Dense Linear Manifolds and Application to Linear Differential Operators*

SUNG J. LEE

Department of Mathematics, University of South Florida, Tampa, Florida 33620

Submitted by K. Fan

1. INTRODUCTION

In an operator theory of perturbed ordinary differential operators, it is important to know whether the perturbations are closed or densely defined. All known general theorems for checking whether they are closed seem to require that the perturbing term is $T$-compact or have a restriction on its $T$-norm (see, for example, Theorems IV.II, IV.II of [5], Lemma V.3.5 of [3], and Proposition 2.5 of [8]). However, these theorems are ineffective or inconvenient to apply to the operator generated by countably many ordinary differential expressions. Thus, the main purpose of this paper is twofold: First we give new characterizations (Theorems 1, 2 below) which are useful for testing the closedness of a perturbed linear operator and do not require any argument on compactness or $T$-norm. Secondly, using these characterizations we show in Theorems 3, 4 below that a large class of operators subject to infinitely many abstract boundary conditions have densely defined domains. The main tool used here is an abstract adjoint theory. We now fix some notations. If $X$ is a Banach space, then $X^*$ will denote the Banach space of all continuous conjugate linear functionals on $X$. If $M$ is a linear manifold, then, $M^*$, $^*M$, $M^c$, $^cM$ will denote the adjoint, the preadjoint, the closure, and the $w^*$-closure of $M$, respectively. If $\mathcal{L}_1$ is a linear operator whose graph is contained in the direct sum $X_1 \oplus X_2$ of Banach spaces $X_1$ and $X_2$, then $\mathcal{L}_1^*$ will denote the (possibly) multi-valued operator whose graph is $\text{(graph } \mathcal{L}_1)^*$. A similar definition applies to $^*\mathcal{L}_1^+$ if $\mathcal{L}_1^+$ is a linear operator whose graph is contained in $X_1^* \oplus X_2^*$. For definitions of adjoint and preadjoint, see [2] or [9]. If $D_1$ and $D_2$ are $m \times n$ and $q \times r$ matrices, then $D_1 \oplus D_2$ will denote the $(m + q) \times (n + r)$ block matrix by joining the lower right corner of $D_1$ to the upper left corner of $D_2$. The algebraic sum of $M_1$ and $M_2$ is denoted by $M_1 + M_2$. The Hilbert space of all $1 \times N$ complex constant matrices $a$ with $aa^* < \infty$ ($a^*$ the conjugate transpose of $a$) is

* Supported in part by Faculty Research and Creative Scholarship Fund of University of South Florida.

0022-247X/83 $3.00

Copyright © 1983 by Academic Press, Inc.
All rights of reproduction in any form reserved.
DENSE LINEAR MANIFOLDS

353

denoted by \( l^N_2 \). When \( N = \infty \), this space is also denoted by \( l_2 \). If \( F \) is a linear or bilinear functional, and if \( X \) is a \( 1 \times N \) matrix \( \{X_1, \ldots, X_N\} \), then \( F(X) \) will denote the \( 1 \times N \) matrix whose \( j \)th entry is \( F(X_j) \).

2. Closedness for Linear Operators

Throughout \( X_1 \) and \( X_2 \) are complex Banach spaces.

**Theorem 1.** Let \( T_1 \) be a closed linear operator such that \( \text{graph } T_1 \subset X_1 \oplus X_2 \). Let \( S \) be a continuous linear operator from the Banach space \( \text{Domain } T_1 \), equipped with the graph topology of \( T_1 \) into \( X_2 \). Let \( \mathcal{L}_1 \) be a linear operator defined on \( \text{Domain } \mathcal{L}_1 = \text{Domain } T_1 \) by

\[
\mathcal{L}_1 Y = T_1 y + S y, \quad y \in \text{Domain } \mathcal{L}_1.
\]

Then we have the following:

(I) If \( \text{Range } S \) is finite dimensional, then \( \mathcal{L}_1 \) is closed, or equivalently \( \text{Domain } \mathcal{L}_1^* \) is \( w^* \)-dense in \( X_2^* \).

(II) Assume that

(i) \( \text{Graph } T_0 \) is complemented in \( \text{Graph } T_1 \), where \( T_0 \) is the closed operator defined by

\[
\text{Domain } T_0 = \{ y \in \text{Domain } T_1 \mid S y = 0 \},
\]

\[
T_0 y = T_1 y, \quad y \in \text{Domain } T_0.
\]

(ii) \( \text{Range } S \) is isomorphic to \( l_2 \).

(iii) \( \{ \{ y, T_1 y + S z \mid y, z \in \text{Domain } T_1 \} \) is closed in \( X_1 \oplus X_2 \).

Then \( \mathcal{L}_1 \) is closed, or equivalently \( \text{Domain } \mathcal{L}_1^* \) is \( w^* \)-dense in \( X_2^* \).

(III) Suppose that \( N = \dim \text{Range } S < \infty \), or if \( N = \infty \) then all the conditions in (II) are satisfied. Then \( \mathcal{L}_1^* \) is a multi-valued operator having the form

\[
(\text{Graph } \mathcal{L}_1)^* = \{ b \mid b = \{ b_2, b_1 \} \in (\text{Graph } T_0)^*,
\]

\[
B^t(b) C^* + b_2(x) = 0_{1 \times N}.
\]

In particular, if \( \text{Domain } T_1 \) is dense, then \( \mathcal{L}_1^* \) is single-valued. Here,

(i) \( \chi \) is the \( 1 \times N \) matrix \( \{ \chi_j \} \) such that \( \{ \chi_j \mid 1 \leq j \leq N \} \) is a Besselian–Hilbertian basis for \( \text{Range } S \).

(ii) \( B^t \) is a \( w^* \)-continuous linear operator from \( (\text{Graph } T_0)^* \) onto \( l_2^N \) with \( \text{Null } B^t = (\text{Graph } T_1)^* \).
(iii) $C$ is the $N \times N$ nonsingular Hilbert matrix such that

$$
\overline{b_2(a_2)} \overline{b_1(a_1)} = iB(a) C(B'(b))^* \tag{vi}
$$

for all $a = \{a_1, a_2\} \in \text{Graph } T_1$, $b = \{b_2, b_1\} \in (\text{Graph } T_0)^*$. Here $B$ is the linear operator defined by

$$
\text{Domain } B = \text{Graph } T_1, \\
B(\{y, T_1 y\}) = \{a_j(S(y))\}_{1 \times N},
$$

where $\{a_j | 1 \leq j \leq N\}$ is the coordinate functionals relative to the basis $\{x_j | 1 \leq j \leq N\}$ (see p. 87 of [12], or p. 17 of [11] for definition).

**Proof.** First we will prove (II). Define an operator $\tilde{S}$ on $\text{Graph } T_1$ by

$$
\tilde{S}(\{y, T_1 y\}) = Sy, \quad \{y, T_1 y\} \in \text{Graph } T_1.
$$

Then $\text{Range } \tilde{S}$ is isomorphic to $l_2$, and

$$\{a + \{0, \tilde{S}(b)\} | a, b \in \text{Graph } T_1\}
$$

is closed.

Let $\chi$ be as in the theorem. Define

$$
\mathcal{N} = \{b_2(\chi) | b_2 \in \text{Domain } \mathcal{L}_1^*\},
$$

where $b_2(\chi)$ denotes the $1 \times \infty$ matrix whose $j$th entry is $b_2(\chi_j)$. The main part of the proof consists of showing that

$$
(\text{Graph } \mathcal{L}_1)^c = (\text{Graph } \mathcal{L}_1) + \{0, \alpha \chi^t\} | \alpha \in l_2 \ominus \mathcal{N}\tag{*}
$$

where the algebraic sum is direct ($\chi^t$ denotes the transpose of $\chi$). The idea is parallel to that used to prove a concrete case (Theorem 2.3 of [10]). Now any $x \in \text{Range } S$ has a unique representation $\sum_1^\infty a_j(x) x_j$, converging in $X_2$.

Let $B$ be as in the theorem. Then, since the map $\{a_1, a_2\} \mapsto S(a_1)$ for $\{a_1, a_2\} \in \text{Graph } T_1$ is continuous, and the map

$$
x \mapsto \{a_1(x), ..., a_j(x), ...\}
$$

defines an isomorphism from $\text{Range } S$ onto $l_2$, it follows that $B$ defines a continuous linear operator from $\text{Graph } T_1$ onto $l_2$ whose kernel is $\text{Graph } T_0$. By assumption, $\text{Graph } T_0$ is complemented in $\text{Graph } T_1$. Thus it follows that $\text{(Graph } T_1)^* = w^*\text{-complemented in } (\text{Graph } T_0)^*$. Since $\text{(Graph } T_0)^*/\text{(Graph } T_1)^*$ is isomorphic to $l_2$, there exists a $w^*$-continuous operator $B^*$ as in the theorem. Thus, by Theorem 1.6 of [9], there exists a $\infty \times \infty$
nonsingular Hilbert matrix $C$ as in the theorem (where in this case $N = \infty$).

We claim that

$$(\text{Graph } \mathcal{L}_1)^* = \{b \mid b = \{b_2, b_1\} \in (\text{Graph } T_0)^*, \quad B^*(b) C^* = -ib_2(\chi)\}.$$ (**)

Take $u = \{u_2, u_1\} \in (\text{Graph } \mathcal{L}_1)^*$. Then for all $a = \{a_1, a_2\} \in \text{Graph } T_1$

$$\overline{u_2}(T_1 a_1 + S a_1) - \overline{u_1}(a_1) = 0.$$ 

This is, in particular, true for $a \in \text{Graph } T_0$. It then follows that $u \in (\text{Graph } T_0)^*$. Returning to the above equation and using Green's formula,

$$iB(a) C(B^*(u))^* - \overline{u_2}(S(a_1)) = 0$$

for all $a = \{a_1, a_2\} \in \text{Graph } T_1$. Now,

$$S(a_1) - \sum_{j=1}^{\infty} c_j(S(a_1)) \chi_j - B(a) \chi^1,$$

converging in $X_1$. Thus

$$0 = iB(a) C(B^*(u))^* - \overline{u_2}(S(a_1))$$

for all $a \in \text{Graph } T_1$. Since $\text{Range } B = l_1$, this implies that

$$B^*(u) C^* + iu_2(\chi) = 0_{1 \times \infty}.$$ 

This proves the claim. Next, we will show that

$$(\text{Graph } \mathcal{L}_1)^c = \{a + \{0, \beta \chi^1\} \mid a \in \text{Graph } T_1, \beta \in l_2 \text{ such that}$$

$$b_2(\chi)(\beta - B(a))^* = 0 \forall a \in \text{Graph } \mathcal{L}_1^c \}.$$ (**')

Take $a = \{a_1, a_2\} \in \text{Graph } T_1$, $\beta \in l_2$ belonging to the right of the above set. Then for any $b = \{b_2, b_1\} \in (\text{Graph } \mathcal{L}_1)^*$,

$$\overline{b_2}(a_2) - \overline{b_1}(a_1) + \overline{b_2}(\beta \chi^1)$$

$$= iB(a) C(B^*(b))^* - (\beta b_2(\chi))^*$$

$$= iB(a) C(-ib_2(\chi) C^{*-1})^* - \beta(b_2(\chi))^*; \quad \text{by (**)},$$

$$- (\beta - B(a))(B_2(\chi))^* = 0.$$
Thus,
\[ a + \{0, \beta \chi^t\} \in \star((\text{Graph } \mathcal{L}_1)^\star) = (\text{Graph } \mathcal{L}_1)^c. \]

Take now \( \{u_1, u_2\} \in \star((\text{Graph } \mathcal{L}_1)^\star). \) Then
\[
\overline{b}_2(u_2) - \overline{b}_1(u_1) = 0, \quad \text{all} \quad b = \{b_2, b_1\} \in (\text{Graph } \mathcal{L}_1)^\star. \quad (***)
\]

This is, in particular, true for all \( b \) in
\[
(\text{Graph } T_1)^\star \cap ((\text{Range } S)^\perp + \{0\}),
\]
and so using the definition of preadjoint, \( \{u_1, u_2\} \) belongs to
\[
\star((\text{Graph } T_1)^\star \cap ((\text{Range } S)^\perp + \{0\})) = \text{Graph } T_1 \uparrow + (\{0\} \oplus \text{Range } S),
\]
where the equality is true because \( \text{Range } S \) is closed, and the set in right is closed by assumption (iii) of (II). It then follows that
\[
\{u_1, u_2\} = a + \{0, \beta \chi^t\}
\]
for some \( a = \{a_1, a_2\} \) in \( \text{Graph } T_1 \) and \( \beta \) in \( l_2 \). Returning to (***') with this \( \{u_1, u_2\}, \)
\[
0 = b_2(a_2 + \beta \chi^t) - b_1(a_1)
= iB(a) C(B^\dagger(b))^\star + \beta(b_2(\chi))^\star
= (\beta - B(a))(b_2(\chi))^\star,
\]
for all \( b = \{b_2, b_1\} \in (\text{Graph } \mathcal{L}_1)^\star. \) Thus \( \{u_1, u_2\} \) belongs to the set right of (**')', and so (**')' is valid. We now show (*). Clearly, the algebraic sum is direct. Thus, we only need to show that sets in the left and the right are the same. Take any \( z \in (\text{Graph } \mathcal{L}_1)^c. \) Then by (**')',
\[
z = a + \{0, \beta \chi^t\}
\]
for some \( a \in \text{Graph } T_1, \) \( \beta \in l_2 \) such that \( b_2(\chi)(\beta - B(a))^\star = 0, \) all \( b_2 \in \text{Domain}(\text{Graph } \mathcal{L}_1)^\star. \) By construction,
\[
\hat{S}(u) = B(u) \chi^t, \quad \text{all} \quad u \in \text{Graph } T_1.
\]
Then
\[
z = a + \{0, S(a_1)\} + \{0, \beta \chi^t - S(a_1)\}
\]
\[
- a + \{0, S(a_1)\} + \{0, (\beta - B(a)) \chi^t\}.
\]
Now, $a \equiv \beta - B(a) \in \mathcal{M}$, if and only if $b_2(\chi)(\beta - B(a))^* = 0$, all $b_2 \in \text{Domain } \mathcal{L}^\ast$. This shows that the left side of $(\ast)$ is contained in the right side of $(\ast)$. We can easily check that $\{0, \alpha \chi \} \in (\text{Graph } \mathcal{L}_1)^\ast$ for any $\alpha \in \mathcal{M}$ (by using again $(**)^\ast$). We have shown that $(\ast)$ is true. Let $s = \dim(\text{Graph } \mathcal{L}_1)^\ast(0)$. Then to show that $\mathcal{L}_1$ is closed, it is sufficient in view of $(\ast)$ to show that $s = 0$. It may be possible at this stage that $s$ is infinite. The idea is to show that $\text{Graph } \mathcal{L}_1$ is the kernel of a bounded linear operator $P\mathcal{U}$ defined on $(\text{Graph } \mathcal{L}_1)^\ast$. Suppose contrary that $s > 1$. Let $\{\phi_j | 1 \leq j \leq s\}$ be a Besselian–Hilbertian basis for $(\text{Graph } \mathcal{L}_1)^\ast(0)$. Then each $x$ in this space has a unique representation $\sum_{i} \zeta_j(x) \phi_j$, converging in $X_2$. Here $\{\zeta_j | 1 \leq j \leq s\}$ is the associated linear coordinate functions for the space with respect to the basis $\{\phi_j | 1 \leq j \leq s\}$. Then the map $a \mapsto (\zeta_j(a))_{1 \times s}$ defines an isomorphism from $(\text{Graph } \mathcal{L}_1)^\ast(0)$ onto $l_2$.

**Case i.** $s < \infty$. Define a bounded linear operator $U$ from $(\text{Graph } \mathcal{L}_1)^\ast(0)$ into $l_2$ by

$$U(a + \{0, x + \hat{S}(a)\}) = \{\zeta_1(x), \ldots, \zeta_s(x), B(a)\}$$

for $a \in \text{Graph } T_1$, $x \in (\text{Graph } \mathcal{L}_1)^\ast(0)$. Let $P$ be the $\infty \times \infty$ composition matrix $[I_{s \times s} : 0_{s \times \infty}]$. Then this is a Hilbert matrix. Moreover, for $a \in \text{Graph } T_1$, $x \in (\text{Graph } \mathcal{L}_1)^\ast(0)$, we have

$$P(U(a + \{0, x + \hat{S}(a)\}))^* = 0_{s \times 1},$$

if and only if $\zeta_j(x) = 0$, all $j = 1, 2, \ldots, s$, or equivalently $x = 0$. It follows that

$$\text{Graph } \mathcal{L}_1 = \{a \in (\text{Graph } \mathcal{L}_1)^\ast | P(U(a))^* = 0_{s \times 1}\}.$$

**Case ii.** $s = \infty$. Define an operator $B \neq \zeta$ on $(\text{Graph } \mathcal{L}_1)^\ast$ by

$$(B \neq \zeta)(a + \{0, x + \hat{S}(a)\}) = \{a_1(a), \zeta_1(x), \ldots, a_s(a), \zeta_s(x), \ldots\}$$

for $a \in \text{Graph } T_1$, $x \in (\text{Graph } \mathcal{L}_1)^\ast(0)$. Then $B \neq \zeta$ defines a bounded linear operator from $(\text{Graph } \mathcal{L}_1)^\ast$ into $l_2$. Let $P$ denote the $\infty \times \infty$ matrix direct sum $\bigoplus_{n=0}^{\infty}(0, 1)$. Then this is a Hilbert matrix. Moreover, for $a \in \text{Graph } T_1$, $x \in (\text{Graph } \mathcal{L}_1)^\ast(0)$, we have that

$$P(U(a + \{0, x + \hat{S}(a)\}))^* = 0_{\infty \times 1},$$

if and only if $x = 0$. It follows that

$$\text{Graph } \mathcal{L}_1 = \{a \in (\text{Graph } \mathcal{L}_1)^\ast | P((B \neq \zeta)(a))^* = 0_{\infty \times 1}\}.$$

Combining the cases i and ii, we see that $\text{Graph } \mathcal{L}_1$ is the kernel of a continuous linear operator, and so is closed. This, together with $(\ast)$, implies
that $s = 0$, a contradiction. Thus, $s = 0$, and so again by (*), $\mathcal{L}_1$ is closed. This proves the first statement of (II). We now prove the last statement of (II). If $\mathcal{L}_1$ is closed, then by a general theorem on page 15 of [2], Domain $\mathcal{L}_1^*$ is w*-dense. Suppose now that Domain $\mathcal{L}_1^*$ is w*-dense. Then, by a general theorem on page 15 of [2] again, (Graph $\mathcal{L}_1$) is an operator. However, (*) still remains valid. Thus,

\[(\text{Graph } \mathcal{L}_1)(0) = \{0, \alpha \chi^i \mid \alpha \in l_2 \ominus \mu \} = \{0\}, \]

as (Graph $\mathcal{L}_1$) is an operator. Therefore, going back to (*) again, Graph $\mathcal{L}_1 = (\text{Graph } \mathcal{L}_1)^c$, and so $\mathcal{L}_1$ is closed. This proves (II). We now prove (I). Since $\dim \text{Range } S = N < \infty$, (Graph $T_1$) is w*-closed. Thus, imitating the proof for (II) (where in this case the convergence problem will not occur, and Besselian–Hilbertian basis becomes just an ordinary basis) we can conclude the proof for (I) provided that

\[(\text{Graph } T_1) \cap (\text{Null } S \oplus \{0\}) \]

is complemented in Graph $T_1$. However, since $N < \infty$, Range $S$ is closed, and so Range $S^*$ is w*-closed. But

\[\dim \text{Range } S = \dim \text{Range } S^* = N < \infty.\]

It follows that $(\text{Graph } T_1)^* = (\{0\} \oplus \text{Range } S^*)$ is w*-closed. Thus

\[((\text{Graph } T_1) \cap (\text{Null } S \oplus \{0\}))^* = (\text{Graph } T_1)^* = (\{0\} \oplus \text{Range } S^*).\]

In particular, (Graph $T_1$) is w*-complemented in the above set, or equivalently (Graph $T_1$) is complemented in Graph $T_1$. This proves (I). Part (III) is already proved in the course of the proof for (I) and (II). This completes the proof.

Remark. In [1] of the above theorem, the closedness of $\mathcal{L}_1$ also follows from Theorem IV.I.11 of [5] as $S$ there is finite dimensional.

The following is a dual version of the above theorem. We will omit its proof as it is similar to that used to prove the above theorem.

**Theorem 2.** Let $T_1^\dagger$ be a w*-closed linear operator whose graph is contained in $X_1^* \oplus X_1^*$. Let $S^\dagger$ be a w*-continuous linear operator from the linear space, Domain $T_1^\dagger$, equipped with the w*-graph topology of $T_1^\dagger$ into $X_1^*$. Define a linear operator $L_1^\dagger$ on the domain of $T_1^\dagger$ by

\[\mathcal{L}_1^\dagger y = T_1^\dagger y + S^\dagger y, \quad y \in \text{Domain } \mathcal{L}_1^\dagger.\]

Then we have the following:
(I) If Range $S^+$ is finite dimensional, then $L_1^+$ is $w^*$-closed, or equivalently the domain of $L_1^+$ is dense in $X_1$.

(II) Assume that

(i) Range $S^+$ is isomorphic to $l_2$,

(ii) $\{\{y, T_0^+ y + S^+ z\} \mid y, z \in \text{Domain } T_1^+\}$ is $w^*$-closed in $X_2^* \oplus X_1^*$,

(iii) $\{\{y, T_1^+ y\} \mid y \in \text{Domain } T_1^+, S^+ y = 0\}$ is $w^*$-complemented in the graph of $T_1^+$. Then $L_1^+$ is $w^*$-closed, or equivalently Domain $L_1^+$ is dense in $X_1$.

Remark. We do not know whether or not the above theorems remain valid when Range $S$ or Range $S^+$ is not isomorphic to $l_2$.

3. Dense Linear Manifolds

In adjoint operator theory of perturbed differential operators, it is important to know whether or not the perturbations are densely defined. We will study this in an abstract setting when infinitely many conditions are involved. Throughout $X_1$ and $X_2$ will denote complex Banach spaces.

**Theorem 3.** Let $T_0^+, T_1^+$ be $w^*$-closed linear manifolds in $X_2^* \oplus X_1^*$ such that

(i) $T_0^+ \subset T_1^+$ and $T_0^+$ is $w^*$-complemented in $T_1^+$,

(ii) The quotient space $T_1^+/T_0^+$ is isomorphic to $l_2^+$ for some extended integer $N$,

(iii) Domain $T_1^+$ is $w^*$-dense in $X_2^*$.

Let $B^+$ be a $w^*$-continuous linear operator from $T_1^+$ onto $l_2^+$ with Null $B^+ = T_0^+$, and let $\Psi$ be a $1 \times N$ matrix whose $j$th entry, $\psi_j$, belongs to $X_2$ for all $j$. Define

$$L_1^+ = \{ \{b_2, b_1\} \in T_1^+ \mid B^+(\{b_2, b_1\}) = -ib_2(\Psi)\},$$

where $b_2(\Psi)$ denotes the $1 \times N$ matrix whose $j$th entry is $b_2(\psi_j)$. Then we have the following:

(I) If $N < \infty$, then Domain $L_1^+$ is $w^*$-dense in $X_2^*$.

(II) Assume $N = \infty$. Suppose further that

(i) $\sum_{j=1}^{\infty} a_j \psi_j$ converges in $X_2$ for all $(a_j) \in l_2$,

(ii) $\{\sum_{j=1}^{\infty} a_j \psi_j \mid (a_j) \in l_2\}$ is isomorphic to a separable Hilbert space,

(iii) $\{a + \{0, \sum_{j=1}^{\infty} a_j \psi_j\} \mid a \in *T_1^+, (a_j) \in l_2\}$ is closed in $X_1 \oplus X_2$. Then Domain $L_1^+$ is $w^*$ dense in $X_2^*$. 
(III) Suppose that the assumptions in (I) or in (II) are satisfied. Then $\mathcal{L}^\dagger_1$ is a closed linear operator (the graph of) given by

$$\mathcal{L}^\dagger_1 = \{ a + \{ 0, B(a) C \Psi^\dagger \} \mid a \in \mathcal{T}_0^\dagger \},$$

where $B$ is any continuous linear operator from $\mathcal{T}_0^\dagger$ onto $l_2^\dagger$ whose kernel is $\mathcal{T}_1^\dagger$, and $C$ is the $N \times N$ nonsingular Hilbert matrix such that

$$\overline{h_2(a_2)} - \overline{h_1(a_1)} = iB(a) C(R^+(b))^*$$

for all $a = \{ a_1, a_2 \} \in \mathcal{T}_0^\dagger$, $b = \{ b_2, b_1 \} \in \mathcal{T}_1^\dagger$.

Proof: We will prove (II) and (III) simultaneously. First we will show that $\mathcal{L}^\dagger_1$ is the adjoint of the graph of a closed linear manifold. Notice that $\mathcal{T}_0^\dagger$ is an operator as Domain $\mathcal{T}_0^\dagger$ is $w^*$-dense. Define operators $\mathcal{S}$ and $\mathcal{N}$ by

$$\mathcal{S}y = -B(\{ y, \mathcal{T}_0^\dagger y \}) C \Psi^\dagger,$$

Domain $\mathcal{S} = \text{Domain}(\mathcal{T}_0^\dagger)$;

$$\mathcal{N}y = \mathcal{T}_0^\dagger y + Sy,$$

Domain $\mathcal{N} = \text{Domain}(\mathcal{T}_0^\dagger)$.

Since

$$\{ B(\{ y, \mathcal{T}_0^\dagger y \}) C \mid y \in \text{Domain } \mathcal{T}_0^\dagger \} = l,$$

by assumption (iii)

$$\{ \{ y, \mathcal{T}_0^\dagger y + Sz \} \mid y, z \in \text{Domain } \mathcal{T}_0^\dagger \}$$

is closed. Since

$$\{ \{ y, \mathcal{T}_0^\dagger y \} \in \mathcal{T}_0^\dagger \mid Sy = 0 \}$$

is closed, and stays between $\mathcal{T}_0^\dagger$ and $\mathcal{T}_0^\dagger$, it is complemented in $\mathcal{T}_0^\dagger$. Thus it follows from Theorem 1 that $\mathcal{L}_1^\dagger$ is closed. Let $\mathcal{N}$ be a closed linear manifold in $X_1$ such that $\{ \{ y, \mathcal{T}_0^\dagger y \} \in \mathcal{T}_0^\dagger \mid Sy = 0 \}$ is the direct sum of $\mathcal{T}_1^\dagger$ and $U \oplus \{ 0 \}$. We will compute $\mathcal{L}_1^\dagger \ast$. Take any $b = \{ b_2, b_1 \}$ in $\mathcal{L}_1^\dagger$. Then for all $y \in \text{Domain } \mathcal{T}_0^\dagger$,

$$0 = \overline{b_2}(\mathcal{T}_0^\dagger y + Sy) - b_1(y).$$

This is, in particular, true for all $\{ y, \mathcal{T}_0^\dagger y \}$ with $Sy = 0$. Thus,

$$\{ b_2, b_1 \} \subset (\mathcal{T}_1^\dagger + (\mathcal{N} \oplus \{ 0 \}))^*,$$

and so

$$\{ b_2, b_1 \} \subset \mathcal{T}_1^\dagger \cap (\mathcal{N} \oplus \{ 0 \})^*.$$
Returning to the above equation with \( \{b_2, b_1\} \in T_1^* \) and using the Green's formula, we obtain that

\[
0 = iB(a) \, C(B^\dagger(b))^* + B(a) \, C(b_2(\Psi))^*
\]

for all \( a \in *T_0^* \). Since \( B \) is onto \( l_2^\infty \), and \( C \) is nonsingular, this implies that \( B^\dagger(b) = ib_2(\Psi) \). Therefore, \( \mathcal{L}_1^* = \mathcal{L}_1^\dagger \). Since \( \mathcal{L}_1^\dagger \) is closed, this implies that \( \mathcal{L}_1^* = *\mathcal{L}_1^* \). In particular, \( *\mathcal{L}_1^* \) is a closed operator, and so by page 15 of [2]. Domain \( \mathcal{L}_1^* \) is \( w^* \)-dense. This proves (II) and (III). Part (I) is clear in the course of the proof for (II). This completes the proof.

The following is a dual of the above theorem. We will omit its proof as it is similar to that used to prove Theorem 3.

**Theorem 4.** Let \( T_0, T_1 \) be closed linear manifolds in \( X_1 \oplus X_2 \) such that

1. \( T_0 \subset T_1 \), and \( T_0 \) is complemented in \( T_1 \),
2. the quotient space \( T_1/T_0 \) is isomorphic to \( l_2^\infty \) for some extended integer \( N \),
3. Domain \( T_0 \) is dense in \( X_1 \).

Let \( B \) be a continuous linear operator from \( T_1 \) onto \( l_2^N \) with \( \text{Null } B = T_0 \), and let \( \Psi^t \) be a \( 1 \times N \) matrix whose \( j \)-th entry, \( \Psi_{tj} \), belongs to \( X_1 \) for all \( j \). Define

\[
\mathcal{L}_1 = \{ \{a_1, a_2\} \in T_1 \mid B(\{a_1, a_2\}) = -\overline{\Psi^t(a_1)} \},
\]

where \( \Psi^t(a_1) \) denote the \( 1 \times N \) matrix whose \( j \)-th entry is \( \overline{\Psi_{tj}(a_1)} \). Then we have the following:

1. If \( N < \infty \), then Domain \( \mathcal{L}_1 \) is dense in \( X_1 \).
2. Assume that \( N = \infty \). Suppose further that
   1. \( \sum_1^{\infty} a_j \psi_j^t \) converges in \( X_1^* \) for all \( \{a_j\} \in l_2 \),
   2. \( \{\sum_1^{\infty} a_j \psi_j^t \mid \{a_j\} \in l_2 \} \) is isomorphic to a separable Hilbert space,
   3. \( \{b + \{0, \sum_1^{\infty} a_j \psi_j^t \} \mid b \in T_0^*, \{a_j\} \in l_2 \} \) is \( w^* \)-closed.

Then Domain \( \mathcal{L}_1 \) is dense in \( X_1 \).

3. Suppose that the assumptions in (I) or in (II) are satisfied. Then \( \mathcal{L}_1^* \) is a closed linear operator given by

\[
\mathcal{L}_1^* = \{ b + \{0, B^\dagger(b) \, C^*(\Psi^t)^t \} \mid b \in T_0^* \},
\]

where \( B^\dagger \) is any \( w^* \)-continuous linear operator from \( T_0^* \) onto \( l_2^N \) with \( \text{Null } B^\dagger = T_1^* \), and \( C \) is the \( N \times N \) nonsingular Hilbert matrix such that

\[
\overline{b_2(a_2)} - \overline{b_1(a_1)} = iB(a) \, C(B^\dagger(b))^*
\]

for all \( a = \{a_1, a_2\} \) in \( T_1 \) and \( b = \{b_1, b_2\} \) in \( T_0^* \).
Remark 2. Part (I) of Theorems 3 and 4 when \( T_1 \) and \( T_1^+ \) are operators coincides with Lemma 2.2 of \cite{7} when \( D \) and \( D^* \) there are provided with suitable graph topologies. However, an infinite dimensional generalization of the lemma by use of the same method used in its proof is not possible as it depends on Lemma 2.1 of \cite{4} (also by a different method in Lemma 5.1 of \cite{7}, and Lemma IV.2.8 of \cite{3}), and this does not hold for an infinite dimensional case. Also, part (I) of Theorems 3 and 4 is related to Lemmas 2.5 and 2.7 in \cite{1} (see also p. 37 of \cite{6}) where a necessary and sufficient condition for a linear manifold subject to \textit{finitely} many conditions to be dense is given. We may consider part (II) of Theorems 3 and 4 as an infinite dimensional generalization of Lemma 2.2 of \cite{7} and Lemmas 2.5 and 2.7 of \cite{1}. The Theorems 3 and 4 are false if the ontoness condition is omitted. Also, if \( X_2 \) is a Hilbert space with an inner product \( ( , ) \), then \( b_2(\Psi) \) is replaced by \( (\Psi, b_2) \). Similarly, if \( X_1 \) is a Hilbert space with an inner product \( ( , ) \), then \( \Psi^T(a_1) \) is replaced by \( (a_1, \Psi^T) \).

4. Example

We will give a nontrivial example of Theorem 4 when \( T_1 \) is the graph of a 2nd-order differential operator. Let \( \{(a_j, b_j) | j \in \mathbb{N}\} \) be a set of disjoint, open, bounded intervals such that for some \( c > 0, M > 0, \)

\[
\varepsilon \leq b_j - a_j \leq M < \infty, \quad \text{all } j \in \mathbb{N}.
\]

Put \( I = \bigcup (a_j, b_j) \), where \( j \) runs through \( \mathbb{N} \). Let

\[
\mathcal{D}_1 = \{ y \in L_2(I) \mid \text{for all } j \in \mathbb{N}, y \text{ is continuously differentiable on } (a_j, b_j), y' \in AC_{loc}(a_j, b_j) \text{ and } y'' \in L_2(I) \},
\]

where \( y' \) denotes the derivative of \( y \). Let \( \phi_{kj}, \Psi_{kj} (k = 0, 1; j \in \mathbb{N}) \) be elements in \( L_2(I) \) such that

(i) for \( k = 0, 1 \) and \( (a_j) \in l_2 \), \( \sum_{j=1}^{\infty} a_j \phi_{kj} \) and \( \sum_{j=1}^{\infty} a_j \Psi_{kj} \) converge in \( L_2(I) \), and the set

\[
\left\{ \sum_{j=1}^{\infty} \left( a_{0j} \Psi_{0j} + a_{1j} \Psi_{1j} + \beta_{0j} \phi_{0j} + \beta_{1j} \phi_{1j} \right) \right\}
\]

\[
(a_{0j}), (a_{1j}), (\beta_{0j}), (\beta_{1j}) \text{ are in } l_2.
\]

is closed in \( L_2(I) \).
(ii) for all but finitely many \( j \), and \( k = 0, 1 \),

\[
\Psi_{kj} \in \mathcal{D}_1, \quad \phi_{kj} \in \mathcal{D}_1,
\]

\[
\Psi_{kj}(a_j+) = \Psi_{kj}'(a_j+) = \phi_{kj}(b_j-) = \phi_{kj}'(b_j-) = 0.
\]

Then the set

\[
\left\{ y \in \mathcal{D}_1 \left| y^{(k)}(a_j+) + \int_I y^\prime \phi_{kj} \, dx = 0, \quad y^{(k)}(b_j-) + \int_I y \Psi_{kj} \, dx = 0, \text{ for all } j \in \mathbb{N}, k = 0, 1 \right. \right\}
\]

is dense in \( L_2(I) \). Moreover, this set is the domain of \( \mathcal{L}^* \) where \( \mathcal{L} \) is the operator defined on \( \mathcal{D}_1 \) by

\[
\mathcal{L} y = y''(x) - \sum_{i=1}^{\infty} (y'(a_j+) \phi_{0j} - y(a_j+) \phi_{ij} \quad - y'(b_j-) \Psi_{0j} + y(b_j-) \Psi_{ij})).
\]

Moreover, \( \mathcal{L} \) is closed.

**Proof.** Define \( T_1 = \{ \{ y, y'' \} \mid y \in \mathcal{D}_1 \} \). Then \( T_1 \) is closed in \( X_1 \oplus X_2 \) where \( X_1 = X_2 = L_2(I) \). Define an operator \( B \) on \( T_1 \) by

\[
B(\{ y, y'' \}) = \{ y(a_j+), y'(a_j+), y(b_j-), y'(b_j-), \ldots, y(a_j+), y'(a_j+), y(b_j-), y'(b_j-), \ldots \}.
\]

Then the domain of the Null of \( B \) is dense and \( B \) defines a bounded linear operator onto \( l_2 \) (Example 4.1 of [8]). Let \( \Psi \) be the \( 1 \times \infty \) matrix

\[
\{ i\phi_{01}, i\phi_{11}, i\psi_{01}, i\psi_{11}, \ldots, i\phi_{0j}, i\phi_{ij}, i\psi_{0j}, i\psi_{ij}, \ldots \}.
\]

Now take \( B^* = B \), and let \( C \) be the \( \infty \times \infty \) unitary matrix

\[
\bigoplus_{i} \left[ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right].
\]

Then

\[
\int_I (zy'' - z''\overline{y}) \, dx = iB(\{ y, y'' \}) C(B(\{ z, z'' \}))^*
\]

for all \( y, z \) in \( \mathcal{D}_1 \).
We now apply Theorem 4 to get the result. Notice that the T-norm of the perturbed term in the definition of $\mathcal{L}_1$ can be arbitrary by choosing $\phi_{kj}$ suitably. Thus, Theorem IV.1.1 of [5] and Lemma V.3.5 of [4] are not applicable to show that $\mathcal{L}_1$ is closed. However, it is closed by our theorem.

REFERENCES