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Positive Solutions of a Nonlinear m-Point Boundary Value Problem

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Abstract—Let $a_i \geq 0$ for $i = 1, \dots, m-3$ and $a_{m-2} > 0$. Let ξ_i satisfy $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$ and $\sum_{i=1}^{m-2} a_i \xi_i < 1$. We study the existence of positive solutions to the boundary-value problem

$$u'' + a(t)f(u) = 0, \qquad t \in (0,1),$$
$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i),$$

where $a \in C([0,1],[0,\infty))$, and $f \in C([0,\infty),[0,\infty))$. We show the existence of at least one positive solution if f is either superlinear or sublinear by applying a fixed-point theorem in cones. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Second-order multipoint BVP, Positive solution, Cone, Fixed point.

1. INTRODUCTION

The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1,2]. Gupta [3] studied three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multipoint boundary value problems have been studied by several authors by using the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, coincidence degree theory, and fixed-point theorem in cones. We refer the reader to [4–12], for some recent results of nonlinear multipoint boundary value problems.

In this paper, we consider the existence of positive solutions to the equation

$$u'' + a(t)f(u) = 0, \qquad t \in (0,1), \tag{1.1}$$

with the boundary condition

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{1.2}$$

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where $a_i \geq 0$ for $i = 1, \dots, m-3$ and $a_{m-2} > 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ is given. We also assume the following.

(A1) $f \in C([0,\infty), [0,\infty))$ and the limits

$$f_0 := \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty := \lim_{u \to \infty} \frac{f(u)}{u}$$

exist. (We note that $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.)

(A2) $a \in C([0,1], [0,\infty))$, and there exists $x_0 \in [\xi_{m-2}, 1]$ such that $a(x_0) > 0$.

(A3) For $i = 1, \dots, m-2$, $a_i \geq 0$, and

$$\sum_{i=1}^{m-2} a_i \xi_i < 1.$$

By the positive solution of (1.1),(1.2), we understand a function $u(t)$ which is positive on $0 < t < 1$ and satisfies the differential equation (1.1) and the boundary conditions (1.2).

Very recently, the author [11] showed the existence of positive solutions for the second-order three-point boundary value problem

$$u'' + a(t)f(u) = 0, \qquad t \in (0,1), \tag{1.3}$$

$$u(0) = 0, \qquad u(1) = \alpha u(\eta), \tag{1.4}$$

which is the special case of (1.1),(1.2). The main result in [11] is the following.

THEOREM A. *Assume (A1) and (A2) hold and $\alpha\eta < 1$. Then problem (1.3),(1.4) has at least one positive solution in the case*

(i) *$f_0 = 0$ and $f_\infty = \infty$ (superlinear), or*

(ii) *$f_0 = \infty$ and $f_\infty = 0$ (sublinear).*

It is well known that for every solutions u of (1.1),(1.2), there exists $\mu_u \in [\xi_1, \xi_{m-2}]$ such that u is a solution of

$$u'' + a(t)f(u) = 0, \qquad t \in (0,1), \tag{1.5}$$

$$u(0) = 0, \qquad u(1) = \alpha u(\mu_u), \tag{1.6}$$

where $\alpha = \sum_{i=1}^{m-2} a_i$. So, by using this fact and the maximal principle established for the three-point boundary value problem in [11], we can easily establish the following result for the m-point boundary value problem (1.3),(1.4).

THEOREM B. *Let (A1) and (A2) hold, and assume the following.*

(A4) $(\sum_{i=1}^{m-2} a_i)\xi_{m-2} < 1$.

Then problem (1.1),(1.2) has at least one positive solution in the case

(i) *$f_0 = 0$ and $f_\infty = \infty$ (superlinear), or*

(ii) *$f_0 = \infty$ and $f_\infty = 0$ (sublinear).*

Clearly, Condition (A3) is weaker than (A4). Our purpose here is to show the existence of positive solutions to the m-point boundary value problem (1.1),(1.2) under (A3). The main result is the following.

THEOREM 1. *Assume (A1)–(A3) hold. Then problem (1.1),(1.2) has at least one positive solution in the case*

(i) *$f_0 = 0$ and $f_\infty = \infty$ (superlinear), or*

(ii) *$f_0 = \infty$ and $f_\infty = 0$ (sublinear).*

Our methods in this paper involve establishing a maximal principle for m-point boundary value problems, but do not use the maximal principle established for the three-point boundary value problem in [11].

The proof of above theorem is based upon an application of the following well-known Guo-Krasnoselskii fixed-point theorem [13].

THEOREM 2. *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \longrightarrow K$$

be a completely continuous operator such that

(i) $\| Au \| \leq \| u \|$, $u \in K \cap \partial\Omega_1$ *and* $\| Au \| \geq \| u \|$, $u \in K \cap \partial\Omega_2$; *or*

(ii) $\| Au \| \geq \| u \|$, $u \in K \cap \partial\Omega_1$ *and* $\| Au \| \leq \| u \|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. THE PRELIMINARY LEMMAS

LEMMA 1. *(See [7].) Let $a_i \geq 0$ for $i = 1, \dots, m-2$, and $\sum_{i=1}^{m-2} a_i\xi_i \neq 1$; then for $y \in C[0,1]$, the problem*

$$u'' + y(t) = 0, \qquad t \in (0,1), \tag{2.1}$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{2.2}$$

has a unique solution

$$u(t) = -\int_0^t (t-s)y(s)\,ds - t\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\,y(s)\,ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} + t\frac{\int_0^1 (1-s)y(s)\,ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i}.$$

LEMMA 2. *Let $a_i \geq 0$ for $i = 1, \dots, m-2$, and $\sum_{i=1}^{m-2} a_i\xi_i < 1$. If $y \in C[0,1]$ and $y \geq 0$, then the unique solution u of problem (2.1),(2.2) satisfies*

$$u \geq 0, \qquad t \in [0,1].$$

PROOF. From the fact that $u''(x) = -y(x) \leq 0$, we know that the graph of $u(t)$ is concave down on (0,1). So, if $u(1) \geq 0$, then the concavity of u together with the boundary condition $u(0) = 0$ implies that $u \geq 0$ for $t \in [0,1]$.

If $u(1) < 0$, then from the concavity of u, we know that

$$\frac{u(\xi_i)}{\xi_i} \geq \frac{u(1)}{1}, \qquad \text{for } i = 1, \dots, m-2. \tag{2.3}$$

This implies

$$u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \geq \sum_{i=1}^{m-2} a_i\xi_i u(1). \tag{2.4}$$

This contradicts the fact that $\sum_{i=1}^{m-2} a_i\xi_i < 1$.

LEMMA 3. *Let* $a_i \geq 0$ *for* $i = 1, \dots, m-3$, $a_{m-2} > 0$, *and* $\sum_{i=1}^{m-2} a_i \xi_i > 1$.

If $y \in C[0,1]$ *and* $y(t) \geq 0$ *for* $t \in (0,1)$, *then (2.1),(2.2) has no positive solution.*

PROOF. Assume that (2.1),(2.2) has a positive solution u, then $u(\xi_i) > 0$ for $i = 1, \dots, m-2$, and

$$\begin{aligned} u(1) &= \sum_{i=1}^{m-2} a_i u(\xi_i) \\ &= \sum_{i=1}^{m-2} a_i \xi_i \frac{u(\xi_i)}{\xi_i} \\ &\geq \sum_{i=1}^{m-2} a_i \xi_i \frac{u(\bar{\xi})}{\bar{\xi}} \\ &> \frac{u(\bar{\xi})}{\bar{\xi}} \end{aligned} \tag{2.5}$$

(where $\bar{\xi} \in \{\xi_1, \dots, \xi_{m-2}\}$ satisfies $(u(\bar{\xi}))/\bar{\xi} = \min\{(u(\xi_i))/\xi_i \mid i = 1, \dots, m-2\}$). This contradicts the concavity of u.

If $u(1) = 0$, then applying $a_{m-2} > 0$, we know that $u(\xi_{m-2}) = 0$. From the concavity of u, it is easy to see that $u(t) \leq 0$ for $t \in [0,1]$.

In the rest of the paper, we assume that $a_i \geq 0$ for $i = 1, \dots, m-3$, $a_{m-2} > 0$, and $\sum_{i=1}^{m-2} a_i \xi_i < 1$. Moreover, we will work in the Banach space $C[0,1]$, and only the sup norm is used.

LEMMA 4. *Let* $a_i \geq 0$ *for* $i = 1, \dots, m-2$, *and* $\sum_{i=1}^{m-2} a_i \xi_i < 1$. *If* $y \in C[0,1]$ *and* $y \geq 0$, *then the unique solution u of problem (2.1),(2.2) satisfies*

$$\inf_{t \in [\xi_{m-2}, 1]} u(t) \geq \Gamma \|u\|,$$

where

$$\Gamma = \min \left\{ \frac{a_{m-2}(1-\xi_{m-2})}{1 - a_{m-2}\xi_{m-2}},\ a_{m-2}\xi_{m-2},\ \xi_1 \right\}.$$

PROOF. We divide the proof into two steps.

STEP 1. We deal with the case that

$$\sum_{i=1}^{m-2} a_i < 1. \tag{2.6}$$

Set

$$u(\bar{t}) = \|u\|. \tag{2.7}$$

If $\bar{t} \leq \xi_{m-2} < 1$, then

$$\min_{t \in [\xi_{m-2}, 1]} u(t) = u(1). \tag{2.8}$$

From the fact that $u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \geq a_{m-2} u(\xi_{m-2})$, we get

$$\begin{aligned} u(\bar{t}) &\leq u(1) + \frac{u(1) - u(\xi_{m-2})}{1 - \xi_{m-2}}(0-1) \\ &= u(1) - \frac{u(1)}{1-\xi_{m-2}} + \frac{u(\xi_{m-2})}{1-\xi_{m-2}} \\ &= u(1)\left[1 - \frac{1}{1-\xi_{m-2}} + \frac{1}{a_{m-2}(1-\xi_{m-2})}\right] \\ &= u(1)\frac{1 - a_{m-2}\xi_{m-2}}{a_{m-2}(1-\xi_{m-2})}. \end{aligned} \tag{2.9}$$

This, together with (2.8), implies that

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq \|u\| \frac{a_{m-2}\left(1-\xi_{m-2}\right)}{1-a_{m-2}\xi_{m-2}}. \tag{2.10}$$

We note that (2.6) implies

$$\frac{a_{m-2}\left(1-\xi_{m-2}\right)}{1-a_{m-2}\xi_{m-2}} > 0.$$

If $\xi_{m-2} < \bar{t} < 1$, then we claim that

$$\min_{t\in[\xi_{m-2},1]} u(t) = u(1). \tag{2.11}$$

In fact, if $\min_{t\in[\xi_{m-2},1]} u(t) = u(\xi_{m-2})$, then we have that $\bar{t} \in [\xi_{m-2}, 1]$ and

$$u\left(\xi_{m-2}\right) \geq \cdots \geq u\left(\xi_2\right) \geq u\left(\xi_1\right).$$

This, together with (2.6), implies that

$$u(1) = \sum_{i=1}^{m-2} a_i u\left(\xi_i\right) \leq \sum_{i=1}^{m-2} a_i u\left(\xi_{m-2}\right) < u\left(\xi_{m-2}\right) \leq u(1),$$

a contradiction! Therefore, (2.11) holds.

From the concavity of u, we know that

$$\frac{u\left(\xi_{m-2}\right)}{\xi_{m-2}} \geq \frac{u\left(\bar{t}\right)}{\bar{t}} \geq u\left(\bar{t}\right). \tag{2.12}$$

Combining (2.12) with the fact that $u(1) \geq a_{m-2}u(\xi_{m-2})$, we conclude that

$$\frac{u(1)}{a_{m-2}\xi_{m-2}} \geq u\left(\bar{t}\right).$$

This, together with (2.11), implies that

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq a_{m-2}\xi_{m-2}\|u\|. \tag{2.13}$$

STEP 2. We deal with the case that

$$\sum_{i=1}^{m-2} a_i \geq 1. \tag{2.14}$$

Set

$$u\left(\bar{t}\right) = \|u\|. \tag{2.15}$$

If $u(\xi_{m-2}) \leq u(1)$, then

$$\min_{t\in[\xi_{m-2},1]} u(t) = u\left(\xi_{m-2}\right). \tag{2.16}$$

It is easy to see from the concavity of u that

$$\bar{t} \in \left[\xi_{m-2}, 1\right]. \tag{2.17}$$

This implies that

$$\frac{u\left(\xi_{m-2}\right)}{\xi_{m-2}} \geq \frac{u\left(\bar{t}\right)}{\bar{t}} \geq u\left(\bar{t}\right).$$

Thus,

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq \xi_{m-2}\|u\|. \tag{2.18}$$

If

$$u(\xi_{m-2}) > u(1), \tag{2.19}$$

then

$$\min_{t\in[\xi_{m-2},1]} u(t) = u(1). \tag{2.20}$$

Furthermore, we have

$$\bar{t} \in [\xi_1, 1]. \tag{2.21}$$

In fact, assume to the contrary that $\bar{t} \in [0, \xi_1)$, then

$$u(\xi_1) \geq u(\xi_2) \geq \cdots \geq u(\xi_{m-2}) > u(1).$$

This implies

$$u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \geq \sum_{i=1}^{m-2} a_i u(\xi_{m-2}) > u(1) \sum_{i=1}^{m-2} a_i \geq u(1),$$

a contradiction! So (2.21) holds.

Since $\sum_{i=1}^{m-2} a_i \geq 1$, we know that there exists $\tilde{\xi} \in \{\xi_1, \dots, \xi_{m-2}\}$ such that

$$u\left(\tilde{\xi}\right) \leq u(1). \tag{2.22}$$

This implies that

$$u(\xi_1) \leq (\xi_2) \leq \cdots \leq u\left(\tilde{\xi}\right) \leq u(1). \tag{2.23}$$

Combining (2.23) and (2.21) with the concavity of u, we can conclude that

$$\frac{u(1)}{\xi_1} \geq \frac{u(\xi_1)}{\xi_1} \geq \frac{u(\bar{t})}{\bar{t}} \geq u(\bar{t}). \tag{2.24}$$

This together with (2.20) implies that

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq \xi_1\|u\|. \tag{2.25}$$

From (2.10), (2.13), (2.18), and (2.25), we know that

$$\inf_{t\in[\xi_{m-2},1]} u(t) \geq \Gamma\|u\|,$$

where

$$\begin{aligned}\Gamma &= \min\left\{\frac{a_{m-2}(1-\xi_{m-2})}{1-a_{m-2}\xi_{m-2}}, a_{m-2}\xi_{m-2}, \xi_{m-2}, \xi_1\right\} \\ &= \min\left\{\frac{a_{m-2}(1-\xi_{m-2})}{1-a_{m-2}\xi_{m-2}}, a_{m-2}\xi_{m-2}, \xi_1\right\}.\end{aligned}$$

3. PROOF OF MAIN THEOREM

PROOF OF THEOREM 1. SUPERLINEAR CASE. Suppose then that $f_0 = 0$ and $f_\infty = \infty$. We wish to show the existence of a positive solution of (1.1),(1.2). Now (1.1),(1.2) has a solution $y = y(t)$ if and only if y solves the operator equation

$$\begin{aligned} y(t) = &-\int_0^t (t-s)a(s)f(y(s))\,ds \\ &- t\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\,a(s)f(y(s))\,ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} + t\frac{\int_0^1 (1-s)a(s)f(y(s))\,ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\ &\stackrel{\text{def}}{:=} Ay(t). \end{aligned} \tag{3.1}$$

Denote

$$K = \left\{ y \mid y \in C[0,1],\ y \geq 0,\ \min_{\xi_{m-2}\leq t\leq 1} y(t) \geq \Gamma\|y\| \right\}, \tag{3.2}$$

where Γ is defined in Lemma 4. It is obvious that K is a cone in $C[0,1]$. Moreover, by Lemma 4, $AK \subset K$. It is also easy to check that $A : K \to K$ is completely continuous.

Now since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(y) \leq \epsilon y$, for $0 < y < H_1$, where $\epsilon > 0$ satisfies

$$\frac{\epsilon \int_0^1 (1-s)a(s)\,ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \leq 1. \tag{3.3}$$

Thus, if $y \in K$ and $\|y\| = H_1$, then from (3.1) and (3.3), we get

$$\begin{aligned} Ay(t) &\leq \frac{t\int_0^1 (1-s)a(s)f(y(s))\,ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\ &\leq \frac{t\int_0^1 (1-s)a(s)\epsilon y(s)\,ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\ &\leq \frac{\int_0^1 (1-s)a(s)\epsilon\,ds\|y\|}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\ &\leq \frac{\int_0^1 (1-s)a(s)\epsilon\,dsH_1}{1-\sum_{i=1}^{m-2} a_i\xi_i}. \end{aligned} \tag{3.4}$$

Now if we let

$$\Omega_1 = \{y \in C[0,1] \mid \|y\| < H_1\}, \tag{3.5}$$

then (3.4) shows that $\|Ay\| \leq \|y\|$, for $y \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\hat{H}_2 > 0$ such that $f(u) \geq \rho u$, for $u \geq \hat{H}_2$, where $\rho > 0$ is chosen so that

$$\rho\Gamma\frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i}\sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_1(1-s)a(s)\,ds \geq 1. \tag{3.6}$$

Let $H_2 = \max\{2H_1, (\hat{H}_2/\Gamma)\}$ and $\Omega_2 = \{y \in C[0,1] \mid \|y\| < H_2\}$, then $y \in K$ and $\|y\| = H_2$ implies

$$\min_{\xi_{m-2}\leq t\leq 1} y(t) \geq \Gamma\|y\| \geq \hat{H}_2,$$

and so

$$Ay(\xi_i) = -\int_0^{\xi_i} (\xi_i - s)\, a(s) f(y(s))\, ds - \xi_i \frac{\sum\limits_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\, a(s) f(y(s))\, ds}{1 - \sum\limits_{i=1}^{m-2} a_i \xi_i} + \xi_i \frac{\int_0^1 (1-s) a(s) f(y(s))\, ds}{1 - \sum\limits_{i=1}^{m-2} a_i \xi_i}.$$

This implies

$$\begin{aligned}
u(1) &= \sum_{i=1}^{m-2} a_i u(\xi_i) \\
&= -\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\, a(s) f(y(s))\, ds \\
&\quad - \sum_{i=1}^{m-2} a_i \xi_i \frac{\sum\limits_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\, a(s) f(y(s))\, ds}{1 - \sum\limits_{i=1}^{m-2} a_i \xi_i} \\
&\quad + \sum_{i=1}^{m-2} a_i \xi_i \frac{\int_0^1 (1-s) a(s) f(y(s))\, ds}{1 - \sum\limits_{i=1}^{m-2} a_i \xi_i} \\
&= \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\, a(s) f(y(s))\, ds \frac{(-1)}{1 - \sum\limits_{i=1}^{m-2} a_i \xi_i} \\
&\quad + \sum_{i=1}^{m-2} a_i \xi_i \frac{\int_0^1 (1-s) a(s) f(y(s))\, ds}{1 - \sum\limits_{i=1}^{m-2} a_i \xi_i} \\
&= \frac{1}{1 - \sum\limits_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \left[-\int_0^{\xi_i} \xi_i a(s) f(y(s))\, ds + \int_0^{\xi_i} s a(s) f(y(s))\, ds \right. \qquad (3.7) \\
&\quad \left. + \int_0^1 \xi_i a(s) f(y(s))\, ds - \int_0^1 \xi_i s a(s) f(y(s))\, dsx \right] \\
&\geq \frac{1}{1 - \sum\limits_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \left[\int_{\xi_i}^1 \xi_i a(s) f(y(s))\, ds - \xi_i \int_{\xi_i}^1 s a(s) f(y(s))\, ds \right] \\
&\quad \text{(we have used, in fact, that } 1 > \xi_i) \\
&= \frac{1}{1 - \sum\limits_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \xi_i (1-s) a(s) f(y(s))\, ds \\
&\geq \frac{1}{1 - \sum\limits_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \xi_1 (1-s) a(s) f(y(s))\, ds \\
&\geq \frac{1}{1 - \sum\limits_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_1 (1-s) a(s) f(y(s))\, ds.
\end{aligned}$$

Hence, for $y \in K \cap \partial\Omega_2$,

$$\|Ay\| \geq |u(1)| \geq \rho\Gamma \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_1(1-s)a(s)\,ds\|y\| \geq \|y\|.$$

Therefore, by the first part of the fixed-point theorem, it follows that A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, such that $H_1 \leq \|u\| \leq H_2$. This completes the superlinear part of the theorem.

SUBLINEAR CASE. Suppose next that $f_0 = \infty$ and $f_\infty = 0$. We first choose $H_3 > 0$ such that $f(y) \geq My$ for $0 < y < H_3$, where

$$M\Gamma \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_1(1-s)a(s)\,ds \geq 1. \tag{3.8}$$

For $y \in K$ and $\|y\| = H_3$, by using the method to get (3.7), we can get that

$$\begin{aligned} Ay(1) &= \sum_{i=1}^{m-2} a_i Ay(\xi_i) \\ &\geq \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \xi_i(1-s)a(s)f(y(s))\,ds \\ &\geq \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_1(1-s)a(s)My(s)\,ds \\ &\geq \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_1(1-s)a(s)M\Gamma\,ds\|y\| \\ &\geq H_3. \end{aligned} \tag{3.9}$$

Thus, we may let $\Omega_3 = \{y \in C[0,1] \mid \|y\| < H_3\}$, so that

$$\|Ay\| \geq \|y\|, \qquad y \in K \cap \partial\Omega_3.$$

Now, since $f_\infty = 0$, there exists $\hat{H}_4 > 0$ so that $f(y) \leq \lambda y$ for $y \geq \hat{H}_4$, where $\lambda > 0$ satisfies

$$\frac{\lambda \int_0^1 (1-s)a(s)\,ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \leq 1. \tag{3.10}$$

We consider two cases.

CASE (i). Suppose f is bounded, say $f(y) \leq N$ for all $y \in [0,\infty)$. In this case, choose

$$H_4 = \max\left\{ 2H_3, \frac{N\int_0^1 (1-s)a(s)\,ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \right\},$$

so that, for $y \in K$ with $\|y\| = H_4$, we have

$$\begin{aligned} Ay(t) &= -\int_0^t (t-s)a(s)f(y(s))\,ds \\ &\quad - t\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\, a(s) f(y(s))\,ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} + t\frac{\int_0^1 (1-s)a(s)f(y(s))\,ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\ &\le \frac{\int_0^1 (1-s)a(s)f(y(s))\,ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\ &\le \frac{\int_0^1 (1-s)a(s)N\,ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\ &\le H_4, \end{aligned}$$

and therefore, $\|Ay\| \le \|y\|$.

Case (ii). If f is unbounded, then we know from (A1) that there is $H_4 : H_4 > \max\{2H_3, (1/\Gamma)\hat{H}_4\}$ such that

$$f(y) \le f(H_4), \qquad \text{for } 0 < y \le H_4.$$

(We are able to do this, since f is unbounded.) Then for $y \in K$ and $\|y\| = H_4$, we have

$$\begin{aligned} Ay(t) &= -\int_0^t (t-s)a(s)f(y(s))\,ds \\ &\quad - t\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\, a(s) f(y(s))\,ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} + t\frac{\int_0^1 (1-s)a(s)f(y(s))\,ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\ &\le t\frac{\int_0^1 (1-s)a(s)f(y(s))\,ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\ &\le \frac{\int_0^1 (1-s)a(s)f(H_4)\,ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\ &\le \frac{\int_0^1 (1-s)a(s)\lambda H_4\,ds}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\ &\le H_4. \end{aligned}$$

Therefore, in either case, we may put

$$\Omega_4 = \{y \in C[0,1] \mid \|y\| < H_4\},$$

and for $y \in K \cap \partial\Omega_4$, we may have $\|Ay\| \le \|y\|$. By the second part of the fixed-point theorem, it follows that BVP (1.1),(1.2) has a positive solution. Therefore, we have completed the proof of Theorem 1.

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