueness assertion:

If $(\lambda, y) \approx (\lambda_0, 0)$ belongs to the intersection of the locking cones corresponding to the solutions to (4.15) (this intersection is a locking cone, again) and if $x \approx \mathcal{O}(x_0)$ is a solution to (2.11) with this control parameter (λ, y), then x is a member of one of the families of solutions to (2.11) corresponding to the solutions to (4.15). In particular, if (4.15) (with fixed $(\mu_0, z_0) \in \mathcal{S}$) is not solvable, then there do not exist any solutions to $F(x, \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu) = \varepsilon z$ with $x \in X$ near $\mathcal{O}(x_0)$, $\varepsilon \in \mathbb{R}$ near zero, $\lambda_1 \in \Lambda_1$ near λ_{01}, $\mu \in \Lambda_2$ near μ_0 and $z \in Y$ near z_0.

Remark 4.5. This is a remark concerning assertion (ii) of Theorem 4.3. If the control parameter $(\lambda, y) \in K(\varepsilon_0, \mu_0, z_0, W)$ tends to $(\lambda_1 + \hat{\lambda}_2(\lambda_1), 0)$ (with fixed $(\lambda_1, \mu_0, z_0) \in W$), but not along a straight line $\{(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \varepsilon z) : \varepsilon \in \mathbb{R}\}$, the solution $\hat{x}(\lambda, y)$ does not converge, in general (because the right-hand side of (4.14) with $\varepsilon = 0$ depends on μ and z, in general). In other words, in general it is not possible to continue the family $(\lambda, y) \in K(\varepsilon_0, \mu_0, z_0, W) \mapsto \hat{x}(\lambda, y) \in X$ of solutions to (2.11) continuously onto the closure of $K(\varepsilon_0, \mu_0, z_0, W)$, for example. This phenomenon is well-known in the theory of damping and forcing of second order ordinary differential equations (cf. [20, 34]).

Remark 4.6. Theorem 4.3 describes families of solutions to (2.11) that are smoothly parametrized by control parameters (λ, y) belonging to locking cones $K(\varepsilon_0, \mu_0, z_0, W)$. But Theorem 4.3 does not state any assertion about the questions whether these families have a unique smooth continuation outside of $K(\varepsilon_0, \mu_0, z_0, W)$ or not (with the exception of the assertion on the impossibility of a continuous continuation into the points $(\lambda, y) = (\lambda_1 + \hat{\lambda}_2(\lambda_1), 0)$, cf. Remark 4.5), whether there exists a maximal domain of definition of such a continuation or not and how the solution x behaves if (λ, y) tends to the boundary of such a maximal domain of continuation.

Remark 4.7. Let $(\lambda, y) \in K(\varepsilon_0, \mu_0, z_0, W)$. Then, because of Theorem 4.3(i), $\gamma \cdot \hat{x}(\lambda, y) = \hat{x}(\lambda, \gamma \cdot y)$ for all γ near the unit element. Hence, the map $\gamma \in \Gamma \mapsto \gamma \cdot \hat{x}(\lambda, y) \in X$ is C^k-smooth (cf. assumption (2.9)). This is a kind of "abstract solution regularity" result for (2.11): The group Γ does not act smoothly on each element $x \in X$, in general, but it does so on solutions to (2.11).

Remark 4.8. If assumption (4.7) is not satisfied, then, at a first glance, the parametrization (2.10) seems to be inappropriate for solving (2.11) because we have

$$H(\gamma\delta, \varepsilon, \lambda_1, \mu, z) = \delta^{-1} \cdot H(\gamma, \varepsilon, \lambda_1, \mu, z) \qquad \text{for all} \quad \delta \in \Gamma x_0.$$

Hence, the solutions $\gamma \in \Gamma$ of the scaled bifurcation Eq. (4.4) (with fixed control parameters $\varepsilon, \lambda_1, \mu$ and z) are not isolated, but appear as Γx_0-orbits. On the other hand, because of the uniqueness assertion of Proposition 2.1, each such Γx_0-orbit of solutions $\gamma \in \Gamma$ to (4.4) corresponds to an isolated solution $x \in X$ of (2.11) (with control parameters $(\lambda, y) \in \Lambda \times Y$ determined by (4.1)). Therefore, it should be possible to apply our results of Section 3 (especially a second Lyapunov–Schmidt reduction) in order to obtain families of Γx_0-orbits of solutions to (4.4) and, hence, families of isolated solutions to (2.11). These families would be parametrized by the control parameters of the corresponding equations, which have to belong to certain submanifolds in the control parameter spaces of codimension less than or equal to dim Γx_0. This is a topic of future research.

Remark 4.9. We have $H(\delta\gamma, \varepsilon, \lambda_1, \mu, z) = H(\gamma, \varepsilon, \lambda_1, \mu, z)$ for all $\delta \in \Gamma z$. Therefore, the solutions $\gamma \in \Gamma$ to the scaled bifurcation equation (4.4) (with fixed control parameters ε, λ_1, μ and z) appear as Γz-orbits, and, hence, are not isolated if dim $\Gamma z > 0$. In contrast to the situation, considered in Remark 4.8, such a Γz-orbit of solutions to (4.4) corresponds to a Γz-orbit of, in general, non-isolated solutions $x \in X$ to (2.11) (with fixed control parameters λ and y, determined by (4.1)).

Now, suppose $H_0(\gamma_0, \mu_0, z_0) = 0$ and dim $\Gamma z_0 > 0$. Then (4.6) is not an isomorphism, and Theorem 4.3 is not applicable. But (2.11) with $z \approx z_0$ is a forced symmetry breaking problem, again (the parameter $z - z_0$ breaks the Γz_0-symmetry). Therefore, on principle one can apply the results of this section to (4.4) with $z \approx z_0$ (especially a second Lyapunov–Schmidt reduction and a second scaling in order to obtain a "scaled bifurcation equation for the scaled bifurcation equation"). Then Theorem 4.3 yields families of isolated solutions to (2.11) that are parametrized by control parameters (λ, y) belonging to certain "locking cones of second kind".

Remark 4.10. Theorem 4.3 has the advantage that the reduced bifurcation Eq. (4.5) does not depend on the implicitly given map $\hat{u}$ (cf. (4.3)). Moreover, if (3.12) holds then the reduced bifurcation equation reads $-(I - \tilde{P})\,\gamma^{-1} \cdot z = 0$. Hence, it depends on the map F (the left hand side of the original problem (2.11)) via the projector $\tilde{P}$, only. Finally, in certain cases the reduced bifurcation equation does not depend on F at all: For example, if $X = \tilde{X}$ are Hilbert spaces, if Γ works unitarily and if $F(\cdot, \lambda)$ is a gradient map for each λ, then (3.12) is satisfied, and $I - \tilde{P}$ is the orthogonal projector onto $T_{x_0}\mathcal{O}(x_0)$, which does not depend on F but only on the action of the group Γ on x_0. Chillingworth, Marsden and Wan used this property in their study of the dead load traction problem in three-dimensional elastostatics [10, 39, 8].

Remark 4.11. The generalization of Theorem 4.3 to problems of the type (2.17) with (2.18) (cf. Remark 2.3) is straightforward. In this case one has to use the following more general form of the reduced bifurcation equation

$$(I-\tilde{P})(\partial_\lambda \mathscr{F}(x_0, \lambda_0, 0)\, \mu_0 + \partial_y \mathscr{F}(x_0, \lambda_0, 0)\, \gamma^{-1} \cdot z_0) = 0, \qquad (4.16)$$

and the matrix (4.11) has to be replaced by the matrix

$$-\left[\left\langle \partial_y \mathscr{F}(x_0, \lambda_0, 0) \frac{d}{dt} [\exp(ta_i)\, \gamma_0^{-1} \cdot z_0]_{t=0}, v_j^* \right\rangle\right]_{i,\, j=1}^{n}. \qquad (4.17)$$

5. STABILITY

Theorem 4.3 states that regular solutions to the reduced bifurcation Eq. (4.5) generate families of solutions to the original Eq. (2.11). In this section we show that, moreover, the matrix representation (4.11) of the linearization of (4.5) in such a solution determines the linearized stability of all the corresponding solutions to (2.11). For related results concerning bifurcations from isolated solutions see [16, 24, 29, 38].

In this section we assume (2.8) and (4.9) to be satisfied.

As usually, we denote by spec L the set of all complex numbers ρ such that the operator $L-\rho J$ is not an isomorphism of the complexification of X onto the complexification of $\tilde{X}$. Here $J \in \mathscr{L}(X; \tilde{X})$ is the embedding operator. Because of (2.13) we have

$$c := \inf\{|\rho| : \rho \in \operatorname{spec} L,\ \rho \neq 0\} > 0.$$

Let (γ_0, μ_0, z_0) be a solution to (4.5) such that the determinant of (4.11) does not vanish. Then (4.14) implies that, for all $(\lambda, y) \in K(\varepsilon_0, \mu_0, z_0, W)$, $\partial_x F(\hat{x}(\lambda, y), \lambda)$ is close to $\gamma_0 \cdot L\gamma_0^{-1}$ in the sense of the operator norm in $\mathscr{L}(X, \tilde{X})$. Therefore, perturbation results for isolated normal eigenvalues of linear operators (cf., e.g., [12, Chapter 14]) yield the following:

Let ε_0 and W be sufficiently small. Then, for all $(\lambda, y) \in K(\varepsilon_0, \mu_0, z_0, W)$, the set

$$\sigma_0(\lambda, y) := \left\{\rho \in \operatorname{spec} \partial_x F(\hat{x}(\lambda, y), \lambda) : |\rho| < \frac{c}{2}\right\}$$

is finite. It consists of eigenvalues only, and the sum of the algebraic multiplicities of all these eigenvalues is equal to dim Γ. The following theorem shows how to verify whether all these eigenvalues have negative real parts or not:

THEOREM 5.1. *Suppose* (2.8), *and let* (γ_0, μ_0, z_0) *be a solution to* (4.5) *such that the determinant of* (4.11) *does not vanish.*

Then the following is true for all $(\lambda, y) \in K(\varepsilon_0, \mu_0, z_0, W)$: *If all eigenvalues of the matrix* (4.11) *have negative real parts* (*resp. one such eigenvalue has a positive real part*), *then* $\max\{\operatorname{Re}\rho : \rho \in \sigma_0(\lambda, y)\}$ *is negative* (*resp. positive*).

Proof. Let $\hat{u}$ be the map determined by Lemma 2.2, $\hat{x}_0$ and $\hat{\lambda}_2$ the maps determined by Theorem 3.1 and $\hat{x}$, (resp. $\check{y}$) the maps determined by (resp. in the proof of) Theorem 4.3. From 2.3 and (4.14) we conclude

$$\check{y}(\varepsilon, \lambda_1, \mu, z)^{-1} \cdot \partial_x F(\hat{x}(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \varepsilon z), \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu)\, \check{y}(\varepsilon, \lambda_2, \mu, z)$$
$$= \partial_x F(x_0 + \hat{u}(\check{y}(\varepsilon, \lambda_1, \mu, z)), \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \varepsilon z), \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu). \tag{5.1}$$

An application of the Implicit Function Theorem (cf. [29, 38]) yields that there exist operators $A(\varepsilon, \lambda_1, \mu, z) \in \mathscr{L}(\ker P)$, $B(\varepsilon, \lambda_1, \mu, z) \in \mathscr{L}(\operatorname{im} P; \operatorname{im} \tilde{P})$, $C(\varepsilon, \lambda_1, \mu, z) \in \mathscr{L}(X)$ and $\tilde{C}(\varepsilon, \lambda_1, \mu, z) \in \mathscr{L}(\tilde{X})$, that depend C^{k-1}-smoothly on $\varepsilon \in \mathbb{R}$ near zero, $\lambda_1 \in \Lambda_1$ near λ_{01}, $\mu \in \Lambda_2$ near μ_0 and $z \in Y$ near z_0, such that

$$\begin{aligned} &A(0, \lambda_{01}, \mu_0, z_0) = 0,\\ &B(0, \lambda_{01}, \mu_0, z_0) = L \text{ on } \operatorname{im} P,\\ &\tilde{C}(0, \lambda_{01}, \mu_0, z_0) = I; \end{aligned} \tag{5.2}$$

and that, for all suitable ε, λ_1, μ and z, we have $C(\varepsilon, \lambda_1, \mu, z) = \tilde{C}(\varepsilon, \lambda_1, \mu, z)$ on X and

$$\partial_x F(x_0 + \hat{u}(\check{y}(\varepsilon, \lambda_1, \mu, z), \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu, \varepsilon z), \lambda_1 + \hat{\lambda}_2(\lambda_1) + \varepsilon\mu)$$
$$= \tilde{C}(\varepsilon, \lambda_1, \mu, z)[A(\varepsilon, \lambda_1, \mu, z) \oplus B(\varepsilon, \lambda_1, \mu, z)]\, C(\varepsilon, \lambda_1, \mu, z)^{-1}. \tag{5.3}$$

In (5.3), $A \oplus B \in \mathscr{L}(X; \tilde{X})$ is the "diagonal" operator, which is defined by $(A \oplus B)(a + b) := Aa + Bb$ for $a \in \ker P$ and $b \in \operatorname{im} P$.

From (4.7) it follows that the dimension of the kernel of $\partial_x F(\hat{x}_0(\lambda_1), \lambda_1 + \hat{\lambda}(\lambda_1))$ (which is the limit for $\varepsilon \to 0$ of the left-hand side of (5.3)) is equal to $\dim \Gamma$. On the other hand, (2.13) and (5.3) imply that $B(\varepsilon, \lambda_1, \mu, z)$ is an isomorphism from $\operatorname{im} P$ onto $\operatorname{im} \tilde{P}$ for $\varepsilon \approx 0$, $\lambda_1 \approx \lambda_{01}$, $\mu \approx \mu_0$ and $z \approx z_0$. Hence, $\dim \ker A(0, \lambda_1, \mu, z) = \dim \Gamma$ for such λ_1, μ and z. But $\dim \Gamma$ is the dimension of the space where $A(0, \lambda_1, \mu, z)$ is defined. Therefore, $A(0, \lambda_1, \mu, z)$ is the zero operator, and we have

$$A(\varepsilon, \lambda_1, \mu, z) = \varepsilon \bar{A}(\varepsilon, \lambda_1, \mu, z), \tag{5.4}$$

where $\bar{A}(\varepsilon, \lambda_1, \mu, z) \in \mathscr{L}(\ker P)$ depends C^{k-2}-smoothly on ε, λ_1, μ and z. Moreover, (5.2)–(5.4) imply

$$\bar{A}(0, \lambda_{01}, \mu_0, z_0)\, v_j$$

$$= (I - \tilde{P}) \frac{d}{d\varepsilon} [\partial_x F(x_0 + \hat{u}(\check{\gamma}(\varepsilon, \lambda_{01}, \mu_0, z_0), \lambda_0 + \varepsilon\mu_0, \varepsilon z_0), \lambda_0 + \varepsilon\mu_0)\, v_j\,]_{\varepsilon=0} \tag{5.5}$$

for all $j = 1, ..., n$. Let us denote

$$\gamma(\varepsilon) := \check{\gamma}(\varepsilon, \lambda_{01}, \mu_0, z_0), \qquad u(\varepsilon) := \hat{u}(\gamma(\varepsilon), \lambda_0 + \varepsilon\mu_0, \varepsilon z_0). \tag{5.6}$$

Theorem 4.3 and (4.14) imply that $F(\gamma(\varepsilon) \cdot (x_0 + u(\varepsilon)), \lambda_0 + \varepsilon\mu_0) = \varepsilon z_0$ for all small $\varepsilon \in \mathbb{R}$. Hence

$$F(\exp(ta_j) \cdot (x_0 + u(\varepsilon)), \lambda_0 + \varepsilon\mu_0) = \varepsilon \exp(ta_j)\, \gamma(\varepsilon)^{-1} \cdot z_0 \tag{5.7}$$

for all small $\varepsilon \in \mathbb{R}$ and $j = 1, ..., n$. We differentiate the identity (5.7) with respect to t and ε in $t = 0$ and $\varepsilon = 0$ and obtain, using (4.8), (5.5) and (5.6),

$$\bar{A}(0, \lambda_0, \mu_0, z_0)\, v_j = (I - \tilde{P}) \frac{d}{dt} [\exp(ta_j)\, \gamma_0^{-1} \cdot z_0\,]_{t=0}.$$

Hence, the matrix (4.11) is the matrix representation of the operator $\bar{A}(0, \lambda_{01}, \mu_0, z_0)$ with respect to the basis $\{v_1, ..., v_n\}$.

Let us summarize. Denote by M the matrix (4.11). Then the spectrum of M is equal to the spectrum of $\bar{A}(0, \lambda_{01}, \mu_0, z_0)$. Hence, (5.4) yields to

$$\operatorname{sgn} \max\{\operatorname{Re} \rho : \rho \in \operatorname{spec} M\} = \operatorname{sgn} \max\{\operatorname{Re} \rho : \rho \in \operatorname{spec} A(\varepsilon, \lambda_1, \mu, z)\} \tag{5.8}$$

for all small $\varepsilon > 0$, $\lambda_1 \in \Lambda_1$ near λ_{01}, $\mu \in \Lambda_2$ near λ_{02} and $z \in Y$ near z_0. Further, (5.2) and the upper-semicontinuity of spectra (cf. [12, Chapter 14]) provide

$$\inf\{|\rho| : \rho \in \operatorname{spec} B(\varepsilon, \lambda_1, \mu, z)\} > \frac{c}{2} \tag{5.9}$$

for all small $\varepsilon > 0$, $\lambda_1 \in \Lambda_1$ near λ_{01}, $\mu \in \Lambda_2$ near λ_{02} and $z \in Y$ near z_0. Now, (5.1), (5.3), (5.8) and (5.9) imply the desired result. ∎

Remark 5.2. Suppose, for the sake of simplicity, that $X = \tilde{X} = \mathbb{R}^m$. Moreover, assume

$$\max\{\operatorname{Re} \rho : \rho \in \operatorname{spec} L, \rho \neq 0\} < 0. \tag{5.10}$$

Then the stationary solution $x = x_0$ of the Γ-equivariant ordinary differential equation

$$\dot{x} = F(x, \lambda_0) \tag{5.11}$$

is usually called linearly orbitally stable. This property implies the so-called asymptotic orbital stability with asymptotic phase, i.e., each solution $x(t)$ to (5.11) with $x(0) \approx x_0$ exists and stays near $\mathcal{O}(x_0)$ for all times $t \geqslant 0$, and there exists a $\gamma_0 \in \Gamma$ such that $x(t) \to \gamma_0 \cdot x_0$ for $t \to \infty$ (cf. [4, 18]).

Now, let (γ_0, μ_0, z_0) be a solution to (4.5) such that the determinant of (4.11) does not vanish. Then (5.10) yields

$$\operatorname{Re} \rho < 0 \text{ for all } \rho \in \operatorname{spec} \partial_x F(\hat{x}(\lambda, y), \lambda) \text{ with } |\rho| > \frac{c}{2} \tag{5.12}$$

and all $(\lambda, y) \in K(\varepsilon_0, \mu_0, z_0, W)$. Hence, Theorem 5.1 implies the following: If all eigenvalues of the matrix (4.11) have negative real parts (resp. one such eigenvalue has a positive real part), then the stationary solution $x = \hat{x}(\lambda, y)$ of $\dot{x} = F(x, \lambda) - y$ is asymptotically stable (resp. unstable).

In case of $\dim X = \dim \tilde{X} = \infty$, the situation is more difficult, of course. In particular, in this case $\operatorname{spec} L$ is not bounded, in general. Consequently, the property $\sup\{\operatorname{Re} \rho : \rho \in \operatorname{spec} L,\ \rho \neq 0\} < 0$ does not imply (5.12), in general.

Remark 5.3. The generalization of Theorem 5.1 to problems of the type (2.17) with (2.18) (cf. Remarks 2.3 and 4.11) is straightforward. In this case the eigenvalues of the matrix (4.17) determine the linearized stability of the solution families to (2.17) corresponding to solutions to the reduced bifurcation Eq. (4.16).

6. FORCED FREQUENCY LOCKING OF MODULATED WAVE SOLUTIONS

In this section $k \geqslant 2$, $m \geqslant 2$ and n are natural numbers, $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^m$, and A is a non-zero real $m \times m$-matrix such that $A^T = -A$ and $e^{2\pi A} = I$. Further, $f\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is a C^k-map, and we assume

$$f(e^{\gamma A}\xi, \lambda) = e^{\gamma A} f(\xi, \lambda) \text{ for all } \gamma \in \mathbb{R},\ \xi \in \mathbb{R}^m \text{ and } \lambda \in \mathbb{R}^n.$$

In other words, the vector fields $f(\cdot, \lambda)$ are equivariant with respect to the unitary $\mathbf{S}^1$-representation $e^{i\gamma} \mapsto e^{\gamma A}$ on $\mathbb{R}^m$. The symbol $\|\cdot\|$ will be used for the Euclidean norms in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. By $C_{2\pi}$ resp. $C^l_{2\pi}$ (for

$l \in \mathbb{N}$) we denote the Banach spaces of all 2π–periodic maps $x: \mathbb{R} \to \mathbb{R}^m$ that are continuous resp. C^l-smooth, equipped with the usual maximum norms.

In this section we consider the differential equation

$$\dot{\xi}(\tau) = f(\xi(\tau), \lambda) - e^{\alpha\tau A} y(\beta\tau). \tag{6.1}$$

In (6.1), $y \in C^k_{2\pi}$ is assumed to be small, α and β are real, and we regard (6.1) as an autonomous parameter depending $\mathbf{S}^1$-equivariant differential equation with small quasiperiodic perturbation $e^{\alpha\tau A} y(\beta\tau)$, which breaks autonomy and equivariance.

Let us introduce new variables

$$t := \beta\tau, \qquad x(t) := e^{-(\alpha/\beta)\, tA} \xi\left(\frac{t}{\beta}\right) = e^{-\alpha\tau A} \xi(\tau). \tag{6.2}$$

Then we obtain (6.1) in the equivalent form

$$\beta\dot{x}(t) = f(x(t), \lambda) - \alpha A x(t) - y(t). \tag{6.3}$$

The aim of this section is to apply the results of the Sections 2–5 to Eq. (6.3) and, after that, to translate the results via (6.2) into results for (6.1).

Let $x_0 \in C^1_{2\pi}$, $\lambda_0 \in \mathbb{R}^m$, $\alpha_0 > 0$ and $\beta_0 > 0$ be fixed such that

$$\xi_0(\tau) := e^{\alpha_0 \tau A} x_0(\beta_0 \tau) \tag{6.4}$$

is a solution to equation $\dot{\xi} = f(\xi, \lambda_0)$. Obviously this is equivalent to

$$\beta_0 \dot{x}_0(t) = f(x_0(t), \lambda_0) - \alpha_0 A x_0(t). \tag{6.5}$$

Vector functions of the type (6.4) are usually called modulated waves. Together with the modulated wave solution ξ_0 its "temporal" phase shifts $\xi_0(\cdot + \delta)$ and its "spatial" phase shifts $e^{\gamma A}\xi_0(\cdot)$ are modulated wave solutions to equation $\dot{\xi} = f(\xi, \lambda_0)$, too. Hence, we will describe the perturbation behavior of a two-parameter family of modulated wave solutions under a forcing of modulated wave type.

This problem appears in the mathematical modeling of the locking behavior of self-pulsating lasers to periodically modulated optical signals, see [3, 17, 23, 27]. In these applications $e^{\alpha\tau A} y(\beta\tau)$ describes the external injected light with optical frequency α and $(2\pi/\beta)$-periodic intensity, the modulated wave solution ξ_0 is a so-called self-pulsation of the laser with optical frequency α_0 and $(2\pi/\beta_0)$-periodic intensity, and λ describes the internal laser parameters. For a description of a possible "origin" of self-pulsations in ring lasers (Hopf bifurcation from rotating waves) see, e.g., [31].

In the following we will apply the Theorems 4.3 and 5.1 to Eq. (6.3). Hence, we introduce an appropriate setting. We set

$$X = C^1_{2\pi}, \qquad \tilde{X} = C_{2\pi}, \qquad Y = C^k_{2\pi},$$

$$\Lambda_1 = \mathbb{R}^n \quad \text{(the } \lambda\text{-space)}, \qquad \Lambda_2 = \mathbb{R}^2 \quad \text{(the } (\alpha,\beta)\text{-space)},$$

$$F(x, \lambda, \alpha, \beta)(t) = -\beta \dot{x}(t) + f(x(t), \lambda) - \alpha A x(t).$$

Obviously, $F(\cdot, \lambda, \alpha, \beta)$ is equivariant with respect to the $\mathbf{T}^2$-action

$$(e^{i\gamma}, e^{i\delta}) \cdot x := e^{\gamma A} x(\cdot + \delta).$$

Hence, (2.3) and (2.8) are satisfied. Remark that this action of $\mathbf{T}^2$ on the space $C_{2\pi}$ (as well as on $C^1_{2\pi}$ or $C^k_{2\pi}$) is not C^1-smooth, but it satisfies (2.4) and (2.9).

The ordinary differential operator

$$\partial_x F(x_0, \lambda_0, \alpha_0, \beta_0) = -\beta_0 \frac{d}{dt} + \partial_x f(x_0, \lambda_0) - \alpha_0 A \tag{6.6}$$

is a Fredholm operator from $C^1_{2\pi}$ into $C_{2\pi}$, therefore (2.2) is satisfied. We assume that

$$\ker \partial_x F(x_0, \lambda_0, \alpha_0, \beta_0) = \operatorname{span}\{v_1, v_2\} \quad \text{with } v_1 := A x_0, \quad v_2 := \dot{x}_0 \tag{6.7}$$

and that there exist vector functions $v_1^*, v_2^* \in C^1_{2\pi}$ such that

$$-\beta_0 \dot{v}_j^* = \partial_x f(x_0, \lambda_0)^T v_j^* + \alpha_0 A v_j^*, \qquad \int_0^{2\pi} \langle v_i(t), v_j^*(t) \rangle \, dt = \delta_{ij}. \tag{6.8}$$

In other words, we suppose that zero is a semi-simple eigenvalue of (6.6) of multiplicity two. Hence, (2.6), (2.7), (4.7) and (4.9) are fulfilled (with λ_0 replaced by $(\lambda_0, \alpha_0, \beta_0)$). Here we identify the functions $v_j^* \in C^1_{2\pi}$ with functionals $x \mapsto \int_0^{2\pi} \langle x(t), v_j^*(t) \rangle \, dt$, i.e. with elements of the dual space to $C_{2\pi}$. Finally, (3.10) is satisfied because of

$$\partial_\alpha F(x_0, \lambda_0, \alpha_0, \beta_0) = -A x_0, \qquad \partial_\beta F(x_0, \lambda_0, \alpha_0, \beta_0) = -\dot{x}_0. \tag{6.9}$$

Remark that the assumptions (6.7) and (6.8) imply that the invariant (with respect to the flow of $\dot{\xi} = f(\xi, \lambda_0)$) manifold

$$\mathcal{T} := \{ e^{\varphi A} x_0(\psi) : \varphi, \psi \in \mathbb{R} \}$$

is diffeomorphic to a two-dimensional torus.

Our last assumption reads

$$\sup\{ \operatorname{Re} \rho : \rho \in \operatorname{spec} \partial_x F(x_0, \lambda_0, \alpha_0, \beta_0),\ \rho \notin 2\pi\mathbb{Z} \} < 0, \tag{6.10}$$

i.e., the Floquet multiplicator one of the 2π-periodic solution x_0 to equation $\beta_0 \dot{x} = f(x, \lambda_0) - \alpha_0 A x$ is semi-simple with multiplicity two and the absolute values of all other Floquet multiplicators are smaller than one. This implies that the solution ξ_0 to equation $\dot{\xi} = f(\xi, \lambda_0)$ is asymptotically orbitally stable with asymptotic phases in the following sense (cf. [30]): For each $\varepsilon > 0$ there exists a $\delta > 0$ such that for all solutions $\xi(\tau)$ with $\|\xi(0) - \xi_0(0)\| < \delta$ we have that $\xi(\tau)$ is defined for all $\tau \geqslant 0$, $\inf\{ \|\xi(\tau) - e^{\varphi A} \xi_0(\tau + \psi)\| : \varphi, \psi \in \mathbb{R}\} < \varepsilon$ for all $\tau \geqslant 0$, and that there exist real φ_0 and ψ_0 such that $\|\xi(\tau) - e^{\varphi_0 A} \xi_0(\tau + \psi_0)\| \to 0$ for $\tau \to \infty$. Hence, the purpose of this chapter is to describe the behavior of an asymptotically orbitally stable modulated wave solution of an autonomous $\mathbf{S}^1$-equivariant differential equation under a small perturbation of modulated wave type. Especially, we look for synchronized modulated wave solutions to (6.1), i.e. for solutions of the type

$$\xi(\tau) = e^{\alpha \tau A} x(\beta \tau) \quad \text{with } x \in C^1_{2\pi} \text{ and } x(\tau) \approx \mathscr{T} \quad \text{for all } \tau. \tag{6.11}$$

An application of Theorem 3.1 provides the following result on the persistence of the modulated wave solution ξ_0 under symmetry and autonomy preserving perturbations:

THEOREM 6.1. *There exist a neighbourhood $W \subset \mathbb{R}^n$ of λ_0 and C^k-maps $\hat{u}: W \to C^1_{2\pi}$, $\hat{\alpha}: W \to \mathbb{R}$ and $\hat{\beta}: W \to \mathbb{R}$ with $\hat{u}(\lambda_0) = 0$, $\hat{\alpha}(\lambda_0) = \alpha_0$, and $\hat{\beta}(\lambda_0) = \beta_0$ such that, for all $\lambda \in W$,*

$$\xi_\lambda(\tau) := e^{\hat{\alpha}(\lambda)\, \tau A}(x_0(\hat{\beta}(\lambda)\, \tau) + [\hat{u}(\lambda)](\hat{\beta}(\lambda)\tau)) \tag{6.12}$$

is an asymptotically orbitally stable modulated wave solution with asymptotic phases to equation $\dot{\xi} = f(\xi, \lambda)$, and $\int_0^{2\pi} \langle [\hat{u}(\lambda)](t), v_j^*(t)\rangle \, dt = 0$ for $j = 1, 2$.

Remark 6.2. It is easy to verify that the map $\lambda \in \mathbb{R}^n \mapsto (\hat{\alpha}(\lambda), \hat{\beta}(\lambda)) \in \mathbb{R}^2$ is a submersion in $\lambda = \lambda_0$ if the linear map

$$\lambda \in \mathbb{R}^n \mapsto \left[\int_0^{2\pi} \langle \partial_\lambda f(x_0(t), \lambda_0)\, \lambda, v_j^*(t)\rangle \, dt \right]_{j=1,2} \in \mathbb{R}^2 \tag{6.13}$$

is surjective. Hence, in this generic case there occurs no locking between the frequencies $\hat{\alpha}(\lambda)$ and $\hat{\beta}(\lambda)$. This is a well-known situation for modulated wave solutions of equivariant autonomous differential equations, cf., e.g., [30, 28, 21, 11].

Let us introduce the locking cones

$$K(\varepsilon_0, \mu_0, v_0, z_0, W)$$
$$:= \{(\lambda, \hat{\alpha}(\lambda) + \varepsilon\mu, \hat{\beta}(\lambda) + \varepsilon v, \varepsilon z) \in \mathbb{R}^{n+2} \times C^k_{2\pi} : 0 < |\varepsilon| < \varepsilon_0, (\lambda, \mu, v, z) \in W\}$$

for $\varepsilon_0 > 0$, $(\mu_0, \nu_0, z_0) \in \mathscr{S}$ and neighbourhoods $W \subset \mathbb{R}^n \times \mathscr{S}$ of $(\lambda_0, \mu_0, \nu_0, z_0)$. Here $\mathscr{S} := \{(\mu, \nu, z) \in \mathbb{R} \times \mathbb{R} \times C^k_{2\pi} : |\mu| + |\nu| + \|z\|_k = 1\}$ is the unit sphere in $\mathbb{R} \times \mathbb{R} \times C^k_{2\pi}$, and $\|\cdot\|_k$ is the norm in $C^k_{2\pi}$.

Let us write the reduced bifurcation Eq. (4.5) in coordinates corresponding to the basis $\{v_1, v_2\}$. Because of (6.7)–(6.9), it is of the form

$$
\begin{aligned}
-\mu_0 - \int_0^{2\pi} \langle e^{-\gamma_0 A} z_0(t - \delta_0), v_1^*(t) \rangle \, dt = 0, \\
-\nu_0 - \int_0^{2\pi} \langle e^{-\gamma_0 A} z_0(t - \delta_0), v_2^*(t) \rangle \, dt = 0
\end{aligned}
\tag{6.14}
$$

for $(\gamma_0, \delta_0) \in \mathbb{R}^2$ and $(\mu_0, \nu_0, z_0) \in \mathscr{S}$.

In order to apply Theorem 5.1, we remark the following: If $F(x, \lambda, \alpha, \beta) = y$, then $\operatorname{spec} \partial_x F(x, \lambda, \alpha, \beta) = \{\rho + 2l\pi i : \rho \in \operatorname{spec} M, \; l \in \mathbb{Z}\}$, where M is the monodromy matrix of the solution x. Thus, assumption (6.10) implies that the real parts of all elements of $\operatorname{spec} \partial_x F(x, \lambda, \alpha, \beta)$ are negative iff this is the case for all elements of $\operatorname{spec} \partial_x F(x, \lambda, \alpha, \beta)$ that are close to zero.

From the Theorems 4.3 and 5.1 we obtain the following theorem, that describes families of synchronized modulated wave solutions to (6.1) near $\mathscr{T}$. It is the main result of this section.

THEOREM 6.3. *Let (γ_0, δ_0) be a solution to (6.14) with parameters $(\mu_0, \nu_0, z_0) \in \mathscr{S}$, and suppose that the determinant of the matrix*

$$
\begin{bmatrix}
\int_0^{2\pi} \langle A e^{-\gamma_0 A} z_0(t - \delta_0), v_1^*(t) \rangle \, dt & \int_0^{2\pi} \langle e^{-\gamma_0 A} \dot{z}_0(t - \delta_0), v_1^*(t) \rangle \, dt \\
\int_0^{2\pi} \langle A e^{-\gamma_0 A} z_0(t - \delta_0), v_2^*(t) \rangle \, dt & \int_0^{2\pi} \langle e^{-\gamma_0 A} \dot{z}_0(t - \delta_0), v_2^*(t) \rangle \, dt
\end{bmatrix}
\tag{6.15}
$$

does not vanish.

Then there exist $\varepsilon_0 > 0$, a neighborhood $W \subset \mathbb{R}^n \times \mathscr{S}$ of $(\lambda_0, \mu_0, \nu_0, z_0)$, C^{k-1}-maps $\hat{\gamma}: W \to \mathbb{R}$ and $\hat{\delta}: W \to \mathbb{R}$ with $\hat{\gamma}(\lambda_0, \mu_0, \nu_0, z_0) = \gamma_0$ and $\hat{\delta}(\lambda_0, \mu_0, \nu_0, z_0) = \delta_0$ and a C^k-map $\hat{x}: K(\varepsilon_0, \mu_0, \nu_0, z_0, W) \to C^1_{2\pi}$ such that the following holds:

(i) *Let $(\lambda, \alpha, \beta, y) \in K(\varepsilon_0, \mu_0 \nu_0, z_0, W)$. Then $\hat{\xi}(\tau, \lambda, \alpha, \beta, y) := e^{\alpha\tau A} \hat{x}(\lambda, \alpha, \beta, y)(\beta\tau)$ is a solution to* (6.1). *It is asymptotically stable* (*resp. unstable*) *if all eigenvalues of* (6.15) *have negative real parts* (*resp. if one such eigenvalue has a positive real part*).

(ii) *Let $(\lambda, \mu, \nu, z) \in W$ be fixed. Then, $\hat{\xi}(\tau, \lambda, \hat{\alpha}(\lambda) + \varepsilon\mu, \hat{\beta}(\lambda) + \varepsilon\nu, \varepsilon z)$ tends to $e^{(\hat{\alpha}(\lambda)\tau + \hat{\gamma}(\lambda, \mu, \nu, z))A} [x_0 + \hat{u}(\lambda)](\hat{\beta}(\lambda)(\tau + \hat{\delta}(\lambda, \mu, \nu, z)))$ for $\varepsilon \to 0$.*

By means of Theorem 6.3, there exists a straightforward procedure to construct control parameters λ, α, β and y such that (6.1) has a synchronized modulated wave solution near $\mathscr{T}$: First, take $z_0 \in C^k_{2\pi}$. Second, take a regular value (μ_0, ν_0) of the map

$$(\gamma, \delta) \mapsto \left[-\int_0^{2\pi} \langle e^{-\gamma A} z_0(t-\delta), v_j^*(t) \rangle \, dt \right]_{j=1,2}$$

Then $\alpha = \alpha_0 + \hat{\alpha}(\lambda) + \varepsilon\mu$, $\beta = \beta_0 + \hat{\beta}(\lambda) + \varepsilon\nu$ and $y = \varepsilon z$ with arbitrary $\varepsilon \in \mathbb{R}$ near zero, $\lambda \in \mathbb{R}^n$ near λ_0, $\mu \in \mathbb{R}$ near μ_0, $\nu \in \mathbb{R}$ near ν_0 and $z \in C^k_{2\pi}$ near z_0 are parameters of the type demanded.

In applications, however, one often has to answer more specific questions about the synchronization behaviour of (6.1). For example, the following question is natural: Given any α near α_0, β near β_0 and y near zero, do there exist parameters λ near zero such that (6.1) has a synchronized modulated wave solution near $\mathscr{T}$? In other words: Is it possible to adjust λ near λ_0 such that the modulated wave solution ξ_0 to the unforced equation locks in with respect to any given small forcing modulated wave, the frequencies of which are close to the corresponding frequencies of ξ_0?

The following corollary gives an answer. Speaking in the language of the applications to laser dynamics, it shows that, for "almost any" external light signal of modulated wave type with frequencies which are close to the corresponding frequencies of the self-pulsation, it is possible to adjust the internal laser parameters (at least two of them) such that synchronization takes place.

COROLLARY 6.4. *Suppose the map* (6.13) *to be surjective. Then, for any* α *near* α_0, β *near* β_0 *and* $y \in C^k_{2\pi}$ *near zero such that the map*

$$(\gamma, \delta) \in \mathbb{R}^2 \mapsto \left[-\int_0^{2\pi} \langle e^{-\gamma A} y(t-\delta), v_j^*(t) \rangle \, dt \right]_{j=1,2} \in \mathbb{R}^2 \qquad (6.16)$$

has regular values, there exist parameters λ *near* λ_0 *such that* (6.1) *has a synchronized modulated wave solution of the type* (6.11) (*cf. Fig.* 1).

Proof. Take $y \in C^k_{2\pi}$ sufficiently small, and let (μ, ν) be a regular value of the map (6.16). Then $|\mu| + |\nu| + \|y\|_k$ is small, and, because of Theorem 6.3, (6.1) has a synchronized modulated wave solution of the type (6.11) for parameters $\alpha = \alpha_0 + \hat{\alpha}(\lambda) + \mu$ and $\beta = \beta_0 + \hat{\beta}(\lambda) + \nu$ with arbitrary λ near λ_0. But $\lambda \mapsto (\hat{\alpha}(\lambda), \hat{\beta}(\lambda))$ is a submersion in λ_0 (cf. Remark 6.2), therefore the corollary is proved. ∎

Remark 6.5. Theorem 6.3(ii) says that, for control parameters $\alpha = \alpha_0 + \varepsilon\mu$, $\beta = \beta_0 + \varepsilon\nu$ and $y(t) = \varepsilon z(t)$, there exists a modulated wave solution to (6.1)

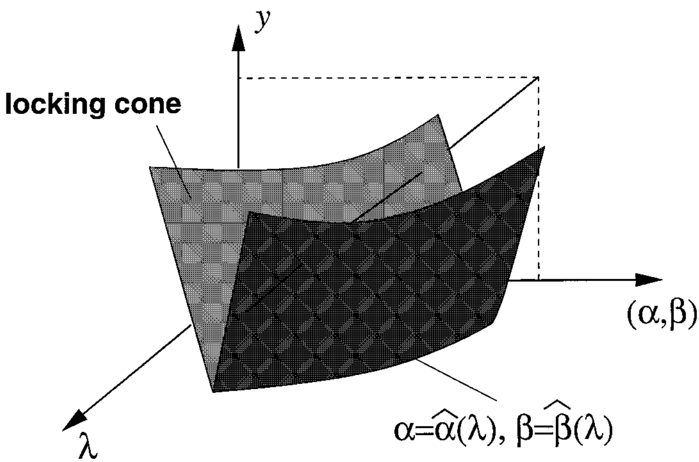

Fig. 1. Crossing of the locking cone by variation of the internal control parameters.

which tends, for $\varepsilon \to 0$ (and λ, μ, v, and z fixed), to $e^{\gamma A}\xi_\lambda(\cdot+\delta)$, where ξ_λ is defined in (6.12). The phase shifts γ and δ are determined by the reduced bifurcation Eq. (6.14). In particular, they depend on λ, μ, v and z, in general. Hence, the frequency locking phenomena considered here do not have the so-called phase locking property. On the contrary, it is possible to control the phases of the locked solutions by changing the control parameters (for questions concerning "phase locking" and "phase regulation" see, e.g., [1]).

Remark 6.6. Let us briefly describe how to obtain results on forced subharmonic frequency locking of a modulated wave solution with modulation frequency β_0 under a forcing of an external modulated wave with modulation frequency $\beta \approx (p/q)\,\beta_0$ (with $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and p and q relatively prime) into modulated wave solutions with modulation frequency (β/q). Hence, let us look for $2\pi q$-periodic solutions x to Eq. (6.3) for given control parameters $\lambda \approx \lambda_0$, $\alpha \approx \alpha_0$, $\beta \approx (p/q)\,\beta_0$ and $y \in C^k_{2\pi}$ near zero.

We define a $2\pi q$-periodic vector function x_{pq} by $x_{pq}(t) := x_0((p/q)\,t)$. Then, because of assumption (6.5),

$$\frac{p}{q}\,\beta_0 \dot{x}_{pq} = f(x_{pq}, \lambda_0) - A x_{pq}.$$

Hence, everything that has been done in this section may be repeated with β_0 and x_0 replaced by $(p/q)\,\beta_0$ and x_{pq}, respectively, and by working in spaces of $2\pi q$-periodic vector functions. Especially, all integrals from 0 to 2π have to be replaced by integrals from 0 to $2\pi q$, and, therefore, $v_1^*(t)$ and

$v_1^*(t)$ have to be replaced by $(1/p)\, v_1^*((p/q)\, t)$ and $(1/q)\, v_2^*((p/q)\, t)$, respectively (cf. (6.8)). The reduced bifurcation Eq. (6.14) takes the form

$$\mu = -\frac{1}{p}\int_0^{2\pi q} \left\langle e^{-\gamma A} z(t-\delta),\, v_1^*\left(\frac{p}{q}\, t\right)\right\rangle dt,$$
$$\nu = -\frac{1}{q}\int_0^{2\pi q} \left\langle e^{-\gamma A} z(t-\delta),\, v_2^*\left(\frac{p}{q}\, t\right)\right\rangle dt, \tag{6.17}$$

and one has to look for solutions $\gamma \in [0, 2\pi)$ and $\delta \in [0, 2\pi q)$.

Of course, the results, obtained in the way described above are by no means uniform with respect to p and q. On the contrary, for (6.17) (with fixed p, q and z_0) to be solvable, the parameter (μ, ν) has to belong to the image of the map

$$(\gamma, \delta) \mapsto \Bigg(-\frac{1}{p}\int_0^{2\pi q} \left\langle e^{-\gamma A} z(t-\delta),\, v_1^*\left(\frac{p}{q}\, t\right)\right\rangle dt,$$
$$-\frac{1}{q}\int_0^{2\pi q} \left\langle e^{-\gamma A} z(t-\delta),\, v_2^*\left(\frac{p}{q}\, t\right)\right\rangle dt \Bigg).$$

Let us show that this image has a diameter of order $(pq)^{-k-1}$ for large p and q (and, hence, that the corresponding locking cone is "thin"). Indeed, if

$$z(t) = \frac{a_0(z)}{2} + \sum_{j=1}^{\infty} (a_j(z) \cos jt + b_j(z) \sin jt),$$
$$v_i^*(t) = \frac{a_0(v_i^*)}{2} + \sum_{j=1}^{\infty} (a_j(v_i^*) \cos jt + b_j(v_i^*) \sin jt),$$

then the Fourier coefficients $a_j(z)$, $b_j(z)$, $a_j(v_i^*)$ and $b_j(v_i^*)$ are of order j^{-k-1} for $j \to \infty$ (because the vector functions z and v_i^* are C^k-smooth). Therefore, the right-hand side of the first equation in (6.17), for example, is of the form

$$-\frac{1}{p}\int_0^{2\pi q} \left\langle e^{-\gamma A} z(t-\delta),\, v_1^*\left(\frac{p}{q}\, t\right)\right\rangle dt$$
$$= -\frac{q}{2p}\left[\langle e^{-\gamma A} a_0(z), a_0(v_1^*)\rangle + \sum_{j=1}^{\infty} (A_j(\gamma) \cos jp\delta + B_j(\gamma) \sin jp\delta)\right]$$

with

$$A_j(\gamma) := \langle e^{-\gamma A} a_{jp}(z), a_{jq}(v_1^*)\rangle - \langle e^{-\gamma A} b_{jp}(z), b_{jq}(v_1^*)\rangle,$$
$$B_j(\gamma) := \langle e^{-\gamma A} a_{jp}(z), b_{jq}(v_1^*)\rangle - \langle e^{-\gamma A} b_{jp}(z), a_{jq}(v_1^*)\rangle,$$

and the coefficients $A_j(\gamma)$ and $B_j(\gamma)$ are of order $(j^2pq)^{-k-1}$ for large p and q.

For results concerning the uniformity with respect to p and q of the Lyapunov–Schmidt reduction, see [5].

Remark 6.7. There are many similarities of the methods used in this section (Lyapunov–Schmidt reduction in a space of periodic vector functions near a family of periodic solutions to the unforced equation, scaling, application of the Implicit Function Theorem) to the generalization of Melnikov's method used by C. Chicone (Lyapunov–Schmidt reduction in the phase space near a submanifold of periodic orbits to the unforced equation, scaling, application of the Implicit Function Theorem) for solving frequency entrainment problems in [6] and [7]. It seems that the information about the locking (number, stability and phase of the locking solutions, location of the locking cones) can be determined by means of Chicone's "reduced bifurcation function" as well as by our "reduced bifurcation equation", although the algorithms seem to be completely different.

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