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Univalent Solutions of Briot–Bouquet Differential Equations

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Let β and γ be complex numbers and let $h(z)$ be regular in the unit disc U. This article studies the Briot–Bouquet differential equation $q(z)+zq'(z)/(\beta q(z)+\gamma)=h(z)$. Sufficient conditions are obtained for both the regularity and univalency of the solution in U. In addition, applications of these results to differential subordinations, integral operators and univalent functions are given. © 1985 Academic Press, Inc.

1. INTRODUCTION

Let β and γ be complex numbers and let $h(z)$ be regular in the unit disc U. In this article we shall be concerned with determining properties of the solutions of the differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=h(z), \tag{1}$$

with $q(0)=h(0)$. Differential equations of this form are said to be of Briot–Bouquet type [3, p. 403].

Several applications of these equations in the theory of univalent functions have recently appeared in [10 and 2]. In the latter article several results involving dominants of differential subordinations are obtained under the assumption that the solutions of (1) are regular and univalent in U. In this paper we will determine sufficient conditions for both the regularity and univalency of these solutions.

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In the special case $\beta=0$, the differential equation (1) has a regular solution given by $q(z)=\gamma z^{-\gamma}\int_0^z h(t)\cdot t^{\gamma-1}\,dt$. If, in addition, h is a convex function and $\operatorname{Re}\gamma\geqslant 0$ then $q(z)$ is a univalent function [5, p. 115]. When $\beta\neq 0$ it is possible to formally obtain a solution of (1). We first consider the case $q(0)=h(0)=c\neq 0$. If we let

$$Q(z)=z\exp\int_0^z \frac{q(t)-c}{ct}\,dt \tag{2}$$

and

$$H(z)=z\exp\int_0^z \frac{h(t)-c}{ct}\,dt,$$

then

$$q(z)=\frac{czQ'(z)}{Q(z)} \qquad \text{and} \qquad h(z)=\frac{czH'(z)}{H(z)}. \tag{3}$$

Multiplying (1) by β and using (3) we obtain

$$\frac{\beta czQ'(z)}{Q(z)}+\frac{\beta zq'(z)}{\beta q(z)+\gamma}=\frac{\beta czH'(z)}{H(z)}$$

and

$$\beta c\left(\frac{Q'(z)}{Q(z)}-\frac{1}{z}\right)+\frac{\beta q'(z)}{\beta q(z)+\gamma}=\beta c\left(\frac{H'(z)}{H(z)}-\frac{1}{z}\right).$$

Integrating from 0 to z we obtain

$$\log\left(\frac{Q(z)}{z}\right)^{\beta c}+\log\left(\frac{\beta q(z)+\gamma}{\beta c+\gamma}\right)=\log\left(\frac{H(z)}{z}\right)^{\beta c}.$$

Simplifying and using (3) leads to

$$[Q(z)]^{\beta c}\left(\frac{\beta czQ'(z)}{Q(z)}+\gamma\right)=(\beta c+\gamma)[H(z)]^{\beta c}. \tag{4}$$

Multiplying by $z^{\gamma-1}$ and integrating from 0 to z we obtain

$$[Q(z)]^{\beta c}z^{\gamma}=(\beta c+\gamma)\int_0^z [H(t)]^{\beta c}\,t^{\gamma-1}\,dt.$$

Using this result with (3) and (4) we obtain the formal solution of (1) given by

$$q(z)=\frac{czQ'(z)}{Q(z)}=\frac{\beta c+\gamma}{\beta}\left(\frac{H(z)}{Q(z)}\right)^{\beta c}-\frac{\gamma}{\beta} \tag{5}$$

$$=z^{\gamma}[H(z)]^{\beta c}\left(\beta\int_0^z [H(t)]^{\beta c}\, t^{\gamma-1}\,dt\right)^{-1}-\gamma/\beta,$$

where $H(z)$ is given by (2).

In the case $q(0)=h(0)=0$, let

$$Q(z)=z\exp\frac{\beta}{\gamma}\int_0^z\frac{q(t)}{t}\,dt \tag{6}$$

and

$$H(z)=z\exp\frac{\beta}{\gamma}\int_0^z\frac{h(t)}{t}\,dt.$$

Using the same technique as above, we obtain the formal solution

$$q(z)=\frac{\gamma}{\beta}\left(\frac{zQ'(z)}{Q(z)}-1\right)=\frac{\gamma}{\beta}\left(\frac{H(z)}{Q(z)}\right)^{\gamma}-\frac{\gamma}{\beta} \tag{7}$$

$$=H^{\gamma}(z)\left(\beta\int_0^z H^{\gamma}(t)\,t^{-1}\,dt\right)^{-1}-\gamma/\beta.$$

In Section 2 we will determine conditions on β, γ, and $h(z)$ so that the formal solutions, given by (5) when $q(0)=h(0)=c\neq 0$, and by (7) when $q(0)=h(0)=0$, are well defined, regular, and univalent. In Section 3 we apply these results to Briot–Bouquet differential subordinations.

We close this section with a lemma that will be used several times in the next section.

LEMMA 1. *Let $\lambda(z)$ be a function defined in U with* $\operatorname{Re}\lambda(z)\geqslant 0$ *for $z\in U$. If $f(z)$ is regular in U and*

$$\operatorname{Re}[f(z)+\lambda(z)\,zf'(z)]>0 \tag{8}$$

for $z\in U$, then $\operatorname{Re}f(z)>0$ *for $z\in U$.*

Proof. Let $f(0)=a$ and $\psi(r,s)=r+\lambda(z)s$. From (8) we obtain $\operatorname{Re}a>0$ and $\operatorname{Re}\psi(f(z),\ zf'(z))>0$. The conclusion of the lemma follows from Theorem 5 of [8] if we can show that $\operatorname{Re}\psi(r_2 i, s_l)\leqslant 0$ when $s_1\leqslant 0$. But in this case we have $\operatorname{Re}\psi(r_2 i, s_1)=[\operatorname{Re}\lambda]\,s_1\leqslant 0$. Hence $\operatorname{Re}f(z)>0$.

2. Regular and Univalent Solutions

THEOREM 1. *Let β and γ be complex numbers with $\beta \neq 0$, and let $h(z) = c + h_1 z + \cdots$, be regular in U. If* $\mathrm{Re}[\beta h(z) + \gamma] > 0$ *then the solution of*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \tag{9}$$

with $q(0) = c$, is regular in U. The solution satisfies $\mathrm{Re}[\beta q(z) + \gamma] > 0$ *and is given by*

$$\begin{aligned} q(z) &= H^{\gamma}(z) \left(\beta \int_0^z H^{\gamma}(t)\, t^{-1}\, dt\right)^{-1} - \gamma/\beta \qquad \text{if} \quad c = 0, \\ &= z^{\gamma}[H(z)]^{\beta c} \left(\beta \int_0^z [H(t)]^{\beta c}\, t^{\gamma - 1}\, dt\right)^{-1} - \gamma/\beta \qquad \text{if} \quad c \neq 0, \end{aligned} \tag{10}$$

where

$$\begin{aligned} H(z) &= z \exp \frac{\beta}{\gamma} \int_0^z \frac{h(t)}{t}\, dt \qquad \text{if} \quad c = 0, \\ &= z \exp \int_0^z \frac{h(t) - c}{ct} \qquad \text{if} \quad c \neq 0. \end{aligned} \tag{11}$$

Proof. Since $H(z)/z \neq 0$, we can define the function f by

$$\begin{aligned} f(z) &= \frac{1}{z^{\gamma}(H(z)/z)^{\gamma}} \int_0^z \left(\frac{H(t)}{t}\right)^{\gamma} t^{\gamma - 1}\, dt \qquad \text{if} \quad c = 0, \\ &= \frac{1}{z^{\beta c + \gamma}(H(z)/z)^{\beta c}} \int_0^z \left(\frac{H(t)}{t}\right)^{\beta c} t^{\beta c + \gamma - 1}\, dt \qquad \text{if} \quad c \neq 0, \end{aligned}$$

where all powers are chosen as principal ones. The function f is regular in U and satisfies

$$\begin{aligned} f(z) &= \frac{1}{H^{\gamma}(z)} \int_0^z H^{\gamma}(t)\, t^{-1}\, dt \qquad \text{if} \quad c = 0, \\ &= \frac{1}{z^{\gamma}[H(z)]^{\beta c}} \int_0^z [H(t)]^{\beta c}\, t^{\gamma - 1}\, dt \qquad \text{if} \quad c \neq 0. \end{aligned} \tag{12}$$

In both cases $f(0) = 1/(\beta c + \gamma)$ and hence $\mathrm{Re} f(0) > 0$. We shall show that $f(z) \neq 0$ in U by using Lemma 1 to show that $\mathrm{Re} f(z) > 0$ in U. By differentiating (12) and logarithmically differentiating (11) we obtain

$$[\beta h(z) + \gamma]\, f(z) + zf'(z) = 1,$$

for any c. If we let $P(z)=\beta h(z)+\gamma$, then $\operatorname{Re} P(z)>0$, $\operatorname{Re}[1/P(z)]>0$, and we obtain

$$f(z)+\frac{1}{P(z)}\,zf'(z)=\frac{1}{P(z)}. \tag{13}$$

Hence

$$\operatorname{Re}\left[f(z)+\frac{1}{P(z)}\,zf'(z)\right]>0,$$

and we can apply Lemma 1 to obtain $\operatorname{Re} f(z)>0$.

Since $f(z)\neq 0$, the function q given by

$$q(z)=\frac{1}{\beta f(z)}-\frac{\gamma}{\beta} \tag{14}$$

is regular in U and satisfies $\operatorname{Re}[\beta q(z)+\gamma]=\operatorname{Re}[1/f(z)]>0$. Combining (11), (12), and (14) we obtain the function $q(z)$ given in (10). Hence this function is regular and, as was shown in Section 1, this function is the solution of (9) with $q(0)=c$. This completes the proof of the theorem.

As an incidental consequence of Theorem 1 we obtain the following corollary. This provides a simple proof of the regularity of an integral operator defined on a class of regular functions.

COROLLARY 1.1. *Let β and γ be complex numbers with $\beta\neq 0$, and let $f(z)=z+c_2z^2+\cdots$, be regular in U. If* $\operatorname{Re}[\beta zf'(z)/f(z)+\gamma]>0$ *in U then the function F defined by*

$$F(z)=\left(\frac{\beta+\gamma}{z^{\gamma}}\int_0^z f^{\beta}(t)\,t^{\gamma-1}\,dt\right)^{1+\beta} \tag{15}$$

is regular in U, $F(z)/z\neq 0$, and $\operatorname{Re}[\beta zF'(z)/F(z)+\gamma]>0$.

Proof. The condition $\operatorname{Re}[\beta zf'(z)/f(z)+\gamma]>0$, together with $f(0)=0$, imply that $f(z)/z\neq 0$ for $z\in U$. The function h, defined by $h(z)=zf'(z)/f(z)$, is regular in U and satisfies $\operatorname{Re}[\beta h(z)+\gamma]>0$ and $h(0)=1$. If we employ this particular h in Theorem 1, with $f=H$ and $c=1$, we conclude that the function

$$q(z)=z^{\gamma}f^{\beta}(z)\left(\beta\int_0^z f^{\beta}(t)\,t^{\gamma-1}\,dt\right)^{-1}-\gamma/\beta=1+q_1z+\cdots, \tag{16}$$

is regular in U and satisfies $\operatorname{Re}[\beta q(z)+\gamma]>0$. If we now define F by

$$F(z)=z\exp\int_0^z\frac{q(t)-1}{t}\,dt, \tag{17}$$

then $F(z)$ is regular in U, $F(z)/z \neq 0$, and $\operatorname{Re}[\beta zF'(z)/F(z)+\gamma] > 0$. Combining (16) and (17) we obtain (15), which completes the proof of the corollary.

If we let $K_{\beta,\gamma} = \{f \mid f$ regular in U, $f(0)=0$, $f'(0)=1$, and $\operatorname{Re}[\beta zf'/f+\gamma] > 0\}$ and define the operator $A(f)=F$, where F is given by (15), then the corollary implies that A maps $K_{\beta,\gamma}$ into $K_{\beta,\gamma}$. Note that $K_{\beta,0}$ with $\operatorname{Re}\beta > 0$ is the set of spiral-like functions [9, p. 172].

We next extend Theorem 1 by determining conditions under which the solutions of the Briot–Bouquet differential equation will be univalent. Recall that the regular function $f(z)$ with $f'(0) \neq 0$, is convex (univalent) in U if and only if $\operatorname{Re} zf''(z)/f'(z) > -1$ [9, p. 44].

THEOREM 2. *Let β and γ be complex numbers with $\beta \neq 0$, and let $h(z)$ be regular in U with $h'(0) \neq 0$. If we set $P(z) = \beta h(z) + \gamma$ and require that*

(i) $\operatorname{Re} P(z) > 0$ *for $z \in U$, and*

(ii) $Q \equiv \log P$ *and $R \equiv 1/P$ are convex in U, then the solution of*

$$q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = h(z) \qquad (h(0) = q(0)),$$

as given by (9), *is univalent in U.*

Proof. We will show that the function $f(z)$ of (12) is a (close-to-convex) univalent function (see [9, p. 51]). This combined with (14) will prove that $q(z)$ is univalent. Note that $R'(0) = -Q'(0)/P(0)$ and $Q'(0) = \beta h'(0)/(\beta h(0)+\gamma) \neq 0$. Since Q and $-Q$ are convex we can prove that $f(z)$ is (close-to-convex) univalent by showing that $\operatorname{Re} f'(z)/[-Q'(z)] > 0$ for $z \in U$. From (13) we have $f(z) + zf'(z)/P(z) = 1/P(z)$. Differentiating this equation and setting $p(z) = -f'(z)/Q'(z)$ we obtain

$$\left[P(z) + 1 + \frac{zR''(z)}{R'(z)}\right] p(z) + zp'(z) = 1. \tag{18}$$

If we let $P_1(z) = P(z) + 1 + zR''(z)/R'(z)$, then from (i) and (ii) we obtain $\operatorname{Re} P_1(z) > 0$, and we can rewrite (18) as

$$p(z) + \frac{1}{P_1(z)}\, zp'(z) = \frac{1}{P_1(z)}.$$

Since $\operatorname{Re}[1/P_1(z)] > 0$ we have

$$\operatorname{Re}\left[p(z) + \frac{1}{P_1(z)}\, zp'(z)\right] > 0,$$

which, by applying Lemma 1, implies that $\operatorname{Re} p(z) > 0$. Hence $f(z)$ is univalent, as is $q(z)$, and this completes the proof of the theorem.

3. Applications to Briot–Bouquet Differential Subordinations

Let $f(z)$ and $F(z)$ be regular in U. The function $f(z)$ is *subordinate* to $F(z)$, written $f(z) \prec F(z)$, if $F(z)$ is univalent, $f(0) = F(0)$ and $f(U) \subset F(U)$.

In [2] the Briot–Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \qquad (p(0) = h(0) = c) \tag{19}$$

was investigated. If the univalent function $q(z)$ has the property that $p(z) \prec q(z)$ for all $p(z)$ satisfying (19) then it is called a *dominant* of (19). If $\tilde{q}(z)$ is a dominant and $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (19) then $\tilde{q}(z)$ is said to be the *best dominant* of the differential subordination. The existence of a best dominant is provided by the following lemma.

Lemma 2 [2, Theorem 2]. *Let β and γ be complex numbers and let $h(z)$ be convex (univalent) in U with* $\operatorname{Re}[\beta h(z) + \gamma] > 0$. *Let $p(z)$ be regular in U and satisfy* (19). *If the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \qquad (q(0) = h(0)) \tag{20}$$

has a univalent solution $q(z)$, then $p(z) \prec q(z) \prec h(z)$, and $q(z)$ is the best dominant of (19).

Lemma 2 requires that $P(z) \equiv \beta h(z) + \gamma$ be convex, while Theorem 2 requires that $1/P$ and $\log P$ be convex. Both results require that $\operatorname{Re} P(z) > 0$. If $\operatorname{Re} P(z) > 0$ then a simple computation shows that P and $1/P$ convex implies that $\log P$ is also convex. Because of this we can combine Lemma 2 and Theorem 2 and obtain the following more definitive result.

Theorem 3. *Let β and γ be complex numbers with $\beta \neq 0$, and let $P(z) = \beta h(z) + \gamma$. If*

(i) $\operatorname{Re} P(z) > 0$ *for $z \in U$, and*

(ii) *P and $1/P$ are convex in U,*

then the solution of (20) *is univalent and is the best dominant of* (19).

The functions $P(z) = e^{\lambda z}$, with $|\lambda| \leqslant 1$, and $P(z) = (1 + Az)/(1 + Bz)$, with $-1 \leqslant B < A \leqslant 1$, are examples of functions satisfying these conditions.

COROLLARY 3.1. *Suppose $r(z)$ and $s(z)$ are regular in U with $r(0)=s(0)=0$, $r'(0)=s'(0)=1$, and*

$$\frac{zr'(z)}{r(z)} \prec \frac{zs'(z)}{s(z)}.$$

If $h(z) \equiv zs'(z)/s(z)$ satisfies the conditions of Theorem 3 and if $R \equiv I(r)$, $S \equiv I(s)$, where I is the operator defined in (15), then

$$\frac{zR'(z)}{R(z)} \prec \frac{zS'(z)}{S(z)} \prec \frac{zs'(z)}{s(z)}.$$

The left subordination result is sharp.

Proof. If we set $p(z)=zR'(z)/R(z)$ and $q(z)=zS'(z)/S(z)$ then $p(z)+zp'(z)/(\beta p(z)+\gamma)=zr'(z)/r(z)$ and $q(z)+zq'(z)/(\beta q(z)+\gamma)=zs'(z)/s(z)=h(z)$. Applying Theorems 2 and 3 we obtain the desired result $p(z) \prec q(z) \prec h(z)$. Since q is the best dominant, $zR'(z)/R(z) \prec zS'(z)/S(z)$ is sharp.

Several cases of this result, for special values of β and γ, have been used to determine the order of starlikeness of classes of univalent functions. An example of this technique in determining the order of spiral-likeness is given in Example 2.

In the rest of this section we restrict our analysis to the case when $h(z)=(1+Az)/(1+Bz)$. This function is convex for $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \leqslant 1$. The dominants arising from the differential subordination corresponding to this particular $h(z)$ have several applications to univalent functions.

COROLLARY 3.2. *Let $A, B, \beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, $|B| \leqslant 1$ and $A \neq B$, and suppose that these constants satisfy*

$$\operatorname{Re}[\beta(1-A)(1-\bar{B})+\gamma|1-B|^2]>0 \tag{21}$$

and

$$\operatorname{Re}[\beta(1-A)(1-\bar{B})+\gamma|1-B|^2]\cdot\operatorname{Re}[\beta(1+A)(1+\bar{B})+\gamma|1+B|^2] - [\operatorname{Im}[\beta(\bar{B}-A)+\gamma(\bar{B}-B)]]^2 \geqslant 0,$$

or

$$\operatorname{Re}[\beta(1+A)(1+\bar{B})+\gamma|1+B|^2] \geqslant 0 \tag{22}$$

and

$$\operatorname{Re}[\beta(1-A)(1-\bar{B})+\gamma|1-B|^2]=\operatorname{Im}[\beta(\bar{B}-A)+\gamma(\bar{B}-B)]=0.$$

Then the differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=\frac{1+Az}{1+Bz}$$

has a univalent solution given by

$$\begin{aligned} q(z) &= \frac{z^{\beta+\gamma}(1+Bz)^{\beta((A-B)/B)}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta((A-B)/B)}\,dt}-\frac{\gamma}{\beta} \qquad \text{if } B\neq 0 \\ &= \frac{z^{\beta+\gamma}e^{\beta Az}}{\beta\int_0^z t^{\beta+\gamma-1}e^{\beta At}\,dt}-\frac{\gamma}{\beta} \qquad \text{if } B=0. \end{aligned} \tag{23}$$

If $p(z)$ is regular in U and satisfies

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\frac{1+Az}{1+Bz}$$

then $p(z)\prec q(z)\prec(1+Az)/(1+Bz)$ and $q(z)$ is the best dominant.

Proof. If we set $h(z)=(1+Az)/(1+Bz)$ and $P(z)=\beta h(z)+\gamma$ then

$$P(z)=\frac{\beta+\gamma+(\beta A+\gamma B)z}{1+Bz}, \tag{24}$$

and P is convex since $|B|\leqslant 1$. We prove this corollary by showing that P satisfies conditions (i) and (ii) of Theorem 3. We first show that $\operatorname{Re} P(z)>0$ for $z\in U$ if and only if conditions (21) or (22) hold. This follows by considering $\operatorname{Re} P(z)$ when $|z|=1$. If we set $z=(1+it)/(1-it)$ with $t\in(-\infty,\infty)$ then a simple calculation shows that $\operatorname{Re} P(z)\geqslant 0$ if and only if

$$\begin{aligned} &\{\operatorname{Re}[\beta(1-A)(1-\bar{B})+\gamma|1-B|^2]\}\,t^2+\{\operatorname{Im}[\beta(\bar{B}-A)+\gamma(\bar{B}-B)]\}t \\ &\quad+\operatorname{Re}[\beta(1+A)(1+\bar{B})+\gamma|1+B|^2]\geqslant 0. \end{aligned} \tag{25}$$

The quadratic $Mt^2+Nt+k\geqslant 0$ for all t if and only if $M>0$ and $N^2-4Mk\leqslant 0$, or if $k\geqslant 0$ and $M=N=0$. These are precisely the conditions given in (21) and (22) for the quadratic in (25). Hence $\operatorname{Re} P(z)>0$ for $z\in U$ and condition (i) of Theorem 3 is satisfied. Using this result together with the fact that P is a linear transformation mapping onto a convex domain we conclude that $1/P$ also maps onto a convex domain. Hence by Theorem 3, $q(z)$ is univalent and is the best dominant of the differential subordination. The formula for $q(z)$ as given in (23) is easily obtained from (10) and (11) with $c=1$, and this completes the proof of the corollary.

We next discuss several examples of this corollary and in the process indicate several applications in the theory of univalent functions.

EXAMPLE 1. Let $\beta > 0$, $\gamma = 0$ and $-1 \leqslant A, B \leqslant 1$ with $A \neq B$. In this case (21) and (22) are satisfied, and hence

$$q(z) + \frac{zq'(z)}{\beta q(z)} = \frac{1 + Az}{1 + Bz} \qquad (q(0) = 1) \tag{26}$$

has the univalent solution $q(z)$ given by (23), with $\gamma = 0$. If, in addition, $p(z)$ is regular and satisfies

$$p(z) + \frac{zp'(z)}{\beta p(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{27}$$

then $p(z) \prec q(z) \prec (1 + Az)/(1 + Bz)$ and $q(z)$ is the best dominant of (27). This result improves the result of Jakubowski and Kaminski [4] which showed that (27) implies $p(z) \prec (1 + Az)/(1 + Bz)$.

If we take $\beta = 1$ in (26), then this differential equation has the univalent solution

$$\begin{aligned} q(z) &= \frac{Az}{(1 + Bz)[1 - (1 + Bz)^{-A/B}]} && A \neq 0, B \neq 0, \\ &= \frac{Bz}{(1 + Bz)\ln(1 + Bz)} && A = 0, B \neq 0, \\ &= \frac{Aze^{Az}}{e^{Az} - 1} && A \neq 0, B = 0. \end{aligned} \tag{28}$$

If in addition $p(z)$ is regular with $p(0) = 1$ then

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1 + Az}{1 + Bz} \Rightarrow p(z) \prec q(z), \tag{29}$$

where $q(z)$ is given by (28). If we set $p(z) = zF'(z)/F(z)$ in (29) we obtain

$$1 + \frac{zF''(z)}{F'(z)} \prec \frac{1 + Az}{1 + Bz} \Rightarrow \frac{zF'(z)}{F(z)} \prec q(z). \tag{30}$$

By further specializing A and B in (30) we can obtain several results about the convex differential operator $1 + zF''/F'$. If we take $A = 2\alpha - 1$,

$0 \leqslant \alpha < 1$, and $B = 1$ in (30) we obtain the following result of MacGregor [6] for convex functions of order α: $\operatorname{Re}(1 + zF''/F') > \alpha$ implies

$$\frac{zF'(x)}{F(z)} \prec q(z) = \frac{(2\alpha - 1)z}{(1+z)[1-(1+z)^{1-2\alpha}]} \qquad \alpha \neq \tfrac{1}{2},$$
$$= \frac{z}{(1+z)\ln(1+z)} \qquad \alpha = \tfrac{1}{2}.$$

If we take $B = -A$ in (30) we obtain

$$1 + \frac{zF''(z)}{F'(z)} \prec \frac{1+Az}{1-Az} \Rightarrow \frac{zF'(z)}{F(z)} \prec \frac{1}{1-Az},$$

where $|A| \leqslant 1$. This is a generalization of results of Marx [7] and Strohhäcker [11] who proved this result for $A = 1$.

If we take $B = 0$ in (30) we obtain

$$1 + \frac{zF''}{F'} \prec 1 + Az \Rightarrow \frac{zF'(z)}{F(z)} \prec \frac{Aze^{Az}}{e^{Az} - 1},$$

where $|A| \leqslant 1$ and $A \neq 0$. If $0 < A \leqslant 1$ we also have

$$\left|\frac{F''(z)}{F'(z)}\right| < A \Rightarrow \frac{zF'(z)}{F(z)} \prec \frac{Aze^{Az}}{e^{Az} - 1}.$$

If we take $A = 0$ in (30) we obtain

$$1 + \frac{zF''(z)}{F'(z)} \prec \frac{1}{1+Bz} \Rightarrow \frac{zF'(z)}{F(z)} \prec \frac{Bz}{(1+Bz)\ln(1+Bz)}.$$

EXAMPLE 2. Let $\beta = e^{i\alpha}$, $-\pi/2 < \alpha < \pi/2$, $\gamma \in \mathbb{C}$, $B = 1$, and $A = \bar{\beta}(2\rho\cos\alpha - \bar{\beta})$, where ρ satisfies

$$\frac{-\operatorname{Re}\gamma}{\cos\alpha} \leqslant \rho < 1. \tag{31}$$

Note that (31) implies that $\operatorname{Re}[\beta + \gamma] > 0$ and that ρ may assume negative values. In addition, (31) implies that (22) is satisfied. Hence, by Corollary 3.1 we obtain

$$p(z) + \frac{zp'(z)}{e^{i\alpha}p(z) + \gamma} \prec h(z) \equiv \frac{1 + [e^{-i\alpha}(2\rho\cos\alpha - e^{-i\alpha})]z}{1+z}$$
$$\Rightarrow p(z) \prec q(z) \prec h(z), \tag{32}$$

with

$$q(z) = e^{-i\alpha}\left\{\frac{z^{e^{i\alpha}+\gamma}(1+z)^{-2(1-\rho)\cos\alpha}}{\int_0^z t^{e^{i\alpha}+\gamma-1}(1+t)^{-2(1-\rho)\cos\alpha}\,dt} - \gamma\right\}.$$

We can use (32) to improve a result of Bajpai [1, Theorem 1] concerning Libera–Bernardi transforms of spiral-like functions. Let $f(z)$ be an α-spiral-like function of order ρ, that is, $f(z)$ is regular in U, $f(0)=f'(0)-1=0$, and

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)>\rho\cos\alpha,$$

where $-\pi/2<\alpha<\pi/2$ and $0\leqslant\rho<1$. Bajpai shows that if $0\leqslant\rho<1$ and $-\operatorname{Re}\gamma/\cos\alpha\leqslant\rho$, then the function F defined by

$$F(z)=\left(\frac{e^{i\alpha}+\gamma}{z^{\gamma}}\int_0^z [f(t)]^{e^{i\alpha}}t^{\gamma-1}\,dt\right)^{e^{-i\alpha}} \tag{33}$$

is also α-spiral-like of order ρ. Hence

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)>\rho\cos\alpha\Rightarrow\operatorname{Re}\left(e^{i\alpha}\frac{zF'(z)}{F(z)}\right)>\rho\cos\alpha,$$

or equivalently

$$e^{i\alpha}\frac{zf'(z)}{f(z)}\prec e^{i\alpha}h(z)\Rightarrow e^{i\alpha}\frac{zF'(z)}{F(z)}\prec e^{i\alpha}h(z),$$

where $h(z)$ is given in (32). If we let $p(z)=zF'(z)/F(z)$ and use (33) then this last result can be written in the form

$$e^{i\alpha}\left(p(z)+\frac{zp'(z)}{e^{i\alpha}p(z)+\gamma}\right)\prec e^{i\alpha}h(z)\Rightarrow e^{i\alpha}p(z)\prec e^{i\alpha}h(z).$$

From (32) we see that we can improve the conclusion of this result to $e^{i\alpha}p(z)\prec e^{i\alpha}q(z)$, that is,

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)>\rho\cos\alpha\Rightarrow e^{i\alpha}\frac{zF'(z)}{F(z)}$$
$$\prec\frac{z^{e^{i\alpha}+\gamma}(1+z)^{-2(1-\rho)\cos\alpha}}{\int_0^z t^{e^{i\alpha}+\gamma-1}(1+t)^{-2(1-\rho)\cos\alpha}\,dt}-\gamma,$$

and this result is the best possible. This result depends only on (31) and not on $0\leqslant\rho<1$, although univalency of f and F may be lost for negative ρ.

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